1 First-order, linear ODE

1.1 Every first-order, linear ODE is of the form

$$\frac{dy}{dt} + p(t)y = q(t) \tag{1.1}$$

Solution of first-order linear equations

1. Determine the integrating factor

$$\mu(t) = \exp\left(\int p(s)\, ds\right). \tag{1.2}$$

   No integration constant is needed here.

2. Multiply (1.1) by $\mu(t)$ to obtain

$$\frac{d}{dt}\Big(\mu(t)y(t)\Big) = \mu(t)q(t)$$

3. Integrate on both sides

$$\mu(t)y(t) = F(t) + c, \tag{1.3}$$

   where $F(t)$ is the antiderivative of the product $\mu(t)q(t)$.

4. Solve for y:

$$y(t) = \frac{F(t) + c}{\mu(t)}.$$

5. If solving an IVP, use the initial value to determine the constant c

1.2 In order to solve an IVP, we can use "definite integration", replacing (1.2) by

$$\mu(t) = \exp\left(\int_{t_0}^{t} p(s)\, ds\right).$$

Furthermore, knowing $\mu(t_0) = 1$, equation (1.3) is replaced by

$$\mu(t)y(t) - y_0 = \int_{t_0}^{t} \mu(t)q(t)\, dt$$

1.3 **Example:** solve the IVP

$$y' + \frac{y}{t} = 1, \qquad y(2) = 2.$$

Solution:

1. The integrating factor is

$$\mu(t) = \exp\left(\int \frac{1}{s}\, ds\right) = e^{\ln|t|} = |t|.$$

2. Since the initial time is positive, we consider $t > 0$, thus $|t| = t$. The modified equation is

$$\frac{d}{dt}\Big(\mu(t)y(t)\Big) = t.$$

3. Integration on both sides yields

$$ty(t) = \frac{1}{2}t^2 + c.$$

4. Dividing by $\mu(t)$ yields

$$y(t) = \frac{t}{2} + \frac{c}{t}.$$

5. Matching the initial condition:

$$2 = y(2) = 1 + \frac{c}{2} \qquad \Rightarrow \quad c = 2 \quad \text{and} \quad y(t) = \frac{t}{2} + \frac{2}{t}.$$

Alternative solution using dfinite integration:

1. The integrating factor is

$$\mu(t) = \exp\left(\int_2^t \frac{1}{s}\, ds\right) = e^{\ln|t| - \ln 2} = \frac{|t|}{2}.$$

2. Since the initial time is positive, we consider $t > 0$, thus $|t| = t$. The modified equation is

$$\frac{d}{dt}\Big(\mu(t)y(t)\Big) = \frac{t}{2}.$$

3. Integration on both sides yields

$$\frac{t}{2}y(t) - 2 = \frac{1}{4}(t^2 - 4).$$

4. Solving for y yields

$$y(t) = \frac{2}{t}\left(\frac{t^2}{4} - 1 + 2\right) = \frac{t}{2} + \frac{2}{t}.$$