1 Eigenvalues and eigenvectors of 2x2 matrices

1.1 Quadratic equations: The solutions of a quadratic equation

$$ax^2 + bx + c = 0$$

are obtained by the solution formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Depending on the discriminant $\Delta = b^2 - 4ac$, we obtain

$\Delta > 0$ **:** two real solutions,

$\Delta = 0$ **:** one real solution,

$\Delta < 0$ **:** two complex conjugate solutions.

1.2 Definition: A number λ is called an eigenvalue of the matrix $\mathbf{A}$ with associated eigenvector $\mathbf{v}$, if the equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

holds. This means, that applying the matrix $\mathbf{A}$ to the eigenvector $\mathbf{v}$ has the same effect as scaling the vector by the factor λ without changing its direction.

Computation of eigenvalues and eigenvectors

The eigenvalues and eigenvectors of a general 2×2-matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

can be computed by the following steps:

1. Set up the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}.$$

2. Using the abbreviations $b = -a_{11} - a_{22}$ and $c = a_{11}a_{22} - a_{12}a_{21}$, the solutions are obtained by the solution formula for quadratic equations. Depending on the discriminant, we obtain

   (a) two real solutions λ_1 and λ_2,

   (b) one real solution λ, or

   (c) two complex conjugate solutions $\mu \pm i\nu$.

3. After computing the eigenvalues, we compute eigenvectors. We distinguish the same cases as above:

   (a) For each eigenvalue λ_1 and λ_2, compute a nonzero eigenvector $\mathbf{v}$ by solving the system of equations

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0. \tag{1.1}$$

   In order to do this, chose one of the rows of the system (they are multiples of each other). Then, choose either the first or the second component of the vector $\mathbf{v}$ and compute the second component from that row.

   (b) Here we have to distinguish two cases: if the matrix is diagonal, it has the form $\lambda\mathbf{I}$. Then, any vector is an eigenvector of $\mathbf{A}$. Therefore, we can just choose two, which are not multiples of each other.

   If the matrix is not diagonal, there is only one eigenvector $\mathbf{v}$ and we have to compute a generalized eigenvector $\mathbf{w}$ in addition:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0,$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}.$$

   (c) For two complex eigenvalues $\mu \pm i\nu$, we compute the (complex) eigenvector for one through equation (1.1). The eigenvector to the other eigenvalue is its complex conjugate. Thus, if the computed eigenvector has the real part $\mathbf{a}$ and the imaginary part $i\mathbf{b}$ ($\mathbf{a}$ and $\mathbf{b}$ are vectors), the two eigenvectors are $\mathbf{a} \pm i\mathbf{b}$

2 First-order, linear, homogeneous 2x2-systems of ODE with constant coefficients

2.1 The general form of these systems is

$$x_1' = a_{11}\,x_1 + a_{12}\,x_2$$
$$x_2' = a_{21}\,x_1 + a_{22}\,x_2$$

or in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{x}$ is a vector function with two components and $\mathbf{A}$ is a 2×2-matrix.

Solution of the ODE systems and IVP

If the matrix $\mathbf{A}$ is diagonal, the system is decoupled and we can solve for each component separately. Else, the following method provides the solution:

1. Compute the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.
2. Depending on the three cases we distinguish with eigenvalues and eigenvectors, we obtain three different sets of two independent solutions each:
   (a) Two exponential solutions
   $$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1,$$
   $$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2.$$
   (b) Two solutions involving the eigenvector and the generalized eigenvector (notation as in section 1):
   $$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v},$$
   $$\mathbf{x}_2(t) = te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}.$$
   (c) Two solutions involving the real and imaginary part of the eigenvectors (notation as in section 1):
   $$\mathbf{x}_1(t) = e^{\mu t}\big(\cos(\nu t)\,\mathbf{a} - \sin(\nu t)\,\mathbf{b}\big),$$
   $$\mathbf{x}_2(t) = e^{\mu t}\big(\sin(\nu t)\,\mathbf{a} + \cos(\mu t)\,\mathbf{b}\big),$$
3. The general solution of the equation has the form
   $$\mathbf{x}(t) = c_1\,\mathbf{x}_1(t) + c_2\,\mathbf{x}_2(t),$$
   with two constants c_1 and c_2.
4. If you solve an IVP, at the very end, identify the constants c_1 and c_2 using the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Since this is an equation between vectors, it amounts to two equations, needed to identify both constants.