Problem 18 (efficiency of Richardson's method) :

(a) Let $\kappa = \operatorname{cond}_2(A)$ be the condition number of the symmetric, positive definite matrix A. Show that the contraction number for Richardson's method with optimal parameter (see last homework) depends on κ like

$$\rho = 1 - \frac{c}{\kappa} + \mathcal{O}(\kappa^{-2}).$$

(b) Show that the number of steps you need to reduce the norm of the error by a certain factor is $\mathcal{O}(\kappa)$.

Problem 19 (Richardson's method for general matrices) : Show that Richardson's method converges for a matrix A, if and only if all Eigenvalues of A have positive real part. Here, "converges" means that you can find a $\theta > 0$, such that the iteration converges.

Hints: Look at the spectral radius $\rho(I - \theta A)$. A visualization of the eigenvalues of A and $I - \theta A$ might help.

Problem 20 (spectrum of the discrete Poisson operator (5-point stencil)) : Consider the matrix A in equation (8.4.5) in the textbook. Below, $N + 1 = 1/h$ refers to the number of intervals of length h in $[0, 1]$ used to define the point grid.

(a) Show with minimal computation, that the matrix is positive semi-definite (all eigenvalues are greater or equal zero).

(b) Verify that the vectors

$$v_{ij}^{(kl)} = \sin\frac{ik\pi}{N+1}\sin\frac{jl\pi}{N+1} \qquad\qquad i, j = 1, \ldots, N$$

are eigenvectors of A and compute the associated eigenvalues $\lambda^{(kl)}$. Here, the number of the eigenvalue is expressed by the pair kl, while ij is the vector index. Both indices are chosen two-dimensional, since this makes for a more convenient writing of λ and v. In particular, we have N^2 pairs and each vector has N^2 entries.

(c) Determine the spectral condition number of A in terms of h (up to higher order terms).

(d) If between two experiments you divide h , how does the number of Richardson steps for solving $Ax = b$ change if we try to achieve the same accuracy?