First Midterm Exam MATH 609-601

Instructor: Guido Kanschat, Blocker 505C

Thursday, October 22nd

Name:____**UIN:**____

Please solve the midterm exam alone, without your study group. You may use the following help to do so:

- The textbook
- Your class notes
- Your graded homework
- You may ask your instructor by email for clarification or for the solution of a substep (the latter implies giving up the credit for this step).

By submitting your solution, you affirm that you prepared your answers only with the help listed above and that you am bound to the Aggie Honor Code “An Aggie does not lie, cheat or steal, or tolerate those who do”.

Problem 1 (17 pts.):

(a) Compute the Lagrange interpolation polynomials for the set of interpolation points

$$x_0 = 0 \qquad x_1 = \frac{\pi}{2} \qquad x_2 = \pi.$$

(b) Compute the interpolating polynomial $p(x)$ for $\sin x$.

(c) Estimate the error at the point $x = \pi/4$ and compare it to the actual error.

(d) Approximate the integral of $\sin(x)$ on $I = [0, \pi]$ by the Simpson rule. Compare to the integral of $p(x)$ and explain.

(e) Interpolate the function $\sin(2x)$. Based on your result, what would you recommend for the interpolation of oscillating functions?

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Problem 2 (15 pts.): Show that a differentiable mapping Φ of the compact, convex set $S \subset \mathbb{R}^n$ into itself is a contraction with contraction number α, that is

$$\|\Phi(x) - \Phi(y)\|_2 \le \alpha \|x - y\|_2$$

if and only if the operator norm of its derivative is bounded by

$$\sup_{x \in S} \|\nabla \Phi(x)\| = \alpha, \tag{a}$$

where $\|.\|$ is the operator norm for the Euclidean norm. In order to do so, proceed through the following steps:

(a) For two points $x, y \in S$, introduce the auxiliary function

$$\varphi(t) = \big\|\Phi(x) - \Phi(x + t(y - x))\big\|_2,$$

and explain why it is related to the problem.

(b) Use $\varphi(t)$ and its derivative $\varphi'(t)$ to estimate $\|\Phi(x) - \Phi(y)\|$ and thus prove the contraction property. A contradiction argument might help.

(c) On the other hand, use the definition of the gradient to show that the contraction number cannot be bounded by any number less than the α in equation (a). To this end, you may use, that for the operator norm of a matrix A, you can always find a vector v with $\|v\| = 1$, such that

$$\|Av\| = \sup_{\|x\|=1} \|Ax\|,$$

thus, the supremum is a maximum.

Problem 3 (18 pts.): Given a symmetric, positive definite $n \times n$-matrix A, we define the scalar product

$$(x, y)_A := x^T A y,$$

and the norm $\|.\|_A$ by $\|x\|_A^2 = (x, x)_A$.

Consider the sequence of vectors constructed by the following process: start with a vector $v^{(0)}$ with $\|v^{(0)}\|_A = 1$. Once the vectors $v^{(0)}$ to $v^{(k)}$ are known, compute $v^{(k+1)}$ by the following rules

$$\begin{aligned} w &:= Av^{(k)}, \\ \tilde{v} &:= w - \sum_{i=0}^{k} (w, v^{(i)})_A \, v^{(i)}, \\ v^{(k+1)} &:= \tilde{v} / \|\tilde{v}\|_A, \end{aligned}$$

(a) Show that $(.,.)_A$ indeed defines a scalar product on $\mathbb{R}^n$. You may look up the definitions of “scalar product” and the matrix properties, but not the proof of this statement.

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(b) Show that the sequence obtained has the orthogonality property

$$(v^{(i)}, v^{(k)})_A = \delta_{ik}.$$

(Show first, that $(v^{(i)}, v^{(i)}) = 1$, which is the easier part.)

(c) Perform the method for the initial values and matrix

$$v^{(0)} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Use of technology is explicitly permitted! Reported numbers may be rounded to four digits.

(d) How many different vectors do you obtain from this method and why not more?

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