Second Midterm Exam MATH 308–514

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Tuesday, April 20th

Name:____

By submitting my solution, I affirm that I prepared my answers only with the help allowed by the instructor and that I am bound to the Aggie Honor Code "An Aggie does not lie, cheat or steal, or tolerate those who do".

If you need the answer to a substep of the solution to continue, you can ask for this solution during the exam, but you give up the credit points for this substep.

These integrals might prove useful:

$$\int t^2 \sin at\, dt = \frac{2t}{a^2} \sin at - \frac{a^2t^2 - 2}{a^3} \cos at, \qquad \int t^2 \cos at\, dt = \frac{2t}{a^2} \cos at + \frac{a^2t^2 - 2}{a^3} \sin at$$

Problem 1 (12 pts.): Use the method of undetermined coefficients to compute a particular solution to

$$y'' + 4y' + 4y = 6e^{-2t}.$$

Problem 2 (16 pts.): Solve the IVP

$$y'' - 2y' - 3y = 10 \cos t, \qquad y(0) = 5, \quad y'(0) = 0.$$

Use the method of undetermined coefficients and the two solutions $y_1(t) = e^{-t}$ and $y_2(t) = e^{3t}$ of the homogeneous equation $y'' - 2y' - 3y = 0$.

Problem 3 (13 pts.): Use variation of parameters to find a particular solution to the equation

$$y'' + 4y = 2t^2.$$

Problem 4 (13 pts.): Find and characterize all critical points of the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y^2 - 1 \\ x - y^2 \end{pmatrix}.$$

Problem 5 (18 pts.): Find and characterize all critical points of the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ \kappa y + \sin x \end{pmatrix},$$

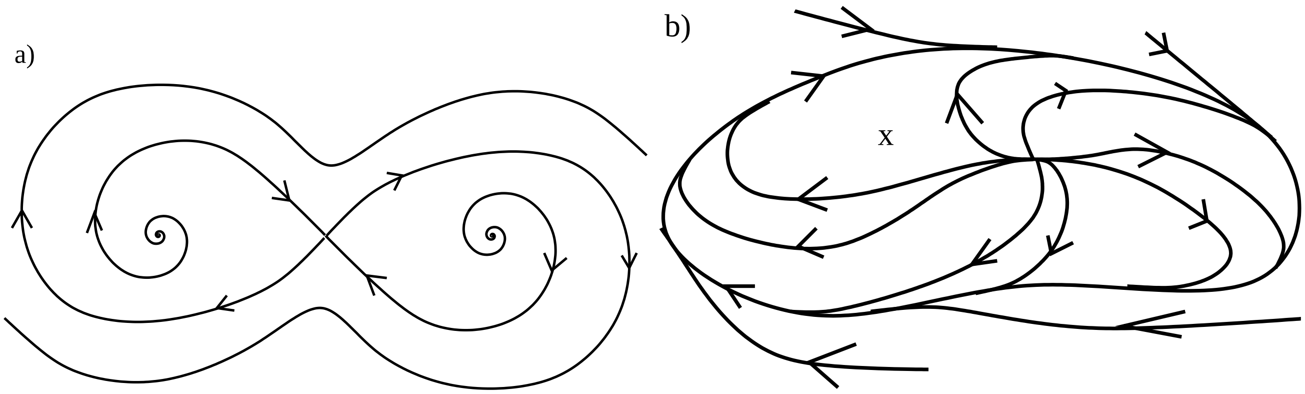
depending on the parameter κ.

Problem 6 (10 pts.): Find the critical points of the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x(x^2+y^2) \\ y(x^2+y^2) \end{pmatrix},$$

and compute the gradient of the right hand side (the Jacobian) at the critical point. Is this critical point stable or unstable and give reasons? In order to answer this question, look at what happens to a solution which starts close to the critical point in the phase plane.

Problem 7 (11 pts.): For each of the following phase portraits, describe the asymptotic properties of the underlying system of ODE (i.e. critical points, limit cycles, stability). For problem b), what will the x and y component of the solution do if t goes to infinity?



Problem 8 (6 pts.): Derive an explicit formula for the Laplace transform $Y(s)$ of the solution to the IVP

$$y''' - 2y' + y = e^t \qquad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -1.$$

Problem 9 (5 pts.): Compute the inverse Laplace transform of

$$Y(s) = \frac{s-19}{(s^2-3s-10)}$$

Problem 10 (8 pts.): Compute the inverse Laplace transform of

$$Y(s) = \frac{s^2+4s+5}{(s^2-1)(s-1)}$$

Problem 11 (8 pts.): Use Laplace transform to transform the following IVP into an algebraic system:

$$x'' - y' + 3x = e^{-t}\cos 4t$$
$$y''' + x' - 2y = t^4,$$

with $x(0) = 2$, $x'(0) = y(0) = 1$, and $y''(0) = y'(0) = 0$.

Total: 120 pts.

Solution 1 (12 pts.)

(a) (3 pts.) The characteristic equation has the single solution $\lambda = -2$.

(b) (2 pts.) Two linearly independent solutions are $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$

(c) (2 pts.) Our guess is $y_p(t) = at^2e^{-2t}$.

(d) (2 pts.) The derivatives are

$$y_p'(t) = a(2t - 2t^2)e^{-2t}$$
$$y_p''(t) = a(2 - 8t + 4t^2)e^{-2t}$$

(e) (2 pts.) Entering in the equation yields $2a = 6$.

(f) (1 pt.) The solution is

$$y_p = 3t^2e^{-3t}.$$

Solution 2 (16 pts.)

(a) (2 pts.) Our guess is $y_p(t) = a\cos t + b\sin t$, which is not a solution to the homogeneous problem.

(b) (2 pts.) The derivatives are

$$y_p'(t) = -a\sin t + b\cos t$$
$$y_p''(t) = -a\cos t - b\sin t$$

(c) (2 pts.) Entering y_p into the equation yields

$$-a\cos t - b\sin t + 2a\sin t - 2b\cos t - 3a\cos t - 3b\sin t =$$
$$(-2b - 4a)\cos t + (2a - 4b)\sin t = 8\cos t$$

(d) (4 pts.) Therefore, $a = 2b$ and $-10b = 10$, thus

$$b = -1 \qquad a = -2.$$

(e) (2 pts.) The general solution and its derivative are

$$y(t) = c_1e^{-t} + c_2e^{3t} - 2\cos t - \sin t,$$
$$y(t)' = -c_1e^{-t} + 3c_2e^{3t} + 2\sin t - \cos t.$$

(f) (2 pts.) Matching the initial condition yields the system

$$5 = c_1 + c_2 - 2$$
$$0 = -c_1 + 3c_2 - 1$$

(g) (2 pts.) The solution is

$$c_1 = 5 \qquad c_2 = 2.$$

Solution 3 (13 pts.)

(a) (3 pts.) The solutions to the homogeneous equation $y'' + 4y = 0$ are $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$.

(b) (1 pt.) The system to determine $\mathbf{u}'$ is

$$\begin{aligned} \cos 2t u_1' + \sin 2t u_2' &= 0 \\ -2\sin 2t u_1' + 2\cos 2t u_2' &= t^2. \end{aligned}$$

(c) (3 pts.) The first equation yields $u_1' = -\sin 2t / \cos 2t u_2'$. Entering into the second equation yields

$$\left(2\frac{\sin 2t}{\cos 2t} + 2\cos 2t \right) u_2' = \frac{2}{\cos 2t} u_2' = 2t^2.$$

(d) (2 pts.) Thus,

$$u_1' = -t^2 \sin 2t, \qquad u_2' = t^2 \cos 2t.$$

(e) (2 pts.) Their integrals are on the top of the problem sheet

$$u_1 = -\frac{t}{2}\sin 2t + \frac{2t^2-1}{4}\cos 2t, \qquad u_2 = \frac{t}{2}\cos 2t + \frac{2t^2-1}{4}\sin 2t,$$

(f) (2 pts.) Entering into the formula for y_p yields

$$y_p = -\frac{t}{2}\sin 2t\cos 2t + \frac{2t^2-1}{4}\cos^2 2t + \frac{t}{2}\sin 2t \cos 2t + \frac{2t^2-1}{4}\sin^2 2t = \frac{2t^2-1}{4}.$$

Solution 4 (13 pts.)

(a) (4 pts.) The critical points are the two points with coordinates and $x = 1$ and $y = \pm 1$.

(b) (5 pts.) The Jacobian and the equation for its eigenvalues are

$$\nabla F = \begin{pmatrix} 0 & 2y \\ 1 & -2y \end{pmatrix}, \qquad \lambda^2 + 2y\lambda - 2y = 0$$

(c) (2 pts.) For $y = 1$, we obtain $\lambda = -1 \pm \sqrt{1+2}$, thus we have a saddle point

(d) (2 pts.) For $y = -1$, we obtain $\lambda = 1 \pm \sqrt{1-2}$, thus we have an unstable spiral point.

Solution 5 (18 pts.)

(a) (2 pts.) The critical points have the coordinates $x = k\pi$ and $y = 0$, where $k = ..., -2, -1, 0, 1, 2, ...$

(b) (4 pts.) The Jacobian is

$$\nabla F = \begin{pmatrix} 0 & 1 \\ \cos x & \kappa \end{pmatrix}.$$

(c) (5 pts.) For $x = ..., -2\pi, 0, 2\pi, 4\pi, ...$, the eigenvalues are the solutions to $\lambda^2 - \kappa\lambda - 1$, namely $(\kappa \pm \sqrt{\kappa^2+4})/2$. Since the root is greater than κ, these are saddle points.

(d) (7 pts.) For $x = ..., -\pi, \pi, 3\pi, ...$, the eigenvalues are the solutions to $\lambda^2 - \kappa\lambda + 1$, namely $(\kappa \pm \sqrt{\kappa^2 - 4})/2$. For $|\kappa| < 2$, the discriminant is negative, thus it is a stable spiral point. For $|\kappa| \geq 2$, it is a node, stable for negative κ, unstable for positive.

Solution 6 (10 pts.)

(a) (2 pts.) The only critical point is the origin.

(b) (4 pts.) The gradient is the zero matrix.

(c) (4 pts.) The critical point is unstable, since a perturbation in any direction will grow.

Solution 7 (11 pts.)

(a) (6 pts.) a) has two unstable spiral points and a saddle point in the center.

(b) (4 pts.) b) has an unstable node in the center and a stable limit cycle.

(c) (1 pt.) The solutions in b) will oscillate periodically as $t \to \infty$.

Solution 8 (6 pts.)

(a) (4 pts.) The equation is transformed to

$$(s^3 - 2s + 1)Y - s + 1 = \frac{1}{s-1}.$$

(b) (2 pts.) Bringing everything to the right yields

$$Y(s) = \frac{1 + (s-1)^2}{(s^3 - 2s + 1)(s-1)} = \frac{s^2 - 2s + 2}{(s^3 - 2s + 1)(s-1)}$$

Solution 9 (5 pts.)

(a) (1 pt.) The denominator factors into $(s+2)(s-5)$.

(b) (1 pt.) The partial fraction expansion is $s - 19 = A(s-5) + B(s+2)$

(c) (3 pts.) The coefficients are $A = 3$ and $B = -2$. Looking up the inverse transform yields

$$y(t) = 3e^{-2t} - 2e^{5t}.$$

Solution 10 (8 pts.)

(a) (1 pt.) The denominator factors into $(s+1)(s-1)^2$.

(b) (1 pt.) The partial fraction expansion is $s^2 + 4s + 5 = A(s+1) + B(s+1)(s-1) + C(s-1)^2$

(c) (1 pt.) Choosing $s = 1$ yields $A = 10/2 = 5$.

(d) (1 pt.) Choosing $s = -1$ yields $C = 2/4 = 1/2$.

(e) (1 pt.) Choosing $s = 0$ yields $-B = 5 - A - C$ and thus $B = -1/2$

(f) (3 pts.) Looking up the inverse transform yields

$$y(t) = 5te^t - \tfrac{1}{2}e^t + \tfrac{1}{2}e^{-t}$$

Solution 11 (8 pts.)

(a) (8 pts.)

$$
\begin{aligned}
s^2X - 2s - 1 - sY + 1 + 3X &= \frac{s+1}{(s+1)^2 + 16} \\
s^3Y - s^2 + sX - 2 - 2Y &= \frac{24}{s^5}
\end{aligned}
$$