Local Solvability of Laplacian Difference Operators Arising From the Discrete Heisenberg Group

A notion of differential operators can be developed using the action of a finitely generated group G on a set S. On a discrete group, these are difference operators with variable differences. If S is equipped with an invariant measure μ, these operators can be associated to unitary representations of G on the Hilbert space $L^2(S, \mu)$. The discrete Heisenberg group G is composed of elements (x, y, z) with $x, y, z \in \mathbb{Z}$, so the natural difference operators will be D_x, D_y, and D_z and the resulting Laplacian is the operator $L = D_x^2 + D_y^2 + D_z^2$. We seek in general to solve the difference equation $Lu = f$. In particular, we give a definition of local solvability of an operator on a discrete space and determine whether L is in fact locally solvable. The procedure utilizes techniques of harmonic analysis to decompose the associated representation of G into subrepresentations on simpler spaces. The Laplacian is similarly decomposed, and under certain circumstances we are able to both solve the factor equations and combine the solutions to construct a solution to the original equation.