EQUIVALENCE CLASSES OF SPHERICAL TIGHT FRAMES

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Abstract: A *frame* is a list of vectors $F = (f_i)_{i\in I}$ in a Hilbert space $\mathcal{H}$ satisfying

$$A\|v\|^2 \leq \sum_{i\in I} |\langle v, f_i\rangle|^2 \leq B\|v\|^2 \qquad (v \in \mathcal{H}) \tag{1}$$

for some constants $0 < A \leq B$; the optimal such constants are called the *frame bounds* of F. The frame F is finite if the index set I is finite, which implies $\mathcal{H}$ is finite dimensional. An example is an orthonormal basis; however, in general a frame may have redundancies, and these are essential in many recent applications of frames (including finite frames) in the communications industry. The frame F is said to be *tight* if the constants A and B in (1) can be taken to be equal to each other. We consider finite frames, in both the real and complex cases, i.e. $\mathcal{H} = \mathbf{E}^n$ for $\mathbf{E} = \mathbf{R}$ or $\mathbf{E} = \mathbf{C}$. We will be primarily interested in frames all of whose vectors f_i lie on the unit sphere of $\mathbf{E}^n$, i.e. the *spherical tight frames*. Our focus will be on the set $\mathcal{F}^{\mathbf{E}}_{k,n}$ of all spherical tight frames of k vectors in $\mathbf{E}^n$, for $k > n$, and in particular on the topological questions of connectedness and the manifold structure of $\mathcal{F}^{\mathbf{E}}_{k,n}$.

The technical key to our results is to consider the orbit space $\mathcal{G}^{\mathbf{E}}_{k,n} = \mathcal{F}^{\mathbf{E}}_{k,n}/\mathcal{O}^{\mathbf{E}}_n$ for the obvious action of the group of inner–product preserving transformations $\mathcal{O}^{\mathbf{E}}_n$ of the Hilbert space $\mathbf{E}^n$. (Thus, $\mathcal{O}^{\mathbf{R}}_n$ is the group of $n\times n$ orthogonal matrices, and $\mathcal{O}^{\mathbf{C}}_n$ is the group of $n\times n$ unitary matrices.) We observe that the quotient map $\mathcal{F}^{\mathbf{E}}_{k,n} \to \mathcal{G}^{\mathbf{E}}_{k,n}$ is a locally trivial fiber bundle (with fibers $\mathcal{O}^{\mathbf{E}}_n$) and that $\mathcal{G}^{\mathbf{E}}_{k,n}$ can be naturally identified with the subset of the Grassman manifold of n–planes in $\mathbf{E}^k$ consisting of projections all of whose diagonal entries are equal to n/k. An important consequence is that $\mathcal{G}^{\mathbf{E}}_{k,n}$ and $\mathcal{G}^{\mathbf{E}}_{k,k-n}$ are homeomorphic.

Both $\mathcal{F}^{\mathbf{E}}_{k,n}$ and $\mathcal{G}^{\mathbf{E}}_{k,n}$ are real algebraic sets. By classical results of Whitney, each of these can, therefore, be written as a disjoint union of finitely many manifolds. We explicitly describe such a decomposition. When k and n are relatively prime, we show that $\mathcal{G}^{\mathbf{E}}_{k,n}$ is itself a real analytic manifold, and, therefore, so is $\mathcal{F}^{\mathbf{E}}_{k,n}$. When n and k are not relatively prime, $\mathcal{G}^{\mathbf{E}}_{k,n}$ is written as a disjoint union of manifolds, corresponding to block diagonal decompositions of projections. We get a similar description of $\mathcal{F}^{\mathbf{E}}_{k,n}$. In particular, we say a tight frame $F = (f_1, \ldots, f_k)$ for $\mathbf{E}^n$ is *orthodecomposable* if the vectors in F can be partitioned into proper sublists which form tight frames for orthogonal subspaces of $\mathbf{E}^n$. Let $\hat{M}^{\mathbf{E}}_{k,n}$ be the set of spherical tight frames in $\mathcal{F}^{\mathbf{E}}_{k,n}$ that are not orthodecomposable. Then $\hat{M}^{\mathbf{E}}_{k,n}$ is a nonempty manifold, and $\mathcal{F}^{\mathbf{E}}_{k,n}$ is the union of $\hat{M}^{\mathbf{E}}_{k,n}$ together with other manifolds (of lower dimension) corresponding to orthodecomposability according to certain partitions.

Another consequence of Whitney's results is that $\mathcal{F}^{\mathbf{E}}_{k,n}$ and $\mathcal{G}^{\mathbf{E}}_{k,n}$ have only finitely many connected components. By considering the rearrangement of chains in $\mathbf{R}^2$, we prove that the space $\mathcal{F}^{\mathbf{R}}_{k,2}$ of tight spherical frames of k vectors in $\mathbf{R}^2$ is connected for all $k \geq 4$, and from this result we obtain that the set $\mathcal{F}^{\mathbf{R}}_{n+2,n}$ of real tight spherical frames with two redundant vectors is connected, for all $n \geq 2$.

We also consider two examples in detail: $\mathcal{G}^{\mathbf{R}}_{4,2}$ and $\mathcal{G}^{\mathbf{R}}_{5,2}$ (the latter of which is homeomorphic to $\mathcal{G}^{\mathbf{R}}_{5,3}$). We find that $\mathcal{G}^{\mathbf{R}}_{4,2}$ is a graph with twelve vertices and twenty–four edges, and $\mathcal{G}^{\mathbf{R}}_{5,2}$ is the oriented surface of genus 25.