On the geometry of projections in von Neumann algebras

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In a C^*-algebra, the norm estimate $\|p-q\| \leq 1$ holds for any pair of projections. If $\|p-q\| < 1$, then p and q are homotopic, i.e. they can be connected by a continuous projection-valued path. This is a well-known folklore result which takes place at the beginning of every introduction to the K-theory of C^*-algebras (e.g. [2]).

Conversely, if p and q are homotopic, then an obvious continuity-compacity argument shows that there exists a sequence $p = p_0, p_1, \ldots, p_n = q$ of projections such that $\|p_i - p_{i+1}\| < 1$ for all i. We denote $\delta(p,q)$ the minimum of all integers n fulfilling the latter condition. Also we set $\Delta(A) := \sup \delta(p,q)$, where the supremum runs over all pairs of homotopic projections.

For a general C^*-algebra A, the constant $\Delta(A)$ - which may be seen as a diameter of the Grassmann space - may either be finite or infinite. But we know from a slight modification of [7] that $\Delta(A)$ is uniformly bounded for every von Neumann algebra A. It was proved in [4] that $\Delta(A) = 3$ for $A = B(H)$ with H infinite-dimensional real or complex Hilbert space.

Actually the estimate $\Delta(A) \leq 3$ holds for any von Neumann algebra and we have the following characterization: $\Delta(A) \leq 2$ if and only if A is finite.

We will mostly explain how these results arose from the pioneering work of Kovarik [5], Zemánek [8], Aupetit [1], Esterle [3] and Trémon [6] on the idempotents of a Banach algebra.

References

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