MATH 220.903
Practice Problems for Examination 2
Fall 2005

1. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be *bounded* if there exists a real number $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Let $\mathbf{B}(\mathbb{R}) = \{f \in \mathbf{F}(\mathbb{R}) \mid f \text{ is bounded on } \mathbb{R}\}$. Prove that $\mathbf{B}(\mathbb{R})$ is closed under addition and multiplication.

2. Determine, with proof, which of the following sets are closed in $\mathbf{F}(\mathbb{R})$ under multiplication.

   (a) $\{f \in \mathbf{F}(\mathbb{R}) \mid f(-78) = 0\}$.

   (b) $\{f \in \mathbf{F}(\mathbb{R}) \mid f(6) = 10\}$.

   (c) $\{f \in \mathbf{F}(\mathbb{R}) \mid f \text{ is differentiable at 1 and } f'(1) = 0\}$.

   (d) $\{f \in \mathbf{F}(\mathbb{R}) \mid f(0) \geq 10\}$.

   (e) $\{f \in \mathbf{F}(\mathbb{R}) \mid f(0) \geq -10\}$.

   (f) $\{f \in \mathbf{F}(\mathbb{R}) \mid f(-x) = f(x) \text{ for all } x \in \mathbb{R}\}$.

   (g) $\{f \in \mathbf{F}(\mathbb{R}) \mid f \text{ is a constant function}\}$.

3. For each of the binary operations given below, determine, with proof, which of the following properties hold: associativity, commutativity, and the existence of an identity element.

   (a) Let $*$ be the binary operation on $\mathbb{Z}$ given by $a * b = 2a + 5b + 1$ for all $a, b \in \mathbb{Z}$.

   (b) Let $*$ be the binary operation on $\mathbb{R}$ given by $x * y = 3^{x+y}$ for all $x, y \in \mathbb{R}$.

   (c) For all $f, g \in \mathbf{F}(\mathbb{R})$ define $f * g$ to be the constant function in $\mathbf{F}(\mathbb{R})$ with value $f(0)g(0)$.

4. Let A, B, and C be nonempty sets, and let $f \in \mathbf{F}(A, B)$ and $g \in \mathbf{F}(B, C)$. Prove that if gf is injective then the restriction $g|_{\mathrm{Im}(f)}$ is injective.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by a quadratic polynomial, i.e., for some $a, b \in \mathbb{R}$ with $a \neq 0$ we have $f(x) = ax^2 + bx + c$ for all $x \in X$. Suppose that the restriction $f|_{[0,1]}$ is injective. Prove that $f'(x) \neq 0$ for all $x \in (0, 1)$.

6. Give an example of a bijective differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = 0$ for some $x \in \mathbb{R}$.

7. Recall that multiplication in $M_2(\mathbb{R})$ is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

Prove that the set $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \,|\, b = c = 0 \right\}$ is closed under multiplication.

8. Determine, with proof, which of the following functions are injective and which are surjective. Also, compute the images.

   (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^8 + 3x^4 + 2$.

   (b) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^9 + 2x^7 + 7x^3 + x$.

   (c) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} x^2 + 1, & \text{if } x \geq 0, \\ x, & \text{if } x < 0. \end{cases}$

   (d) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} x \sin(\pi x), & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$

   (e) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = \begin{cases} \frac{n+4}{2}, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}. \end{cases}$

Solutions

1. Let $f, g \in \mathbf{B}(\mathbb{R})$. Then there exists an $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$ and an $N > 0$ such that $|g(x)| \leq N$ for all $x \in \mathbb{R}$. Then $|(f+g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M + N$ for all $x \in \mathbb{R}$, and so $f + g \in \mathbf{B}(\mathbb{R})$. We also have $|(f \cdot g)(x)| = |f(x)g(x)| = |f(x)||g(x)| \leq MN$ for all $x \in \mathbb{R}$, and so $f \cdot g \in \mathbf{B}(\mathbb{R})$. Thus $\mathbf{B}(\mathbb{R})$ is closed under addition and multiplication.

2. (a) If $f, g \in \mathbf{F}(\mathbb{R})$ and $f(-78) = g(-78) = 0$ then $(f \cdot g)(-78) = f(-78)g(-78) = 0$, and so the given set is closed under multiplication.

   (b) Define $f, g \in \mathbf{F}(\mathbb{R})$ by $f(x) = x + 4$ and $g(x) = 10$ for all $x \in \mathbb{R}$. Then $f(6) = g(6) = 10$ but $(f \cdot g)(6) = f(6)g(6) = 10^2 = 100$, and so the given set is not closed under multiplication.

   (c) If $f, g \in \mathbf{F}(\mathbb{R})$ are both differentiable at 1 and $f'(1) = g'(1) = 0$, then by the product rule $(f \cdot g)'(1) = (f'g + g'f)(1) = f'(1)g(1) + g'(1)f(1) = 0 + 0 = 0$. Thus the given set is closed under multiplication.

   (d) If $f, g \in \mathbf{F}(\mathbb{R})$ and $f(0) \geq 10$ and $g(0) \geq 10$ then $(f \cdot g)(0) = f(0)g(0) \geq 10^2 \geq 10$, and so the given set is closed under multiplication.

   (e) The functions $f, g \in \mathbf{F}(\mathbb{R})$ defined by $f(x) = -10$ and $g(x) = 10$ for all $x \in \mathbb{R}$ satisfy $f(0) \geq -10$ and $g(0) \geq -10$, but $(f \cdot g)(0) = f(0)g(0) = -100 \not\geq -10$. Thus the given set is not closed under multiplication.

   (f) If $f, g \in \mathbf{F}(\mathbb{R})$ and $f(-x) = f(x)$ and $g(-x) = g(x)$ for all $x \in \mathbb{R}$ then $(f \cdot g)(-x) = f(-x)g(-x) = f(x)g(x) = (f \cdot g)(x)$ for all $x \in \mathbb{R}$, and so the given set is closed under multiplication.

   (g) If $f, g \in \mathbf{F}(\mathbb{R})$ and f and g are constant functions then there are $a, b \in \mathbb{R}$ such that $f(x) = a$ and $g(x) = b$ for all $x \in \mathbb{R}$. Then $(f \cdot g)(x) = f(x)g(x) = ab$ for all $x \in \mathbb{R}$, so that $f \cdot g$ is a constant function. Hence the given set is closed under multiplication.

3. (a) Notice that $(1 * 1) * 1 = 8 * 1 = 22$ while $1 * (1 * 1) = 1 * 8 = 43$, and so $*$ is neither associative nor commutative. Now suppose that e is an identity element. Then $1 = e * 1 = 2e + 6$ so that $2e = -5$, which is impossible since -5 is not even. Thus there is no identity element.

   (b) Observe that $(0 * 0) * 1 = 1 * 1 = 3^2 = 9$ while $0 * (0 * 1) = 0 * 3 = 3^3 = 27$, and so $*$ is not associative. Now if $x, y \in \mathbb{R}$ then $x * y = 3^{x+y} = 3^{y+x} = y * x$, showing that $*$ is commutative. Finally, suppose that e is an identity element. Then $0 = e * 0 = 3^e$, which is impossible since $3^x > 0$ for all $x \in \mathbb{R}$. Thus there is no identity element.

   (c) Let $f, g, h \in \mathbf{F}(\mathbb{R})$. Then $f * g$ is the constant function with value $f(0)g(0)$, and so $(f * g) * h$ is the constant function with value $(f * g)(0)h(0) = f(0)g(0)h(0)$. On the other hand, $g * h$ is the constant function with value $g(0)h(0)$, and so $f * (g * h)$ is the constant function with value $f(0)(g * h)(0) = f(0)g(0)h(0)$. Thus $*$ is associative. Notice also that $f * g$ and $g * f$ are constant functions both with value $f(0)g(0)$, and so $*$ is commutative. Finally, suppose that e is an identity element. Let $f \in \mathbf{F}(\mathbb{R})$ be a nonconstant function, e.g., $f(x) = x$ for all $x \in X$. Then $e * f \neq f$ since $e * f$ is a constant function, and so we conclude that $*$ has no identity element.

4. Suppose that gf is injective. Let $b_1, b_2 \in \mathrm{Im}(f)$ and suppose that $g|_{\mathrm{Im}(f)}(b_1) = g|_{\mathrm{Im}(f)}(b_2)$. Since $b_1, b_2 \in \mathrm{Im}(f)$ there exist $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Then $gf(a_1) = g(b_1) = g|_{\mathrm{Im}(f)}(b_1) = g|_{\mathrm{Im}(f)}(b_2) = g(b_2) = gf(a_2)$, and so $a_1 = a_2$ since gf is injective. We conclude that $g|_{\mathrm{Im}(f)}$ is injective.

5. Suppose that $f'(t) = 0$ for some $t \in (0, 1)$. Since $f''(x) = 2a \neq 0$ for all $x \in \mathbb{R}$, either f is increasing on $(-\infty, t)$ and decreasing on (t, ∞) (the case $a < 0$) or f is decreasing on $(-\infty, t)$ and increasing on (t, ∞) (the case $a > 0$). Since f is continuous, by the Intermediate Value Theorem $f(1) \in f([0, t])$ or $f(0) \in f([t, 1])$, depending on whether $f(1) \leq f(0)$ or $f(1) \geq f(0)$. This contradicts the assumption $f|_{[0,1]}$ is injective, and so we conclude that $f'(x) \neq 0$ for all $x \in (0, 1)$.

6. Take for example the function given by $f(x) = x^3$ for all $x \in \mathbb{R}$.

7. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M_2(\mathbb{R})$ and $b = c = f = g = 0$ then $af + bh = a \cdot 0 + 0 \cdot h = 0$ and $ce + dg = 0 \cdot e + d \cdot 0 = 0$, and so the given set is closed under multiplication.

8. (a) The function f is not injective since $f(-1) = f(1) = 6$. It is also not surjective because $x^8 + 3x^4 + 2 \geq 2$ for all $x \in \mathbb{R}$, so that 0 (for example) is not in the image. We have $\text{Im}(f) = [2, \infty)$.

(b) We have $f'(x) = 9x^8 + 14x^6 + 21x^2 + 1 > 0$ for all $x \in \mathbb{R}$, so that f is increasing and hence injective. Now since $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$, for any $y \in \mathbb{R}$ we can find $a, b \in \mathbb{R}$ with $a < b$ such that $f(a) < y$ and $f(b) > y$. Since f is continuous, by the Intermediate Value Theorem there exists an $x \in [a, b]$ such that $f(x) = y$. This proves that f is surjective, i.e., $\text{Im}(f) = \mathbb{R}$.

(c) Let $a, b \in \mathbb{R}$ and suppose that $f(a) = f(b)$. Since $f(x) = x^2 + 1 \geq 1$ for all $x \geq 0$ and $f(x) = x < 0$ for all $x < 0$, we must have either $a, b \geq 0$ or $a, b < 0$. In the first case $a^2 + 1 = b^2 + 1$, which implies that $a^2 = b^2$ and hence $a = b$, while in the second case we immediately obtain $a = b$. Thus f is injective. Note also that f is not surjective because $f(x) = x^2 + 1 \geq 1$ for all $x \geq 0$ and $f(x) = x < 0$ for all $x < 0$, so that 0 (for example) is not in the image of f. We have $\text{Im}(f) = (-\infty, 0) \cup [1, \infty)$.

(d) The function f is not injective since $f(-1) = f(0) = 0$. Now let $y \in \mathbb{R}$. Then we can find an even integer n with $n > |y|$. Note that $f(n) = n$, $f(n+1) = -(n+1)$, and $y \in [-(n+1), n]$. Since f is continuous, it follows by the Intermediate Value Theorem that there exists an $x \in [n, n+1]$ such that $f(x) = y$. Thus f is surjective, i.e., $\text{Im}(f) = \mathbb{R}$.

(e) Since $f(-4) = f(1) = 0$, f is not injective. It is however surjective, for if $m \in \mathbb{Z}$ then $2m - 4$ is even and so $f(2m-4) = m$. Thus $\text{Im}(f) = \mathbb{Z}$.