

MATH 220.903
Practice Problems for the Final Examination
Fall 2005

1. Using the Euclidean Algorithm compute the greatest common divisor of each of the following pairs of integers: (a) 1071 and 1932, (b) 2870 and 5863, (c) 357 and 1331.
2. (a) Prove that there do not exist integers n and m such that $n^2 = 6m + 2$.
(b) Prove that there do not exist integers n and m such that $n^3 - 4m = 2$.
3. For each of the binary operations given below, determine, with proof, which of the following properties hold: associativity, commutativity, and the existence of an identity element.
 - (a) Let $*$ be the binary operation on $\mathbb{R} - \{0\}$ given by $x * y = \frac{x+1}{y}$ for all $x, y \in \mathbb{R}$.
 - (b) Let $*$ be the binary operation on \mathbb{R} given by $x * y = 5$ for all $x, y \in \mathbb{R}$.
4. For each of the relations given below, determine, with proof, which of the following properties hold: reflexivity, the property of being symmetric, transitivity, and the property of being antisymmetric.
 - (a) Let A be a nonempty set and let R be the relation on $P(A)$ defined by XYR if $A - X = Y$.
 - (b) Let R be the relation on \mathbb{Z} defined by nRm if $n^2 = 6m$.
 - (c) Let R be the relation on $\mathbf{F}(\mathbb{R})$ defined by fRg if $f(x) < g(x)$ for all $x \in [0, 1]$.
5. (a) Give an example of a nonempty set A , a binary operation $*$ on A , and a nonempty subset B of A such that $B \neq A$, B is closed under $*$, and $A - B$ is closed under $*$.
(b) Give an example of a nonempty set A , a binary operation $*$ on A , and a nonempty subset B of A such that $B \neq A$, B is closed under $*$, and $A - B$ is not closed under $*$.

6. Let $n, m \in \mathbb{Z}^+$. Consider the following statement P : “If n and m are both divisible by 7 then $nm + 1$ is a prime number.”
- Write the converse of P .
 - Write the contrapositive of P .
7. For each of the following sentences, indicate whether it is an **open sentence** or a **statement**.
- A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is bounded if and only if $f(0)$ is an integer that is divisible by 11.
 - $n = pq$.
 - 4 divides $13^n - 1$.
 - For every even integer n , $2n^3 - 4n^2 + 3$ is odd.
 - $\emptyset \subseteq \emptyset$.
 - $A \subseteq \emptyset$.
8. Let $a, b, c \in \mathbb{R}$ and suppose that $a, b, c > 0$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = ax^7 + bx^3 + cx$ for all $x \in X$. Prove that f is invertible.
9. Determine, with proof, which of the following functions are injective and which are surjective. Also, compute the images.
- $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^{73} + 9x^{37} + 6$.
 - $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = \begin{cases} e^{2x}, & \text{if } x \geq 0, \\ -e^{2x}, & \text{if } x < 0. \end{cases}$
 - $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = \begin{cases} x^2 - 21x + 110, & \text{if } x \in [-10, 10], \\ x - 28, & \text{if } x \notin [-10, 10]. \end{cases}$
10. Prove by induction that $7^{2n+1} > (n + 5)^3$ for all $n \in \mathbb{Z}^+$.

Solutions

1. Applying the Euclidean Algorithm we obtain

$$(a) \quad 1932 = (1071)(1) + 861,$$

$$1071 = (861)(1) + 210,$$

$$861 = (210)(4) + 21,$$

$$210 = (21)(10) + 0,$$

$$(b) \quad 5863 = (2870)(2) + 123,$$

$$2870 = (123)(23) + 41,$$

$$123 = (41)(3) + 0,$$

$$(c) \quad 1331 = (357)(3) - 260,$$

$$357 = (260)(1) + 97,$$

$$260 = (97)(2) + 66,$$

$$97 = (66)(1) + 31,$$

$$66 = (31)(2) + 4,$$

$$31 = (4)(7) + 3,$$

$$4 = (3)(1) + 1,$$

$$1 = (1)(1) + 0,$$

and so $(1071, 1932) = 21$, $(5863, 2870) = 41$, and $(357, 1331) = 1$.

2. (a) Suppose that n and m are integers satisfying $n^2 = 6m + 2$. Then $n \equiv 0, 1, 2, 3, 4$, or $5 \pmod{6}$ and so $n^2 \equiv 0^2, 1^2, 3^2, 4^2$, or $5^2 \pmod{6}$. Since $0^2 \equiv 0$, $1^2 \equiv 5^2 \equiv 1$, $2^2 \equiv 4^2 \equiv 4$, and $3^2 \equiv 3 \pmod{6}$, this means that $n^2 \equiv 0, 1, 3$, or $4 \pmod{6}$. But $6m + 2 \equiv 2 \pmod{6}$, yielding a contradiction.

(b) Suppose that n and m are integers satisfying $n^3 - 4m = 2$. Then $n \equiv 0, 1, 2$, or $3 \pmod{4}$ and so $n^3 \equiv 0^3, 1^3, 2^3$, or $3^3 \pmod{4}$. Since $0^3 \equiv 0$, $1^3 \equiv 1$, $2^3 \equiv 0$, and $3^3 \equiv 3 \pmod{4}$ and $4m \equiv 0 \pmod{4}$, we see that $n^3 - 4m \equiv 0, 1$, or $3 \pmod{4}$. Since 2 is not congruent to 0, 1, or 3 modulo 4, we obtain a contradiction.

3. (a) Observe that $(1 * 1) * 1 = 2 * 1 = 3$ while $1 * (1 * 1) = 1 * 2 = 1$, so that $*$ is neither associative nor commutative. Now suppose that e is an identity element. Then for each $x \in \mathbb{R} - \{0\}$ we have $\frac{e+1}{x} = x$ and hence $e = x^2 - 1$. Considering $x = 1$ and then $x = 2$ we obtain $e = 0 = 3$, which is impossible. Thus $*$ has no identity element.

(b) For any $x, y, z \in \mathbb{R}$ we have $(x * y) * z = 5 * z = 5$, $x * (y * z) = x * 5 = 5$, and $y * x = 5 = x * y$, so that $*$ is associative and commutative. Now suppose that e is an identity element. Then $e * 1 = 1$ by the definition of an identity element. But $e * 1 = 5$, producing a contradiction. Thus $*$ has no identity element.

4. (a) Since $A - A = \emptyset \neq A$, R is not reflexive. Since $AR\emptyset$ and $\emptyset RA$ but $A\not RA$, R fails to be transitive. Since $AR\emptyset$ and $\emptyset RA$, R is not antisymmetric. Finally, let $X, Y \in P(A)$ and suppose that XRY . Then $A - X = Y$, which implies that $A - Y = X$, so that YRX . Thus R is symmetric.
- (b) Since $1\not R1$, R is not reflexive. Since $12R24$ but $24\not R12$, R is not symmetric. Since $12R24$ and $24R288$ but $12\not R288$, R is not transitive. Finally, let $n, m \in \mathbb{Z}$ and suppose that nRm and mRn . Then $n^2 = 6m$ and $m^2 = 6n$, and hence $m^4 = (6n)^2 = 36n^2 = 216m$, so that $m = 0$ or 6 . By a similar argument, $n = 0$ or 6 . Since $n^2 = 6m$ we see that n and m must both be either 0 or 6 . Thus R is antisymmetric.
- (c) Since for any $f \in \mathbf{F}(\mathbb{R})$ we have $f(0) \not\prec f(0)$, R is not reflexive. Let $f, g, h \in \mathbf{F}(\mathbb{R})$. Suppose that fRg . Then $f\not Rg$, since $g(x) > f(x)$ for all $x \in [0, 1]$. Thus R is not symmetric. Suppose next that fRg and gRf . Then $f(x) > g(x) > f(x)$ for all $x \in [0, 1]$, which is impossible. Thus R is antisymmetric. Finally, suppose that fRg and gRh . Then $f(x) > g(x) > h(x)$ for all $x \in [0, 1]$, from which we see that fRh , so that R is transitive.
5. (a) Let $*$ be the binary operation of addition on \mathbb{Z} . Then \mathbb{Z}^+ is a nonempty proper subset of \mathbb{Z} which is closed under $*$, and $\mathbb{Z} - \mathbb{Z}^+$ is also closed under the $*$.
- (b) Let $*$ be the binary operation of multiplication on \mathbb{Z} . Then \mathbb{Z}^+ is a nonempty proper subset of \mathbb{Z} which is closed under $*$. However, $\mathbb{Z} - \mathbb{Z}^+$ is not closed under the $*$, since for example $-1 \in \mathbb{Z} - \mathbb{Z}^+$ while $(-1) * (-1) = 1 \in \mathbb{Z}^+$.
6. (a) If $nm + 1$ is a prime number, then n and m are both divisible by 7 .
 (b) If $nm + 1$ is not a prime number, then one of n and m is not divisible by 7 .
7. (a) Statement. (b) Open sentence. (c) Open sentence. (d) Statement. (e) Statement.
 (f) Open sentence.
8. We will show that f is bijective, which is equivalent to invertibility. Since f is a polynomial function, it is differentiable, and $f'(x) = 7ax^6 + 3bx^2 + c \geq c > 0$ for all $x \in \mathbb{R}$. Thus f is an increasing function, which implies that it is injective. Now we will establish surjectivity. Let $y \in \mathbb{R}$. Since $\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow \infty} f(x) = \infty$, we can find $r, s \in \mathbb{R}$ with $r < s$ such that $f(r) < y$ and $f(s) > y$. Since f is continuous, by the Intermediate Value Theorem there exists an $x \in [r, s]$ such that $f(x) = y$. Thus f is surjective, and so we conclude that it is bijective.

9. (a) Since $f'(x) = 73x^{72} + 333x^{36} > 0$ for all $x \neq 0$, f is an increasing function and hence injective. It is also surjective by an argument similar to the one used for Problem 8, and hence $\text{Im}(f) = \mathbb{R}$.

(b) Since $f(x) \neq 0$ for all $x \in \mathbb{R}$, f is not surjective. For $x > 0$ we have $f'(x) = 2e^{2x} > 0$ and for $x < 0$ we have $f'(x) = -2e^{2x} < 0$, and so f is decreasing on $(-\infty, 0)$ and increasing on $[0, \infty)$. Since $\lim_{x \rightarrow -\infty} -e^{2x} = 0$, $\lim_{x \rightarrow \infty} e^{2x} = \infty$, $\lim_{x \rightarrow 0^-} -e^{2x} = -1$, $\lim_{x \rightarrow 0^+} e^{2x} = f(0) = 1$, and f is continuous on $(-\infty, 0)$ and $[0, \infty)$, this implies that $f((-\infty, 0)) = (-1, 0)$ and $f([0, \infty)) = [1, \infty)$, so that $\text{Im}(f) = (-1, 0) \cup [1, \infty)$. It also follows from these observations that f is injective.

(c) For $x \in (-10, 10)$ we have $f'(x) = 2x - 21 < 0$, and thus since f is continuous on $[-10, 10]$ it is decreasing on $[-10, 10]$. Hence $f([-10, 10]) = [f(10), f(-10)] = [0, 420]$. Since $\lim_{x \rightarrow -\infty} f(x) = -\infty$, $\lim_{x \rightarrow \infty} f(x) = \infty$, $\lim_{x \rightarrow -10^-} f(x) = -38$, $\lim_{x \rightarrow 10^+} f(x) = -18$, and f is continuous and increasing on both $(-\infty, -10)$ and $(10, \infty)$, we see that $f((-\infty, -10)) = (-\infty, -38)$ and $f((10, \infty)) = (-18, \infty)$. Thus $\text{Im}(f) = [0, 420] \cup (-\infty, -38) \cup (-18, \infty) = (-\infty, -38) \cup (-18, \infty)$. Hence f is not surjective. It is also not injective, since $f(0) = 110 = f(138)$.

10. For $n \in \mathbb{Z}^+$ let $P(n)$ be the statement that $7^{2n+1} > (n+5)^3$. Since $7^3 > 6^3$, $P(1)$ is true. Now let $k \in \mathbb{Z}^+$ and suppose that $P(k)$ is true. Since $k+5 \geq \frac{1}{2}(k+6)$ we have

$$7^{2(k+1)+1} = 7^{2k+1}7^2 > (k+5)^37^2 \geq 2^{-3}(k+6)^37^2 > (k+6)^3,$$

showing that $P(k+1)$ is true. We thus conclude by induction that $P(n)$ is true for all $n \in \mathbb{Z}^+$.