MATH 251.504
Final Examination
December 11, 2006

Name: ______________________________

ID#: ______________________________

The exam consists of 9 questions. The point value for a question is written next to the question number. There is a total of 100 points.

1. [16] For each the following statements, indicate whether it is **true** or **false**.

(a) Consider the vector field $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Then $\operatorname{div}\mathbf{F} = 3$.

Answer: __________

(b) Let C be a smooth curve in $\mathbb{R}^2$ with a certain orientation and let $-C$ be the same curve with the opposite orientation. Then $\int_C x^2 dx + y^2\, dy = -\int_{-C} x^2 dx + y^2\, dy$.

Answer: __________

(c) Consider the vector field $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} - 7xz\,\mathbf{j} + 4z\,\mathbf{k}$. Then $\iint_S \mathbf{F} \cdot d\mathbf{S} \geq 0$ for every sphere S in $\mathbb{R}^3$.

Answer: __________

(d) The vector field $\mathbf{F}(x, y, z) = e^{y^2 z}\,\mathbf{i} + xyz\,\mathbf{j} + (y^4 + xz^2)\,\mathbf{k}$ is the curl of another vector field.

Answer: __________

(e) The vector field $\mathbf{F}(x, y, z) = x^2 y\,\mathbf{i} + x^5 z^2\,\mathbf{j} + yz\,\mathbf{k}$ is conservative.

Answer: __________

(f) The area of the disk $\{(x, y) : x^2 + y^2 \leq 1\}$ is equal to $\frac{1}{2}\int_C x\,dy - y\,dx$ where C is the circle $x^2 + y^2 = 1$ with clockwise orientation.

Answer: __________

(g) The surface area of the sphere $x^2 + y^2 + z^2 = 1$ is equal to $\iint_D |\mathbf{r}_\theta \times \mathbf{r}_\phi|\,dA$ where $\mathbf{r}(\theta, \phi) = \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$ and $D = \{(\theta, \phi) : 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \phi \leq \pi\}$.

Answer: __________

(h) If f is a function of three variables such that $f(x, y, z) \geq 0$ for all $(x, y, z) \in \mathbb{R}^3$, then $\iint_S f(x, y, z)\,dS \geq 0$ for every smooth surface S in $\mathbb{R}^3$.

Answer: __________

2. [10] Find curl $\mathbf{F}$ and div $\mathbf{F}$.

(a) $\mathbf{F}(x, y, z) = \sin(yz)\,\mathbf{i} + \cos y\,\mathbf{j} + e^z\,\mathbf{k}$.

(b) $\mathbf{F} = \operatorname{curl} \mathbf{G}$ where $\mathbf{G}(x, y, z) = (x^2 + y^2)\,\mathbf{i} + z\,\mathbf{j} + (x + z)\,\mathbf{k}$.

3. [12] Let C be a smooth curve in $\mathbb{R}^3$ with initial point $(0, 1, 3)$ and terminal point $(\pi, -2, 0)$. Use the Fundamental Theorem for Line Integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (3x^2yz^2 + \cos x)\,\mathbf{i} + (x^3z^2 + 1)\,\mathbf{j} + 2x^3yz\,\mathbf{k}$.

4. [10] Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where

$$\mathbf{F}(x, y, z) = \frac{y}{1+z^4}\,\mathbf{i} + (x^3 \sin z)\,\mathbf{j} + (x^2 + y^2 + z^2)^3\,\mathbf{k}$$

and S is the surface of the hemispherical solid consisting of all points $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 + z^2 \leq 1$ and $z \geq 0$.

5. [10] Use Green's Theorem to evaluate $\int_C (xy^2 - xy)\,dx + (x^2y + e^{y^2})\,dy$ where C is the triangle with vertices $(0, -3)$, $(0, 3)$, and $(3, 0)$ with clockwise orientation.

6. [12] Evaluate $\iint_S f(x,y,z)\,dS$ where $f(x,y,z) = 2x^2 + 2y^2 + z^2$ and S is the surface consisting of the cylinder $\{(x,y,z) : x^2 + y^2 = 1 \text{ and } -3 \leq z \leq 3\}$ and its base $\{(x,y,z) : x^2 + y^2 \leq 1 \text{ and } z = -3\}$.

7. [10] Find the area of the part of the surface $z = x^2 + \sqrt{3}y$ that lies above the triangle in the xy-plane with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

8. [10] Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where

$$\mathbf{F}(x, y, z) = \sin((x^2 - x)(y^2 - 1))\,\mathbf{i} + (y^2 - z)\,\mathbf{j} + y(x^2 - x)\,\mathbf{k}$$

and S is the part of the parabolic cylinder $z = y^2$ that lies below the plane $z = 1$ and between the planes $x = 0$ and $x = 1$ with upward orientation.

9. [10] Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y) = \frac{y}{x^2 + y^2}\,\mathbf{i} - \frac{x}{x^2 + y^2}\,\mathbf{j}$$

and C is the unit circle $x^2 + y^2 = 1$ with clockwise orientation.