MATH 251.504
Solutions for the Final Examination
Fall 2006

1. (a) true, (b) true, (c) true, (d) false, (e) false, (f) true, (g) true, (h) true.

2. (a) $\operatorname{curl}\mathbf{F} = \langle 0, y\cos(yz), -z\cos(yz)\rangle$ and $\operatorname{div}\mathbf{F} = -\sin y + e^z$.
   (b) $\operatorname{curl}\mathbf{G} = \langle -1, -1, -2y\rangle$, and so $\operatorname{curl}\mathbf{F} = \langle -2, 0, 0\rangle$ and $\operatorname{div}\mathbf{F} = 0$.

3. We have $\mathbf{F} = \nabla f$ where $f(x,y,z) = x^3yz^2 + \sin x + y$, and so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(\pi,-2,0) - f(0,1,3) = -3$ by the Fundamental Theorem for Line Integrals.

4. We have $\operatorname{div}\mathbf{F} = 6z(x^2+y^2+z^2)^2$ and so by the Divergence Theorem the given surface integral is equal to

$$\iiint_E 6z(x^2+y^2+z^2)^2\,dV = \int_0^1\int_0^{\frac{\pi}{2}}\int_0^{2\pi}(6\rho^5\cos\varphi)\rho^2\sin\varphi\,d\theta\,d\varphi\,d\rho$$
$$= 6\int_0^1\rho^7\,d\rho\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin 2\varphi\,d\varphi\int_0^{2\pi}d\theta = \frac{3}{4}\pi.$$

5. By Green's Theorem the given line integral is equal to

$$\int_0^3\int_{-3+x}^{3-x} x\,dy\,dx = \int_0^3(2x^2-6x)\,dx = -9.$$

6. The cylindrical part S_1 of the surface can be parametrized by $\mathbf{r}(\theta,z) = \langle\cos\theta,\sin\theta,z\rangle$ where $0\le\theta\le 2\pi$ and $-3\le z\le 3$, and since $|\mathbf{r}_\theta\times\mathbf{r}_z| = |\langle\cos\theta,\sin\theta,0\rangle| = 1$ we have

$$\iint_{S_1} f(x,y,z)\,dS = \int_0^{2\pi}\int_{-3}^3(2+z^2)\,dz\,d\theta = 60\pi.$$

   The base S_2 can be parametrized by $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, -3\rangle$ where $0\le r\le 1$ and $0\le\theta\le 2\pi$, and since $|\mathbf{r}_r\times\mathbf{r}_\theta| = |\langle r\cos\theta, r\sin\theta, 0\rangle| = r$ we have

$$\iint_{S_2} f(x,y,z)\,dS = \int_0^{2\pi}\int_0^1(2r^2+9)r\,dr\,d\theta = 10\pi.$$

   Thus $\iint_S f(x,y,z)\,dS = \iint_{S_1} f(x,y,z)\,dS + \iint_{S_2} f(x,y,z)\,dS = 70\pi$.

7. The area is equal to

$$\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = 2\int_0^1 \int_0^x \sqrt{1+x^2}\,dy,\,dx = \frac{2}{3}(2^{3/2}-1).$$

8. The line integral around each of the four smooth parts of the boundary of S is zero, and so the given surface integral is equal to zero.

9. Using the parametrization $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \le t \le 2\pi$ (which gives the opposite orientation) we have $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$ and hence

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_0^{2\pi} \langle \sin t, -\cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt = \int_0^{2\pi} dt = 2\pi.$$