MATH 251.504
Practice Problems for Examination 2
Fall 2006

1. Find the absolute minimum and maximum values of the function $f(x, y) = xy - 6x^2$ on the closed region bounded by the parabola $y = 9x^2 - 1$ and the x-axis.

2. Calculate the double integral.

   (a) $\iint_R 3xe^{x^2}\,dA$, where $R = [-1, 0] \times [0, 1]$.

   (b) $\iint_R 5\cos(y^2)\,dA$, where R is the region bounded by the lines $y = -x$, $x = 0$, and $y = -1$.

   (c) $\iint_R e^{x^2+y^2} \tan^{-1}(\frac{y}{x})\,dA$, where R is the region described in polar coordinates by $\{(r, \theta) : 0 \le \theta \le \frac{\pi}{4},\ 0 \le r \le \theta\}$.

   (d) $\iint_R (5x^2y + 2xy^5)\,dA$, where R is the region between the two parabolas $y = x^2 + 10$ and $y = 2x^2 + 6$.

3. Find the volume of the solid which lies between the paraboloid $z = 6x^2 + y^2$ and the xy-plane and above the region $\{(x, y) : 0 \le y \le 1,\ y^2 \le x \le y\}$.

4. Find the center of mass of a lamina occupying the region $\{(x, y) : 0 \le x \le 5,\ 0 \le y \le x^2\}$ with density function $\rho(x, y) = x^2$.

5. Determine whether the following is true or false: $-2 \le \iint_R x\sin(e^{xy^2} + x^5)\,dA \le 2$, where $R = [0, 1] \times [-2, 0]$.

6. Compute $\int_{-1}^{0}\int_{-x}^{1}(e^{-y^2} + xy)\,dy\,dx$.

7. Find the volume of the solid bounded by the parabolic cylinder $z = y^2$ and the planes $z = 4y$, $x = 1$, and $x = 2$.

8. Consider a solid object which occupies the space between the parabolic cylinders $z = y^2$ and $z = y^2 + 2$ and over the region bounded by the lines $x = 0$, $y = 0$, and $y = 1 - x$. Suppose the object has density function $\rho(x, y, z) = xyz$. Find the moments of inertia of the object about the x- and y-axes.

Solutions

1. We have $f_x(x,y) = y - 12x$ and $f_y = x$, and both of these are zero at the point $(x,y) = (0,0)$, which is on the boundary of the region. For the part of the boundary of the region that lies on the x-axis we have the function $f(x,0) = -6x^2$ for $-\frac{1}{3} \le x \le \frac{1}{3}$, which has maximum value 0 at 0 and minimum value $-\frac{2}{3}$ at $\frac{1}{3}$ and $-\frac{1}{3}$. On the other part of the boundary we have the function $g(x) = f(x, 9x^2 - 1) = 9x^3 - 6x^2 - x$ for $-\frac{1}{3} \le x \le \frac{1}{3}$. Then $g'(x) = 27x^2 - 12x - 1$, which is zero at $x_0 = \frac{12-\sqrt{252}}{54}$. Since $g''(x_0) < 0$ the value $g(x_0) \approx 0.03754$ is maximum for g, while $g(\frac{1}{3}) = g(-\frac{1}{3}) = -\frac{2}{3}$ is the minimum. Thus for f the absolute minimum value is $-\frac{2}{3}$ while the absolute maximum value is approximately 0.03754.

2. (a)
$$\int_0^1 \int_{-1}^0 3xe^{x^2}\,dx\,dy = \int_0^1 \left[\frac{3}{2}e^{x^2}\right]_{x=-1}^{x=0} dy = \frac{3}{2}(1-e).$$

(b)
$$\int_{-1}^0 \int_0^{-y} 5\cos(y^2)\,dy = \int_{-1}^0 \left[5x\cos(y^2)\right]_{x=0}^{x=-y} dy = \int_{-1}^0 -5y\cos(y^2)\,dy$$
$$= -\frac{5}{2}\sin(y^2)\Big]_{-1}^0 = \frac{5}{2}\sin(1).$$

(c)
$$\int_0^{\frac{\pi}{4}} \int_0^{\theta} \theta r e^{r^2}\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \left[\frac{1}{2}\theta e^{r^2}\right]_{r=0}^{r=\theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}\theta(e^{\theta^2} - 1)\,d\theta$$
$$= \frac{1}{4}e^{\theta^2} - \frac{1}{4}\theta^2\Big]_0^{\frac{\pi}{4}} = \frac{1}{4}e^{\frac{\pi^2}{16}} - \frac{\pi^2}{64} - \frac{1}{4}.$$

(d)
$$\int_{-2}^2 \int_{2x^2+6}^{x^2+10} (5x^2y + 2xy^5)\,dy\,dx = \int_{-2}^2 \left[\frac{5}{2}x^2y^2 + \frac{1}{3}xy^6\right]_{y=2x^2+6}^{y=x^2+10} dx$$
$$= \int_{-2}^2 \left(\frac{5}{2}x^2((x^2+10)^2 - (2x^2+6)^2)\right)dx + \int_{-2}^2 \left(\frac{1}{3}x((x^2+10)^6 - (2x^2+6)^6)\right)dx.$$

Expand to evaluate the first integral, and for the second use substitution.

3. The volume is given by

$$V = \int_0^1 \int_{y^2}^{y} (6x^2 + y^2)\,dx\,dy = \int_0^1 \Big[2x^3 + y^2 x\Big]_{x=y^2}^{x=y} dy = \int_0^1 (3y^3 - 2y^6 - y^4)\,dy$$
$$= \frac{3}{4}y^4 - \frac{2}{7}y^7 - \frac{1}{5}y^5\Big]_0^1 = \frac{37}{140}.$$

4. The mass and moments are

$$m = \int_0^5 \int_0^{x^2} x^2\,dy\,dx = \int_0^5 x^4\,dx = \frac{1}{5}x^5\Big]_0^5 = 5^4,$$
$$M_x = \int_0^5 \int_0^{x^2} x^2 y\,dy\,dx = \int_0^5 \Big[\frac{1}{2}x^2 y^2\Big]_{y=0}^{y=x^2} dx = \int_0^5 \frac{1}{2}x^4\,dx = \frac{1}{10}x^5\Big]_0^5 = \frac{5^4}{2}$$
$$M_y = \int_0^5 \int_0^{x^2} x^3\,dy\,dx = \int_0^5 x^5\,dx = \frac{1}{6}x^6\Big]_0^5 = \frac{5^6}{6},$$

and so the center of mass is $(\frac{25}{6}, \frac{1}{2})$.

5. The statement is true. On R the values of the function are bounded between -1 and 1, and the area of R is 2. Therefore $-2 = -1 \times \text{area}(R) \leq \iint_R x \sin(e^{xy^2} + x^5)\,dA \leq 1 \times \text{area}(R) = 2$.

6. Using Fubini's theorem,

$$\int_{-1}^0 \int_{-x}^1 (e^{-y^2} + xy)\,dy\,dx = \int_0^1 \int_{-y}^0 (e^{-y^2} + xy)\,dx\,dy = \int_0^1 \Big[xe^{-y^2} + \frac{1}{2}x^2 y\Big]_{x=-y}^{x=0} dy$$
$$= \int_0^1 \Big(ye^{-y^2} - \frac{1}{2}y^3\Big)\,dy = -\frac{1}{2}e^{-y^2} - \frac{1}{8}y^4\Big]_0^1 = -\frac{1}{2e} + \frac{3}{8}.$$

7. The volume is given by

$$\int_1^2 \int_0^4 (4y - y^2)\,dy\,dx = \int_1^2 \Big[2y^2 - \frac{1}{3}y^3\Big]_{y=0}^{y=4} dx = \frac{32}{3}.$$

8. The moment of inertia about the y-axis is given by

$$
\begin{aligned}
I_y = \iiint_E (x^2+z^2)xyz\,dV &= \int_0^1 \int_0^{1-x} \int_{y^2}^{y^2+2} (x^3yz + xyz^3)\,dz\,dy\,dx \\
&= \int_0^1 \int_0^{1-x} \left[\frac{1}{2}x^3yz^2 + \frac{1}{4}xyz^4\right]_{z=y^2}^{z=y^2+2} dy\,dx \\
&= \int_0^1 \int_0^{1-x} \left(\frac{1}{2}x^3y(y^2+2)^2 + \frac{1}{4}xy(y^2+2)^4 - \frac{1}{2}x^3y^5 - \frac{1}{4}xy^9\right) dy\,dx \\
&= \int_0^1 \left[\frac{1}{12}x^3(y^2+2)^3 + \frac{1}{40}x(y^2+2)^5 - \frac{1}{12}x^3y^6 - \frac{1}{40}xy^{10}\right]_{y=0}^{y=1-x} dx \\
&= \int_0^1 \Big(\frac{1}{12}x^3((1-x)^2+2)^3 + \frac{1}{40}x((1-x)^2+2)^5 \\
&\qquad - \frac{1}{12}x^3(1-x)^6 - \frac{1}{40}x(1-x)^{10} - \frac{2}{3}x^3 - \frac{4}{5}x\Big)\,dx.
\end{aligned}
$$

Now expand to evaluate. The moment of inertia I_x about the x-axis can be similarly computed as $\int_0^1 \int_0^{1-x} \int_{y^2}^{y^2+2} (y^2+z^2)xyz\,dz\,dy\,dx$.