MATH 251
Practice Problems for the Final Examination
Spring 2008

1. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where

$$\mathbf{F}(x, y, z) = 2xz\,\mathbf{i} + (2e^y + xz^3)\,\mathbf{j} + (-z^2 + e^y \cos x)\,\mathbf{k}$$

   and S is the surface of the solid E that lies in the region $x \geq 0$ and is bounded by the parabolic cylinder $y = 1 - x^2$ and the planes $y = 0$, $z = 0$, and $z = x$.

2. Find $\operatorname{curl} \mathbf{F}$ and $\operatorname{div} \mathbf{F}$.

   (a) $\mathbf{F}(x, y, z) = e^{xyz}\,\mathbf{i} + z^3\,\mathbf{j} + \cos(xy)\,\mathbf{k}$.

   (b) $\mathbf{F}(x, y, z) = (y - z)\,\mathbf{i} + (z - x)\,\mathbf{j} + (y - z)\,\mathbf{k}$.

   (c) $\mathbf{F}(x, y, z) = 4\,\mathbf{i} + (x - z)^2\,\mathbf{j} + 4\,\mathbf{k}$.

3. Determine whether or not $\mathbf{F}$ is conservative, and if it is conservative find a function f such that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F}(x, y) = 2xye^{x^2}\,\mathbf{i} + (e^{x^2} + \cos y)\,\mathbf{j}$.

   (b) $\mathbf{F}(x, y, z) = (x + y \sin z)\,\mathbf{i} + x \sin z\,\mathbf{j} + (5 + xy \cos z)\,\mathbf{k}$.

   (c) $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + \dfrac{2y}{y^2 + 1}\,\mathbf{j} + xy\,\mathbf{k}$.

4. Evaluate $\int_C y^2 z^2\, ds$ where C is the line segment from $(0, 3, 0)$ to $(-1, 2, 1)$.

5. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \ln x\,\mathbf{i} + yz\,\mathbf{j} + 3\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = 3\,\mathbf{i} + t\,\mathbf{j} + t^2\,\mathbf{k}$ for $-2 \leq t \leq 2$.

6. Use Green's Theorem to evaluate $\int_C (3x^2y + 12xy^2)\, dx + (x^3 + \sin y)\, dy$ where C is the boundary of the trapezoid with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$, and $(2, 4)$ with clockwise orientation.

7. Use the Fundamental Theorem for Line Integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = y^4 z\,\mathbf{i} + (4xy^3 z + e^y z)\,\mathbf{j} + (e^y + xy^4)\,\mathbf{k}$$

and C is given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{j}$ for $0 \le t \le 7\pi$.

8. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ where

$$\mathbf{F}(x, y, z) = \frac{x^2 + y^2}{z}\,\mathbf{i} + (y^2 - x^2 z)\,\mathbf{j} + e^z\,\mathbf{k}$$

and S is the part of the sphere $x^2 + y^2 + z^2 = 10$ that lies below the plane $z = -1$ with upward orientation.

Solutions

1. Using the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F}\, dV = \int_0^1 \int_0^{1-x^2} \int_0^x 2e^y\, dz\, dy\, dx \\ &= \int_0^1 \int_0^{1-x^2} \Big[2ze^y\Big]_{z=0}^{z=x} dy\, dx = \int_0^1 \int_0^{1-x^2} 2xe^y\, dy\, dx = \int_0^1 \Big[2xe^y\Big]_{y=0}^{y=1-x^2} dx \\ &= \int_0^1 (2xe^{1-x^2} - 2x)\, dx = -e^{1-x^2} - x^2\Big]_0^1 = e - 2.\end{aligned}$$

2. (a) $\operatorname{curl} \mathbf{F} = (-x\sin(xy) - 3z^2)\,\mathbf{i} + (xye^{xyz} + y\sin(xy))\,\mathbf{j} - xze^{xyz}\,\mathbf{k}$; $\operatorname{div} \mathbf{F} = yze^{xyz}$.

   (b) $\operatorname{curl} \mathbf{F} = -\mathbf{j} - 2\mathbf{k}$; $\operatorname{div} \mathbf{F} = -1$.

   (c) $\operatorname{curl} \mathbf{F} = 2(x - z)\,\mathbf{i} + 2(x - z)\,\mathbf{k}$; $\operatorname{div} \mathbf{F} = 0$.

3. (a) Since $\frac{\partial}{\partial y}(2xye^{x^2}) = 2xe^{x^2} = \frac{\partial}{\partial x}(e^{x^2} + \cos y)$ and $\mathbf{F}$ is defined on all of $\mathbb{R}^2$, $\mathbf{F}$ is conservative, i.e., $\mathbf{F} = \nabla f$ for some function f. Since $f_x(x,y) = 2xye^{x^2}$ we must have $f(x,y) = ye^{x^2} + g(y)$ for some single-variable function g. Then $f_y(x,y) = e^{x^2} + g'(y)$, so that $g'(y) = \cos y$, in which case we may take $g(y) = \sin y$ so that $f(x,y) = ye^{x^2} + \sin y$ (note that f is unique only up to addition of a constant term).

   (b) Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$ and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, $\mathbf{F}$ is conservative, i.e., $\mathbf{F} = \nabla f$ for some function f. Since $f_x(x,y,z) = x + y\sin z$ we have $f(x,y,z) = \frac{1}{2}x^2 + xy\sin z + g(y,z)$ for some two-variable function g. Then $f_y(x,y,z) = x\sin z + \frac{\partial}{\partial y}g(y,z)$ and so $\frac{\partial}{\partial y}g(y,z) = 0$, which means that $g(y,z) = h(z)$ for some single-variable function h. Then $f_z(x,y,z) = xy\cos z + h'(z)$ and thus $h'(z) = 5$, so that we may take $h(y) = 5z$ and hence $f(x,y,z) = \frac{1}{2}x^2 + xy\sin z + 5z$ (note that f is unique only up to addition of a constant term).

   (c) Since $\operatorname{curl}\mathbf{F} = \langle x, 0, -z\rangle$, which is not zero everywhere, $\mathbf{F}$ is not conservative.

4. Parametrize C by $\mathbf{r}(t) = \langle -t, 3-t, t\rangle$ for $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle -1, -1, 1\rangle$ so that $|\mathbf{r}'(t)| = \sqrt{3}$ and hence

$$\int_C y^2 z^2\, ds = \int_0^1 (3-t)^2 t^2 \sqrt{3}\, dt = \sqrt{3}\Big(3t^3 - \frac{3}{2}t^4 + \frac{1}{5}t^5\Big)\Big]_0^1 = \frac{17\sqrt{3}}{10}.$$

5. We have $\mathbf{r}'(t) = \langle 0, 1, 2t\rangle$ and hence

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-2}^2 \langle \ln 3, t^3, 3\rangle \cdot \langle 0, 1, 2t\rangle\, dt = \int_{-2}^2 (t^3 + 6t)\, dt = \frac{1}{4}t^4 + 3t^2\Big]_{-2}^2 = 0.$$

6. Using Green's Theorem,

$$\begin{aligned}\int_C (3x^2y + 12xy^2)\,dx + (x^3 + \sin y)\,dy &= \int_0^2 \int_0^{x+2} -24xy\,dy\,dx = \int_0^2 -12xy^2\Big]_{y=0}^{y=x+2} dx \\ &= \int_0^2 (-12x^3 - 48x^2 - 48x)\,dx \\ &= -3x^4 - 16x^3 - 24x^2\Big]_0^2 = -272.\end{aligned}$$

7. Since $\mathbf{F} = \nabla f$ where $f(x,y,z) = xy^4z + e^y z$ (as can be found by a procedure similar to that in the solution of 3(b)), we have, by the Fundamental Theorem for Line Integrals,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(7\pi)) - f(\mathbf{r}(0)) = f(-1,0,14\pi) - f(1,0,0) = 14\pi$$

8. The boundary C of S is a circle which we parametrize by $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, -1\rangle$ for $0 \le t \le 2\pi$. Then $\mathbf{r}'(t) = \langle -3\sin t, 3\cos t, 0\rangle$. Thus, by Stokes' Theorem,

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle -9, 9, 1/e\rangle \cdot \langle -3\sin t, 3\cos t, 0\rangle \, dt \\ &= \int_0^{2\pi} 27(\sin t + \cos t)\, dt = 27(-\cos t + \sin t)\Big]_0^{2\pi} = 0.\end{aligned}$$