MATH 251
Practice Problems for Examination 1
Spring 2008

1. For which values of a are the vectors $\langle a, 3, -9\rangle$ and $\langle a, -4a, -4\rangle$ orthogonal?
2. Determine whether or not the points $P(3, 7, 1)$, $Q(4, 2, 1)$, $R(-1, 3, 1)$, and $S(0, 0, 2)$ lie in the same plane.
3. Find symmetric equations for the line that is contained in both the plane $z = x$ and the tangent plane to the surface $z = 2x^2 + 5y^2$ at the point $(1, 1, 7)$.
4. Give an example of three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ such that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ but $\mathbf{b} \neq \mathbf{c}$.
5. Determine whether the following statement is true or false: For every function $f : \mathbb{R}^2 \to \mathbb{R}$ which is continuous at $(0, 0)$ the function $g(x, y) = (x - 1)(y - 1)f(x, y)$ is continuous at $(1, 1)$.
6. Show that the limit doesn't exist in each of the following two cases.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x(y + 3x)}{5x^2 + y^2}$ (b) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x^2 - 2x - y^2 + 1}{x^2 - 2x + y^2 + 1}$

7. Find f_{xx}, f_{xy} and f_{yy}.
   (a) $f(x, y) = (1 - x^2)(1 - 2x^3)$
   (b) $f(x, y) = \ln(x^2 + e^{2y})$
   (c) $f(x, y) = ye^{-2x} + \cos(y^2)$
8. Find an equation for the tangent plane to the surface $z = xye^x$ at the point $(1, 1, e)$.
9. Find the partial derivatives $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ where $w = 5xyz + \ln(x - y)$ and $x = st$, $y = t^2$, and $z = 4t$.
10. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f_x(0, 0)$ and $f_y(0, 0)$ exist but f is not continuous at $(0, 0)$.

11. Use differentials to approximate the number $\sqrt{35}\sqrt[3]{215}$.

12. In which direction does the function $f(x, y, z) = ze^y + xy^2$ decrease the fastest at the point $(1, 1, 1)$?

13. Classify the quadric surfaces $x^2 - y^2 - 2y - z^2 = 5$ and $y - 4 = z^2 - x^2 + 4x$.

14. Find an equation for the normal line to the surface $z = \cos(y)\sin(x)$ at the point $(\frac{\pi}{2}, \frac{\pi}{4}, 0)$.

Solutions

1. The two vectors are orthogonal precisely when the dot product $\langle a, 3, -9\rangle \cdot \langle a, -4a, -4\rangle = a^2 - 12a + 36 = (a - 6)^2$ is zero, that is, when $a = 6$.

2. It is equivalent to determine whether or not the vectors $\overrightarrow{PQ} = \langle 1, -5, 0\rangle$, $\overrightarrow{PR} = \langle -4, -4, 0\rangle$, and $\overrightarrow{PS} = \langle -3, -7, 1\rangle$ are coplanar. They are not coplanar because the scalar triple product

$$\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) = \begin{vmatrix} 1 & -5 & 0 \\ -4 & -4 & 0 \\ -3 & -7 & 1 \end{vmatrix} = 1(-4) - (-5)(-4) + 0 = -24$$

is not equal to zero.

3. Set $f(x, y) = 2x^2 + 5y^2$. Then $f_x(x, y) = 4x$ and $f_y(x, y) = 10y$, and in particular $f_x(1, 1) = 4$ and $f_y(1, 1) = 10$. Therefore the equation of the tangent plane at $(1, 1, 7)$ is

$$z - 7 = f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) = 4(x - 1) + 10(y - 1)$$

or $4x + 10y - z = 7$, and it has $\langle 4, 10, -1\rangle$ as a normal vector. The plane $z = x$ has $\langle -1, 0, 1\rangle$ as a normal vector, and so the line of intersection of the two planes has direction vector $\langle 4, 10, -1\rangle \times \langle -1, 0, 1\rangle = \langle 10, -3, 10\rangle$. Setting $x = z = 0$ we see from the equation $4x + 10y - z = 7$ that $(0, \frac{7}{10}, 0)$ is a point on the line of intersection of the two planes. Thus symmetric equations for this line of intersection are

$$\frac{x}{10} = \frac{y - \frac{7}{10}}{-3} = \frac{z}{10}.$$

4. Take for example $\mathbf{a} = \mathbf{b} = \mathbf{i}$ and $\mathbf{c} = -\mathbf{i}$.

5. The statement is false. Consider for example the function

$$f(x,y) = \begin{cases} \dfrac{1}{(x-1)(y-1)} & \text{if } (x,y) \neq (1,1), \\ 0 & \text{if } (x,y) = (1,1). \end{cases}$$

6. (a) Approaching $(0,0)$ along the x-axis we have the limit $\lim_{x\to 0} \frac{3x^2}{5x^2} = \frac{3}{5}$, while approaching $(0,0)$ along the y-axis we have the limit $\lim_{y\to 0} \frac{0}{y^2} = 0$, and so the given limit doesn't exist.

   (b) Approaching $(1,0)$ along the x-axis we have the limit $\lim_{x\to 1} \frac{x^2-2x+1}{x^2-2x+1} = 1$, while approaching $(1,0)$ along the line $x = 1$ we have the limit $\lim_{y\to 0} \frac{-y^2}{y^2} = -1$, and so the given limit doesn't exist.

7. (a) $f_{xx}(x,y) = -2 - 12x + 40x^3$, $f_{xy}(x,y) = 0$, $f_{yy}(x,y) = 0$.

   (b) $f_{xx}(x,y) = \frac{2}{x^2+e^{2y}} - \frac{4x^2}{(x^2+e^{2y})^2}$, $f_{xy}(x,y) = \frac{-4xe^{2y}}{(x^2+e^{2y})^2}$, $f_{yy}(x,y) = \frac{4e^{2y}}{x^2+e^{2y}} - \frac{4e^{4y}}{(x^2+e^{2y})^2}$.

   (c) $f_{xx}(x,y) = 4ye^{-2x}$, $f_{xy}(x,y) = -2e^{-2x}$, $f_{yy}(x,y) = -2\sin(y^2) - 4y^2\cos(y^2)$.

8. Set $f(x,y) = xye^x$. Then $f_x(x,y) = (x+1)ye^x$ and $f_y(x,y) = xye^x$, and in particular $f_x(1,1) = 2e$ and $f_y(1,1) = e$. Therefore the equation of the tangent plane at $(1,1,e)$ is

$$z - e = f_x(1,1)(x-1) + f_y(1,1)(y-1) = 2e(x-1) + e(y-1)$$

   or $2ex + ey - z = 2e$.

9. We have

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= \left(5yz + \frac{1}{x-y}\right)t \\ &= 20t^4 + \frac{t}{t(s-t)} \end{aligned}$$

   and

$$
\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\
&= \left(5yz + \frac{1}{x-y}\right)s + 2\left(5xz - \frac{1}{x-y}\right)t + 20xy \\
&= 80st^3 + \frac{s-2t}{t(s-t)}.
\end{aligned}
$$

10. An example is the function

$$
f(x,y) = \begin{cases} 0 & \text{if } xy = 0, \\ 1 & \text{if } xy \neq 0. \end{cases}
$$

The partial derivatives $f_x(0,0)$ and $f_y(0,0)$ are both zero. However, f is not continuous at $(0,0)$, since approaching $(0,0)$ along the x-axis we have zero as the limit while approaching $(0,0)$ along the line $y = x$ we have 1 as the limit, so that $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist.

11. Define the function $f(x,y) = x^{1/2}y^{1/3}$. Then $f_x(x,y) = \frac{1}{2}x^{-1/2}y^{1/3}$ and $f_y(x,y) = \frac{1}{3}x^{1/2}y^{-2/3}$, and in particular $f_x(36,216) = \frac{1}{2}$ and $f_y(36,216) = \frac{1}{18}$. Then

$$
\begin{aligned}
\sqrt{35}\sqrt[3]{215} \approx f(36,216) + dz &= f(36,216) + f_x(36,216)dx + f_y(36,216)dy \\
&= 36 + \frac{1}{2}(-1) + \frac{1}{18}(-1) = 36 - \frac{5}{9}.
\end{aligned}
$$

12. We have $\nabla f(x,y,z) = \langle y^2, ze^y + 2xy, e^y\rangle$ and in particular $\nabla f(1,1,1) = \langle 1, e+2, e\rangle$. So f is decreasing the fastest at the point $(1,1,1)$ in the direction of the vector $-\langle 1, e+2, e\rangle$.

13. Completing the square we have $\frac{x^2}{4} - \frac{(y+1)^2}{4} - \frac{z^2}{4} = 1$ and $y - 8 = z^2 - (x-2)^2$. The first is a hyperboloid of two sheets and the second is a hyperbolic paraboloid.

14. Define the function $f(x,y,z) = z - \cos(y)\sin(x)$. Then $\nabla f(x,y,z) = \langle -\cos(y)\cos(x), \sin(y)\sin(x), 1\rangle$ and in particular $\nabla f(\frac{\pi}{2}, \frac{\pi}{4}, 0) = \langle 0, \frac{1}{\sqrt{2}}, 1\rangle$. So parametric equations for the normal line to the surface $z = \cos(y)\sin(x)$ at $(\frac{\pi}{2}, \frac{\pi}{4}, 0)$ are

$$
x = \frac{\pi}{2}, \qquad y = \frac{\pi}{4} + \frac{1}{\sqrt{2}}t, \qquad z = t.
$$