MATH 302.903
Practice Problems for Examination 1
Spring 2006

1. For each of the following sentences, indicate whether it is a proposition, and in the case that it is a proposition also indicate its truth value.

   (a) 7 divides $13^n - 1$.

   (b) A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable if and only if $f(0)$ is a rational number.

   (c) For every even integer n, $12n^3 - 2n^2 + 18$ is even.

   (d) $\emptyset \in \emptyset$.

   (e) $\emptyset \subseteq A$.

   (f) $f(x)$ is of order e^x.

   (g) $\lfloor \pi \rfloor = 3$.

2. Let n and m be positive integers. Consider the following proposition P: "If n and m are both divisible by 3 then $n(m+1)$ is even." Write the converse, inverse, and contrapositive of P.

3. Incorporate each of the following sentences into a **false** proposition.

   (a) $\emptyset \subseteq A$.

   (b) f is bounded.

   (c) $|A| = 8$.

   (d) Every differentiable function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

   (e) $A \subseteq (B - C)$.

4. Determine, with proof, the least integer n such that $x^4 \ln x - 3x^2 + 4$ is $O(x^n)$.

5. Determine which of the following pairs of compound propositions are logically equivalent.

   (a) $\neg(p \wedge (\neg q))$ and $p \to \neg q$.

   (b) $(p \to q) \wedge r$ and $(p \to r) \wedge (r \to q)$.

   (c) $(p \to q) \to r$ and $(p \vee r) \wedge (q \to r)$.

6. For each of the following compound propositions, indicate whether it is a tautology, a contradiction, or a contingency.

   (a) $(p \to q) \vee p$, (b) $(p \wedge q) \vee (\neg p)$, (c) $p \to (q \to r)$, (d) $(p \vee q) \wedge (\neg p \vee \neg q)$.

7. Show the following.

   (a) $4x^7 + 3x^5 + 9$ is of order x^7.

   (b) $(x^4 + 3x + 2)/(x^2 - 4)$ is of order x^2.

   (c) $(x \ln x + 3x)/(2x + 7)$ is $O(\ln x)$ but not $O(1)$.

   (d) $2x^2 - 3x - 9$ is of order x^2.

8. Determine which of the following functions are injective and which are surjective. Also, determine the range of each function.

   (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^{10} + 2$.

   (b) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^7 + 2x^5 + 7x^3 + 2x$.

   (c) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} x^2 + 5, & \text{if } x \geq 0, \\ x, & \text{if } x < 0. \end{cases}$

   (d) $f : \{0, 1, 2\} \to \{0, 1, 2, 3, 4\}$ given by $f(0) = 3$, $f(1) = 4$, and $f(2) = 0$.

9. Let A be a nonempty set and let $f : A \to A$ and $g : A \to A$ be functions. Prove that if the composition $g \circ f$ is surjective, then g is surjective.

10. Prove that, for all integers n, n is odd if and only if n^3 is odd.

Solutions

1. (a) Not a proposition. (b) False proposition. (c) True proposition. (d) False proposition. (e) Not a proposition. (f) Not a proposition. (g) True proposition.

2. Converse: "If $n(m+1)$ is even then n and m are both divisible by 3."
   Inverse: "If one of n and m is not divisible by 3 then $n(m+1)$ is odd."
   Contrapositive: "If $n(m+1)$ is odd then one of n and m is not divisible by 3."

3. (a) There does not exist a set A such that $\emptyset \subseteq A$.
   (b) For every function $f : \mathbb{R} \to \mathbb{R}$, f is bounded.
   (c) For every finite set A, $|A| = 8$.
   (d) If every differentiable function $f : \mathbb{R} \to \mathbb{R}$ is continuous then $0 = 1$.
   (e) For all subsets A, B, and C of $\mathbb{R}$, $A \subseteq (B - C)$.

4. For all $x > 1$ we have $\ln(x) < x$, $x^2 \leq x^4$, and $1 \leq x^4$, from which it follows using the triangle inequality that

   $$|x^4 \ln x - 3x^2 + 4| \leq x^4 \ln(x) + 3x^2 + 4 \leq x^5 + 3x^5 + 4x^5 = 8x^5.$$

   Thus $x^4 \ln x - 3x^2 + 4$ is $O(x^5)$. Now suppose that $x^4 \ln x - 3x^2 + 4$ is $O(x^4)$. Then we can find constants $C, k > 0$ such that $|x^4 \ln x - 3x^2 + 4| \leq Cx^4$, or equivalently $|\ln x - 3x^{-2} + 4x^{-4}| \leq C$, for all $x > k$. But $\lim_{x\to\infty} |\ln x - 3x^{-2} + 4x^{-4}| = \infty$, which produces a contradiction. We conclude that 5 is the least integer n such that $x^4 \ln x - 3x^2 + 4$ is $O(x^n)$.

5. (a) Not equivalent. (b) Not equivalent. (c) Equivalent.

6. (a) Tautology. (b) Contingency. (c) Contingency. (d) Contingency.

7. (a) For all $x \geq 1$ we have $x^5 \leq x^7$ and $1 \leq x^7$ so that

   $$|f(x)| = 4x^7 + 3x^5 + 9 \leq 4x^7 + 3x^7 + 9x^7 = 16x^7,$$

   which shows that $f(x)$ is $O(x^7)$. Since $f(x) \geq 4x^7$ for all $x \in \mathbb{R}$, f is $\Omega(x^7)$, and so we conclude that f is of order x^7.

(b) For $x \geq 3$ we have $x^2 - 4 \geq \frac{1}{2}x^2$, $x^{-1} \leq x^2$, and $x^{-2} \leq x^2$, so that

$$|f(x)| = \frac{x^4 + 3x + 2}{x^2 - 4} \leq \frac{x^4 + 3x + 2}{\frac{1}{2}x^2} = 2x^2 + 6x^{-1} + 2x^{-2} \leq 2x^2 + 6x^2 + 2x^2 = 10x^2.$$

Thus f is $O(x^2)$. On the other hand, for $x > 2$ we have

$$|f(x)| = \frac{x^4 + 3x + 2}{x^2 - 4} \geq \frac{x^4 + 3x + 2}{x^2} = x^2 + 3x^{-1} + 2x^{-1} \geq x^2,$$

which shows that $f(x)$ is $\Omega(x^2)$. Therefore $f(x)$ is of order x^2.

(c) For $x > e$ we have $1 \leq \ln x$ in which case

$$|f(x)| = \frac{x \ln x + 3x}{2x + 7} \leq \frac{x \ln x + 3x}{2x} = \frac{1}{2} \ln x + \frac{3}{2} \leq \frac{1}{2} \ln x + \frac{3}{2} \ln x = 2 \ln x,$$

so that $f(x)$ is $O(\ln x)$. Since

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\ln x + 3}{2 + \frac{7}{x}} = \infty,$$

$f(x)$ is not $O(1)$.

(d) For all $x \geq 1$ we have $x \leq x^2$ and $1 \leq x^2$, in which case it follows using the triangle inequality that

$$|f(x)| = |2x^2 - 3x - 9| \leq 2x^2 + 3x + 9 \leq 2x^2 + 3x^2 + 9x^2 = 14x^2,$$

which shows that $f(x)$ is $O(x^2)$. On the other hand, since $\lim_{x \to \infty}(3x + 9)/x^2 = 0$, we can find a $k > 0$ such that for all $x > k$ we have $3x + 9 \leq x^2$ and hence

$$f(x) = 2x^2 - (3x + 9) \geq 2x^2 - x^2 = x^2,$$

which shows that $f(x)$ is $\Omega(x^2)$. Therefore f is of order x^2.

8. (a) This function is neither injective (since $f(-1) = f(1)$) nor surjective (since 0 is not in the range). Its range is $[2, \infty)$.

   (b) Since $f'(x) > 0$ for all $x \in \mathbb{R}$, f is strictly increasing and hence injective. It is also surjective, as can be shown using the fact that $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$ and the Intermediate Value Theorem. Being surjective, its range is $\mathbb{R}$.

   (c) This function is injective but not surjective. Its range is $(-\infty, 0) \cup [5, \infty)$.

   (d) This function is injective but not surjective. Its range is $\{0, 3, 4\}$.

9. Let $a \in A$. We need to find an $b \in A$ such that $g(b) = a$. Since by assumption $g \circ f$ is surjective, there is a $c \in A$ such that $(g \circ f)(c) = a$. But then setting $b = f(c)$ we have $g(b) = g(f(c)) = a$, as desired.

10. Let n be an integer. Suppose first that n is odd. Then we can write $n = 2s + 1$ for some $s \in \mathbb{Z}$. Then $n^3 = (2s+1)^3 = 8s^3 + 12s^2 + 6s + 1 = 2(4s^3 + 6s^2 + 3s) + 1$, which is odd since $4s^3 + 6s^2 + 3s$ is an integer.

    Now suppose that n is even. Then we can write $n = 2s$ for some $s \in \mathbb{Z}$. Then $n^3 = (2s)^3 = 8s^3 = 2(4s^3)$, which is even since $4s^3$ is an integer. This shows that if n^3 is odd then n must be odd.

    We conclude that n is odd if and only if n^3 is odd.