MATH 302.903
Practice Problems for Examination 2
Spring 2006

1. Compute the following: (a) $\sum_{j=0}^{12}(2^{j+2}-3)$, (b) $\sum_{j=5}^{15}(5^j-(-1)^{3j})$.

2. Prove that the set of all nonempty bit strings of finite length is countable.

3. Let n be a positive integer.

   (a) What is the number of bit strings of length n?

   (b) What is the number of bit strings of length at most n, including the empty string?

   (c) Assuming that $n \geq 75$, what is the number of bit strings of length n with exactly 75 1s?

   (d) Assuming that $n \geq 75$, what is the number of strings of 0s, 1s, and 2s of length n with exactly 75 1s?

4. Let n be an integer greater than 6.

   (a) What is the number of functions $f : \{1, 2, \ldots, n\} \to \{0, 1, 2\}$ such that $f(\{4, 5, 6\}) \subseteq \{0, 1\}$?

   (b) What is the number of functions $f : \{1, 2, \ldots, n\} \to \{0, 1, 2\}$ such that $f(\{4, 5, 6\}) = \{0, 1\}$?

5. Determine how many strings can be constructed from the letters in MATHEMATICS (a) using all of the letters, and (b) using exactly four of the letters.

6. Consider the sequence $\{a_n\}$ defined recursively by $a_n = 2a_{n-1} + a_{n-2} + 11a_{n-3}$ for $n \geq 4$ with initial conditions $a_1 = 1$, $a_2 = 4$, and $a_3 = 6$. Prove that $a_n < 5^n$ for all $n \geq 1$.

7. Prove by induction that $5^n > (n+1)^2$ for all positive integers n.

8. Prove by induction that $3^n + 7^n + 6$ is divisible by 8 for all positive integers n.

9. In how many ways can 15 identical balls be placed into 20 numbered boxes (a) if each box can hold at most one ball, (b) if each box can hold at most 2 balls, and (c) if each box can hold any number of balls?

10. Let n and k be positive integers. Determine the number of solutions in nonnegative integers of the inequality $x_1 + x_2 + \cdots + x_n \leq k$.

Solutions

1. (a)
$$\sum_{j=0}^{12}(2^{j+2}-3) = 2^2\sum_{j=0}^{12}2^j - 13\cdot 3 = 4(2^{13}-1) - 13\cdot 3.$$
(b)
$$\sum_{j=5}^{15}(5^j-(-1)^{3j}) = 5^5\sum_{j=0}^{10}5^j + \sum_{j=0}^{10}(-1)^j = 5^5\left(\frac{5^{11}-1}{4}\right)+1.$$

2. For every positive integer n there are 2^n bit strings of length n, so that we can construct a bijection f_n from the set of bit strings of length n to the set $\{2^n-1, 2^n, 2^n+1, 2^n+2, \ldots, 2^{n+1}-2\}$. For a nonempty bit string S of length n, set $f(S) = f_n(S)$. Then f defines a bijective function from the set of all nonempty bit strings of finite length to the set of positive integers. Thus the set of all nonempty bit strings of finite length is countable.

3. (a) 2^n.

   (b) Sum the number of bit strings of length k over $k = 0, 1, \ldots, n$ to obtain the answer $\sum_{k=0}^{n} 2^k = 2^{n+1}-1$.

   (c) $\binom{n}{75}$.

   (d) There are $\binom{n}{75}$ ways to choose the positions of the 1s, and then 2^{n-75} ways to fill in the remaining positions with 0s and 2s. The answer is thus $\binom{n}{75}2^{n-75}$.

4. (a) There are 2^3 ways to choose the values of f at 4, 5, and 6, and then 3^{n-3} ways to choose the values elsewhere, yielding the answer $2^3 3^{n-3}$.

   (b) There are 2^3-2 ways to choose the values of f at 4, 5, and 6 (this differs from part (a) in that we must discard the case in which 4, 5, and 6 are all assigned 0 or all assigned 1), and then 3^{n-3} ways to choose the values elsewhere, yielding the answer $(2^3-2)3^{n-3}$.

5. (a) $\frac{11!}{2!2!2!} = \frac{11!}{8}$.

   (b) If all letters are different then there are $8 \cdot 7 \cdot 6 \cdot 5$ possibilities. If there are two Ms and the other two letters are distinct then there are $\binom{4}{2} \cdot 7 \cdot 6$ possibilities, and we get the same number of possibilities if M is replaced here by A or T. Finally, if the string contains two sets of identical letters then there are $\binom{3}{2}$ ways of choosing the two distinct letters and $\binom{4}{2}$ ways of placing them. The answer is thus $8 \cdot 7 \cdot 6 \cdot 5 + 3 \cdot \binom{4}{2} \cdot 7 \cdot 6 + \binom{3}{2}\binom{4}{2}$.

6. Let $P(n)$ be the statement that $a_n < 5^n$. Then $P(1)$, $P(2)$, and $P(3)$ are all true, since $1 < 5^1$, $4 < 5^2 = 25$, and $6 < 5^3 = 125$. Now let $n \geq 3$ and suppose that $P(k)$ is true for every $k = 1, \ldots, n$. Then

$$\begin{aligned} a_{n+1} = 2a_n + a_{n-1} + 11a_{n-2} &< 2 \cdot 5^n + 5^{n-1} + 11 \cdot 5^{n-2} \\ &\leq 2 \cdot 5^n + 5^n + \frac{11}{5^2} 5^n \\ &\leq 5 \cdot 5^n = 5^{n+1}, \end{aligned}$$

   so that $P(n+1)$ is true. We conclude by strong induction that $P(n)$ is true for all $n \geq 1$.

7. Let $P(n)$ be the statement that $5^n > (n+1)^2$. Then $P(1)$ is true, as $5 = 5^1 > 2^2 = 4$. Now let $n \geq 1$ and suppose that $P(n)$ is true. Then

$$5^{n+1} = 5 \cdot 5^n > 5(n+1)^2 = 5n^2 + 10n + 5 \geq n^2 + 4n + 4 = (n+2)^2,$$

   so that $P(n+1)$ is true. We conclude by induction that $P(n)$ is true for all $n \geq 1$.

8. Let $P(n)$ be the statement that $3^n + 7^n + 6$ is divisible by 8. Then $P(1)$ is true, since $3^1 + 7^1 + 6 = 16 = 2 \cdot 8$. Now let $n \geq 1$ and suppose that $P(n)$ is true. Then $3^n + 7^n + 6 = 8s$ for some integer s. We then have

$$3^{n+1} + 7^{n+1} + 6 = 3(3^n + 7^n + 6) + 4 \cdot 7^n - 12 = 8 \cdot 3s + 4(7^n - 3),$$

   and since $7^n - 3$ is always even (this needs a proof, which can be done by induction) we conclude that $3^{n+1} + 7^{n+1} + 6 = 8t$ is divisible by 8. Therefore $P(n+1)$ is true. It follows by induction that $P(n)$ is true for all $n \geq 1$.

9. (a) $C(20, 15)$.

(b) If $0 \leq k \leq 7$ and there are exactly k boxes which contain two balls, then there are $\binom{20}{k}$ ways of choosing these boxes and then $\binom{20-k}{15-2k}$ ways of distributing the remaining balls. The answer is thus $\sum_{k=0}^{n} \binom{20}{k}\binom{20-k}{15-2k}$.

(c) $C(20 + 15 - 1, 15) = C(34, 15)$.

10. $C(n + k, k)$.