

MATH 302.903
Practice Problems for the Final Examination
Spring 2006

1. Solve the following recurrence relations.
 - (a) $a_n = 6a_{n-1} + 2^n + 5$ with initial condition $a_0 = 6$.
 - (b) $a_n = 17a_{n-1} + 4n + 1$ with initial condition $a_0 = 0$.
 - (c) $a_n = 3a_{n-1} + 3^n$ with initial condition $a_0 = 1$.
 - (d) $a_n = a_{n-1} + a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = 2$, and $a_2 = 1$.
 - (e) $a_n = -6a_{n-1} + 7a_{n-2} + n5^n$ with initial conditions $a_0 = -1$ and $a_1 = 2$.

2. Define a relation R on the set of functions from $(0, \infty)$ to \mathbb{R} by fRg if $f(x)$ is $\Theta(g(x))$.
 - (a) Show that R is an equivalence relation.
 - (b) Which of the following pairs of functions are contained in the same equivalence class?
 - (i) $f(x) = x^3$ and $g(x) = 7x^3 + x^2 \ln x$.
 - (ii) $f(x) = (\ln x)^2$ and $g(x) = (\ln x)^3$.
 - (iii) $f(x) = e^{2x^2}$ and $g(x) = e^{x^2+7}$.
 - (iv) $f(x) = xe^x$ and $g(x) = e^x \ln x$.
 - (c) Let $f : (0, \infty) \rightarrow \mathbb{R}$ be an increasing function that satisfies the recurrence relation $f(x) = 2f(x/2) - 5x^2$ whenever $x = 2^k$ for a positive integer k , and let $g : (0, \infty) \rightarrow \mathbb{R}$ be defined by $g(x) = x^3 + 2x + 1$. Give necessary and sufficient conditions on f for f and g to be in the same equivalence class.

3. Determine the general form of the solutions to the recurrence relation $a_n = 7a_{n-1} - 15a_{n-2} + 9a_{n-3}$.

4. Determine with proof which of the following are equivalence relations:
- (a) the relation R on $\mathbb{Z} \times \mathbb{Z}$ defined by $(a_1, a_2)R(b_1, b_2)$ if $a_1b_2 = a_2b_1$,
 - (b) the relation R on the set of all functions from \mathbb{R} to \mathbb{R} defined by fRg if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$,
 - (c) the relation R on the set of all functions from \mathbb{R} to \mathbb{R} defined by fRg if $f(0) \leq g(0)$,
 - (d) the relation R on \mathbb{Z} defined by nRm if $0 \leq n, m \leq 5$ or $n = m$.
5. How many strings can be constructed from the letters in REARRANGEMENT (a) using all of the letters, (b) using exactly four of the letters, and (c) using exactly four of the letters with the requirement that exactly one N be used?
6. Determine how many strings with five or more characters can be formed from the letters in (a) SYMMETRY, (b) REFLEXIVE, and (c) RELATION.
7. Let A be the set of all relations on $\{0, 1, 2\}$ and let B be the set of all relations on $\{0, 1, 2, 3\}$. (a) How many functions are there from A to B ? (b) How many injective functions are there from A to B ? (c) How many surjective functions are there from A to B ? (d) How many functions are there from the power set of A to the power set of B ?
8. Prove by induction that $4^n + 11$ is divisible by 3 for all positive integers n .
9. Determine which of the following pairs of compound propositions are logically equivalent.
- (a) $q \rightarrow (p \vee q)$ and $(\neg q) \rightarrow (\neg(p \wedge q))$.
 - (b) $p \rightarrow (q \rightarrow r)$ and $r \vee (\neg(p \wedge (\neg q)))$.
 - (c) $(p \rightarrow q) \rightarrow q$ and $p \vee q$.
10. Define the functions $f, g : (0, \infty) \rightarrow \mathbb{R}$ by $f(x) = x^2 \ln x$ and $g(x) = x^2 + 1$. Determine, with proof, the least integer n such that $(f \circ g)(x)$ is $O(x^n)$.

Solutions

1. (a) Note that method of Section 6.2 won't work here, since the expression $2^n + 5$ added to the linear homogeneous part is not of the appropriate form. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the associated generating function. Then $xG(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$ and so

$$\begin{aligned} G(x) - 6xG(x) &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 6a_{n-1} x^n = a_0 + \sum_{n=1}^{\infty} (a_n - 6a_{n-1}) x^n \\ &= 6 + \sum_{n=1}^{\infty} 2^n x^n + \sum_{n=1}^{\infty} 5x^n = 6 + \sum_{n=0}^{\infty} 2^n x^n - 1 + \sum_{n=0}^{\infty} 5x^n - 5 \\ &= \frac{1}{1-2x} + \frac{5}{1-x}. \end{aligned}$$

Therefore $G(x) = \frac{1}{(1-6x)(1-2x)} + \frac{5}{(1-6x)(1-x)}$. Using partial fractions, we obtain

$$\begin{aligned} G(x) &= \frac{\frac{3}{2}}{1-6x} + \frac{-\frac{1}{2}}{1-2x} + \frac{6}{1-6x} + \frac{-1}{1-x} = \frac{\frac{15}{2}}{1-6x} - \frac{\frac{1}{2}}{1-2x} - \frac{1}{1-x} \\ &= \sum_{n=0}^{\infty} \left(\frac{15}{2} 6^n - \frac{1}{2} 2^n - 1 \right) x^n. \end{aligned}$$

Thus $a_n = \frac{15}{2} 6^n - \frac{1}{2} 2^n - 1$.

(b) We will take the approach of Section 6.2 (we could also use generating functions). The associated linear homogeneous recurrence relation is $a_n = 17a_{n-1}$, which has the characteristic equation $r - 17 = 0$ and hence has the general solution $a_n = \alpha 17^n$ where α is a constant. We then know that the given recurrence relation (without specifying the initial condition) has a particular solution of the form $p_1 n + p_0$. Substituting this into the recurrence relation we get

$$p_1 n + p_0 = 17(p_1(n-1) + p_0) + 4n + 1,$$

that is,

$$(16p_1 + 4)n - 17p_1 + 16p_0 + 1 = 0.$$

Since this holds for every $n \geq 1$, we must have $16p_1 + 4 = 0$ and $-17p_1 + 16p_0 + 1 = 0$. Thus $p_1 = -\frac{1}{4}$ and $p_0 = -\frac{21}{64}$. Therefore the general solution is $a_n = \alpha 17^n - \frac{1}{4}n - \frac{21}{64}$ where α is a constant. Considering now the initial condition, we have $0 = a_0 = \alpha - \frac{21}{64}$ and so $\alpha = \frac{21}{64}$. Thus the solution is $a_n = \frac{21}{64}17^n - \frac{1}{4}n - \frac{21}{64}$.

(c) We will solve this using generating functions (we could also take the approach of Section 6.2). Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the associated generating function. Then $xG(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$ and so

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 3a_{n-1} x^n = a_0 + \sum_{n=1}^{\infty} (a_n - 3a_{n-1}) x^n \\ &= 1 + \sum_{n=1}^{\infty} 3^n x^n = 1 + \sum_{n=0}^{\infty} 3^n x^n - 1 \\ &= \frac{1}{1-3x}. \end{aligned}$$

Thus $G(x) = \frac{1}{(1-3x)^2} = \sum_{n=0}^{\infty} (n+1)3^n x^n$, and so $a_n = (n+1)3^n$.

(d) This is a linear homogeneous recurrence relation with characteristic equation $r^3 - r^2 - r + 1 = (r-1)^2(r+1) = 0$. We thus know that the general solution (without specifying the initial conditions) has the form $a_n = b + cn + d(-1)^n$ where b, c, d are constants. Using the initial conditions we have

$$\begin{aligned} 1 &= a_0 = b + d, \\ 2 &= a_1 = b + c - d, \\ 1 &= a_2 = b + 2c + d \end{aligned}$$

and so $b = \frac{3}{2}$, $c = 0$, and $d = -\frac{1}{2}$. Thus the solution is $a_n = \frac{3}{2} - \frac{1}{2}(-1)^n$.

(e) We will follow the method of Section 6.2 (we could also proceed using generating functions). The associated linear homogeneous recurrence relation is $a_n = -6a_{n-1} + a_{n-2}$, which has the characteristic equation $r^2 + 6a_{n-1} - 7a_{n-2} = (r+7)(r-1) = 0$ and thus has the general solution $a_n = \alpha_1(-7)^n + \alpha_2$ where α_1, α_2 are constants. We then know that the given recurrence relation (without specifying the initial conditions) has a particular solution of the form $(p_1 n + p_0)5^n$ for some constants p_0, p_1 . Substituting this into the recurrence relation we get

$$(p_1n + p_0)5^n = -6(p_1(n-1) + p_0)5^{n-1} + 7(p_1(n-2) + p_0)5^{n-2} + n5^n,$$

from which it can be deduced that $p_0 = \frac{25}{144}$ and $p_1 = \frac{25}{48}$. Therefore the general solution is $a_n = \alpha_1 7^n + \alpha_2 + \left(\frac{25}{48}n + \frac{25}{144}\right)5^n$ where α_1 and α_2 are constants. Finally, the initial conditions yield

$$\begin{aligned} -1 &= a_0 = \alpha_1 + \alpha_2 + \frac{25}{144}, \\ 2 &= a_1 = -7\alpha_1 + \alpha_2 + 5\left(\frac{25}{48} + \frac{25}{144}\right). \end{aligned}$$

from which we get $\alpha_1 = \frac{43}{1152}$ and $\alpha_2 = -\frac{1395}{1152}$. Thus the solution is $a_n = \frac{43}{1152}7^n + \frac{1395}{1152} + \left(\frac{25}{48}n + \frac{25}{144}\right)5^n$.

2. (a) Let f, g, h be functions from $(0, \infty)$ to \mathbb{R} . Since $|f(x)| \leq |f(x)|$ for all $x \in (0, \infty)$, we see that R is reflexive. If $f(x)$ is $\Theta(g(x))$ then there exist constants $C, D > 0$ such that $C|g(x)| \leq |f(x)| \leq D|g(x)|$ for all sufficiently large x , so that $D^{-1}|f(x)| \leq |g(x)| \leq C^{-1}|f(x)|$ for all sufficiently large x , showing that $g(x)$ is $\Theta(f(x))$ and hence that R is symmetric. Now suppose that $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $\Theta(h(x))$. Then there are constants $C_1, C_2, D_1, D_2 > 0$ such that $C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$ and $D_1|h(x)| \leq |g(x)| \leq D_2|h(x)|$ for all sufficiently large x . It follows that $C_1D_1|h(x)| \leq |f(x)| \leq C_2D_2|h(x)|$ for all sufficiently large x , showing that $f(x)$ is $\Theta(h(x))$ and hence that R is transitive. Thus R is an equivalence relation.

(b) Only the functions in the first pair are contained in the same equivalence class. This can be seen by computing $\lim_{n \rightarrow \infty} \frac{f(x)}{g(x)}$ to be $\frac{1}{7}$ in (i), zero in (ii), and infinity in (iii) and (iv).

(c) By the Master Theorem, $f(x)$ is $O(x^2)$. Hence there exists a constant $C > 0$ such that $|f(x)| \leq Cx^2$ for all sufficiently large x . Now suppose that $g(x)$ is $\Theta(f(x))$. Then there exists a constant $D > 0$ such that $|g(x)| \leq D|f(x)|$ for all sufficiently large x . But then $|g(x)| \leq DCx^2$ for all sufficiently large x . This is impossible since $\lim_{x \rightarrow \infty} \frac{g(x)}{DCx^2} = \infty$. Therefore f and g are never in the same equivalence class.

3. This is a linear homogeneous recurrence relation with characteristic equation $r^3 - 7r^2 + 15r - 9 = (r-1)(r-3)^2 = 0$. Therefore the general form of the solutions is $a_n = \alpha_1 + (\alpha_2n + \alpha_3)3^n$ where $\alpha_1, \alpha_2, \alpha_3$ are constants.

4. (a) This relation is not transitive. For example, $(1, 2)R(0, 0)$ and $(0, 0)R(2, 1)$ but $(1, 2) \not R(2, 1)$.
- (b) This relation is not symmetric. For example, if $f(x) = 0$ and $g(x) = 1$ for all $x \in \mathbb{R}$ then fRg but $g \not Rf$.
- (c) This relation is not symmetric. Use the same example as in (b).
- (d) By the definition of the relation we see that nRm if and only if n and m are contained in a common member of the partition $\{\dots, \{-3\}, \{-2\}, \{-1\}, \{0, 1, 2, 3, 4, 5\}, \{6\}, \{7\}, \{8\}, \dots\}$ of \mathbb{Z} . Since equivalence relations on a set correspond to partitions of that set, we conclude that R is an equivalence relation.
5. (a) $\frac{13!}{3!3!2!2!}$.
- (b) The number of possibilities with all letters different is $7 \cdot 6 \cdot 5 \cdot 4$, with three Rs or three Es $2 \binom{4}{3} 6$, with two pairs of identical letters $\binom{4}{2} \binom{4}{2}$, and with exactly one pair of identical letters $\binom{4}{1} \binom{4}{2} 6 \cdot 5$. Summing these numbers yields the answer.
- (c) The number of possibilities with all letters different is $\binom{4}{1} 6 \cdot 5 \cdot 4$, with three Rs or three Es $2 \binom{4}{3}$, and with exactly one pair of identical letters $\binom{3}{1} \binom{4}{2} \binom{2}{1} 5$. Now sum these numbers to obtain the answer.
6. (a) Suppose $5 \leq n \leq 8$. Then the number of strings that can be formed using n letters with no repeated letters is $6!/(6-n)!$ if $n \leq 6$ and zero otherwise, with two pairs of repeated letters $\binom{n}{2} \binom{(n-2)}{2} 4!/(8-n)!$, and with exactly one pair of repeated letters $\binom{2}{1} \binom{n}{2} 5!/(7-n)!$ if $n \leq 7$ and zero otherwise. Summing all of these numbers over $n = 5, 6, 7, 8$ yields the answer.
- (b) Suppose $5 \leq n \leq 9$. Then the number of strings that can be formed using n letters with no repeated letters is $7!/(7-n)!$ if $n \leq 7$ and zero otherwise, with exactly two Es $\binom{n}{2} 6!/(8-n)!$ if $n \leq 8$ and zero otherwise, and with exactly three Es $\binom{n}{3} 6!/(9-n)!$. Summing all of these numbers over $n = 5, 6, 7, 8, 9$ we obtain the answer.
- (c) Since there are no repeated letters, the number of strings that can be formed using n letters for $0 \leq n \leq 8$ is $8!/(8-n)!$, and so the answer is $8! \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \right)$.

7. A relation on a set X is by definition a subset of $X \times X$. Thus there are $2^{3^2} = 2^9$ possible relations on $\{0, 1, 2\}$ and $2^{4^2} = 2^{16}$ possible relations on $\{0, 1, 2, 3\}$. Hence there are $|B|^{|A|} = (2^{16})^{2^9} = 2^{16 \cdot 2^9}$ functions from A to B , and $2^{16!}/(2^{16} - 2^9)!$ injective functions from A to B . Since the cardinality of B is larger than the cardinality of A , there are no surjective functions from A to B . Finally, since the power set $P(A)$ of A has $2^{|A|} = 2^{2^9}$ elements and the power set $P(B)$ of B has $2^{|B|} = 2^{2^{16}}$ elements, the number of functions from $P(A)$ to $P(B)$ is $|P(B)|^{|P(A)|} = (2^{2^{16}})^{2^{2^9}} = 2^{2^{16} \cdot 2^{2^9}}$.
8. The assertion holds for $n = 1$ since $4^1 + 11 = 15 = 5 \cdot 3$. Now suppose we are given a positive integer n such that the assertion holds for n . Then $4^n + 11 = 3k$ for some integer k , and so

$$4^{n+1} + 11 = 4 \cdot (4^n + 11) - 33 = 4(3k) - 33 = 3(4k - 11).$$

Thus the assertion holds for $n + 1$, and we obtain the result by induction.

9. The pairs of propositions in (a) and (c) are logically equivalent.
10. We have $(f \circ g)(x) = (x^4 + 2x^2 + 1) \ln(x^2 + 1)$ for all $x \in (0, \infty)$. Since $\lim_{x \rightarrow \infty} \frac{\ln(x^2 + 1)}{x} = 0$, we have $\ln(x^2 + 1) \leq x$ for all sufficiently large x . Also, $x^4 + 2x^2 + 1 \leq x^4 + 2x^4 + x^4 = 4x^4$ for $x \geq 1$. Thus for all sufficiently large x we have

$$|(f \circ g)(x)| = (x^4 + 2x^2 + 1) \ln(x^2 + 1) \leq 4x^5$$

and so $(f \circ g)(x)$ is $O(x^5)$. On the other hand, $\lim_{x \rightarrow \infty} \frac{(f \circ g)(x)}{x^4} = \lim_{x \rightarrow \infty} (1 + 2x^{-2} + x^{-4}) \ln(x^2 + 1) = \infty$ and so $(f \circ g)(x)$ is not $O(x^4)$. Therefore 5 is the least integer n such that $(f \circ g)(x)$ is $O(x^n)$.