1. Is \(4x + y^3 - 5y = 0\) an implicit solution to the differential equation \(\frac{dy}{dx} = \frac{8}{3y^2 - 5}\)?

2. Find the general solution to each of the following differential equations.
   (a) \(y'' + 4y' - 5y = 0\)
   (b) \(y' = 4xy^4\) for \(x > 0\)
   (c) \(xy' + 3y = x\) for \(x > 0\)
   (d) \(y'' - 2y + 5 = 0\)
   (e) \(y'' - y = e^x - x - 1\)

3. Construct a second-order linear differential equation that has both \(y_1(x) = 2xe^{4x} + \sin x\) and \(y_2(x) = 7e^{4x} + \sin x\) as solutions.

4. Given that \(f(x) = x^{-1}\) is a solution to the differential equation \(x^2y'' + 4xy' + 2y = 0\), find another linearly independent solution.

5. Compute the Wronskian for each of the following pairs of functions.
   (a) \(y_1(x) = 1,\ y_2(x) = \sin x\)
   (b) \(y_1(x) = e^{3x},\ y_2(x) = x\)
   (c) \(y_1(x) = 3,\ y_1(x) = -2\)
   (d) \(y_1(x) = e^{x} \sin x,\ y_2(x) = e^{x} \cos x\)

6. Find a particular solution to the differential equation
   \[
   \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = \cos x.
   \]
7. Solve the following initial value problems.

(a) \(y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0\)

(b) \(y' = 2xy, \quad y(0) = 3\)

(c) \(y'' - 3y' = 18x, \quad y(1) = 0, \quad y'(1) = 0\)

8. Which of the following pairs of functions are solutions to the same second-order linear homogeneous differential equation? Justify your answer.

(a) \(y_1(x) = e^x, \quad y_2(x) = e^x \sin x\)

(b) \(y_1(x) = e^x \cos 2x, \quad y_1(x) = e^x \sin 2x\)

(c) \(y_1(x) = x^3, \quad y_2(x) = x^4\)

(d) \(y_1(x) = 1, \quad y_2(x) = e^{-4x}\)

**Solutions**

1. Implicitly differentiating the equation \(4x + y^3 - 5y = 0\) gives \(4 + 3y^2y' - 5y' = 0\), or \(y'(3y^2 - 5) = -4\). Since \(-4 \neq 8\) such a \(y\) is not a solution to the given differential equation.

2. (a) This is a linear second-order homogeneous equation with constant coefficients. The auxiliary equation is \(r^2 + 4r - 5 = (r - 1)(r + 5) = 0\), which has roots 1 and -5. So the general solution is \(y(x) = c_1e^x + c_2e^{-5x}\) where \(c_1\) and \(c_2\) are arbitrary constants.

(b) Here we can separate the variables:

\[
\int \frac{dy}{y^2} = \int 4x \, dx.
\]

Thus \(-\frac{1}{3}y^{-3} = 2x^2 + C\) and so \(y = (-6x^2 + D)^{-1/3}\) where \(D\) is an arbitrary constant.
(c) Put the equation in the standard form $y' + \frac{3}{x}y = 1$ and compute the integrating factor (recall that $x > 0$)

$$
\mu(x) = \exp\left[\int \frac{3}{x} \, dx\right] = e^{3 \ln x} = x^3.
$$

Then

$$
y(x) = \frac{1}{\mu(x)} \left[ \int \mu(x) \cdot 1 \, dx + C \right] = \frac{1}{x^3} \left[ \int x^3 \, dx + C \right]
$$

$$
= \frac{1}{x^3} \left( \frac{1}{4} x^4 + C \right) = \frac{1}{4} x + \frac{C}{x^3}
$$

where $C$ is an arbitrary constant.

(d) The corresponding homogeneous equation is $y'' - 2y = 0$, which has auxiliary equation $r^2 - 2 = 0$ with roots $\pm \sqrt{2}$, so that its general solution is $y(x) = c_1 e^{\sqrt{2} x} + c_2 e^{-\sqrt{2} x}$ where $c_1$ and $c_2$ are arbitrary constants. As a particular solution to the given equation $y'' - 2y = -5$, try $y_p(x) = A$ where $A$ is a constant. Then $y_p'' - 2y_p = -2A = -5$ and hence $A = \frac{5}{2}$. Thus the general solution to the given equation is

$$
y(x) = \frac{5}{2} + c_1 e^{\sqrt{2} x} + c_2 e^{-\sqrt{2} x}
$$

where $c_1$ and $c_2$ are arbitrary constants.

(e) The corresponding homogeneous equation is $y'' - y = 0$, which has auxiliary equation $r^2 - 1 = 0$ with roots $\pm 1$, so that its general solution is $y(x) = c_1 e^x + c_2 e^{-x}$ where $c_1$ and $c_2$ are arbitrary constants. Since $y(x) = e^x$ is a solution to the homogeneous equation, let us try $y_p(x) = Ax e^x + Bx + C$ as a particular solution to the given equation. We have $y_p'(x) = A x e^x + A e^x + B$ and $y_p''(x) = A x e^x + 2A e^x$ so that

$$
y_p'' - y_p = A x e^x + 2A e^x - (A x e^x + Bx + C) = 2A e^x - Bx - C,
$$

and this is equal to $e^x - x - 1$ when $A = \frac{1}{2}$, $B = 1$, and $C = 1$. Thus the general solution to the given equation is

$$
y(x) = \frac{1}{2} x e^x + x + 1 + c_1 e^x + c_2 e^{-x}
$$

where $c_1$ and $c_2$ are arbitrary constants.
3. Notice that a second-order linear homogeneous differential equation with constant coefficients whose auxiliary equation is \((r-4)^2 = r^2 - 8r + 16 = 0\) has the particular solutions \(w_1(x) = 2xe^{4x}\) and \(w_2(x) = 7e^{4x}\). This differential equation is \(y'' - 8y' + 16y = 0\). Now if we compute \(L[y] := y'' - 8y' + 16y\) for the function \(y(x) = \sin x\), then \(L[y] = 15\sin x - 8\cos x\). Since \(L\) is a linear operator, we see that the given functions \(y_1\) and \(y_2\) are both solutions to the equation \(y'' - 8y' + 16y = 15\sin x - 8\cos x\).

4. Put the equation in the standard form \(y'' + \frac{4}{5}y' + \frac{2}{x^2}y = 0\) and use the reduction of order formula:

\[
y(x) = f(x) \int \frac{e^{-\int 4x^{-1} \, dx}}{[f(x)]^2} \, dx = x^{-1} \int x^{-2} \, dx = x^{-1}(-x^{-1}) = -x^{-2}.
\]

5. (a) \(W[y_1, y_2](x) = 1 \cdot \cos x - 0 \cdot \sin x = \cos x\).

(b) \(W[y_1, y_2](x) = e^{3x} \cdot 1 - 3e^{3x} \cdot x = (1 - 3x)e^{3x}\).

(c) \(W[y_1, y_2](x) = 3 \cdot 0 - 0 \cdot (-2) = 0\).

(d) \(W[y_1, y_2](x) = e^x \sin x(e^x \cos x - e^x \sin x) - (e^x \sin x + e^x \cos x)e^x \cos x = -e^{2x}\).

6. Using the method of undetermined coefficients, we look for a particular solution of the form \(y_p(x) = A\sin x + B\cos x\). Substituting into the differential equation yields

\[
y''_p - y'_p + y_p = (-A\sin x - B\cos x) - (A\cos x - B\sin x) + (A\sin x + B\cos x)
\]

so that \(A = -1\) and \(B = 0\). Thus \(y_p(x) = -\sin x\) is a particular solution.

7. (a) This is a linear second-order homogeneous equation with constant coefficients. The auxiliary equation is \(r^2 + r - 2 = (r - 1)(r + 2) = 0\), which has roots 1 and -2. So the general solution is \(y(x) = c_1e^x + c_2e^{-2x}\) where \(c_1\) and \(c_2\) are arbitrary constants. To solve the initial value problem we compute \(y'(x) = c_1e^x - 2c_2e^{-2x}\) and require that \(1 = y(0) = c_1 + c_2\) and \(0 = y'(0) = c_1 - 2c_2\), which together yield \(c_1 = \frac{2}{3}\) and \(c_2 = \frac{1}{3}\). Thus the solution is \(y(x) = \frac{2}{3}e^x + \frac{1}{3}e^{-2x}\).
(b) Here we can separate the variables:
\[ \int \frac{dy}{y} = \int 2x \, dx. \]

Thus \( \ln |y| = x^2 + C \) and so the general solution to the differential equation is \( y = De^{x^2} \) where \( D \) is an arbitrary constant. For the initial value problem, we require that \( 3 = y(0) = D \), and so the solution is \( y(x) = 3e^{x^2} \).

(c) The corresponding homogeneous equation is \( y'' - 3y' = 0 \), which has auxiliary equation \( r^2 - 3r = r(r-3) = 0 \) with roots 0 and 3, so that its general solution is \( y(x) = c_1 + c_2e^{3x} \) where \( c_1 \) and \( c_2 \) are arbitrary constants. Using the method of undetermined coefficients, let us look for a particular solution to the given equation of the form \( y_p(x) = Ax^2 + Bx \) (we use this instead of \( Ax + B \) because constant functions are solutions to the homogeneous equation). We have \( y_p''(x) = 2A \) so that
\[ y_p'' - 3y_p' = 2A - 3(2Ax + B) = -6Ax + (2A - 3B) = -18x. \]
Therefore \( A = 3 \) and \( B = 2 \) and a particular solution is \( y_p(x) = 3x^2 + 2x \). Thus the general solution to the given equation is
\[ y(x) = 3x^2 + 2x + c_1 + c_2e^{3x} \]
where \( c_1 \) and \( c_2 \) are arbitrary constants. For the initial value problem we compute \( y'(x) = 6x + 2 + 3c_2e^{3x} \) and require that \( 0 = y(1) = 3 + 2 + c_1 + c_2e^3 \) and \( 0 = y'(1) = 6 + 2 + 3c_2e^3 \). Then \( c_2 = -\frac{8}{3}e^{-3} \) and \( c_1 = -\frac{7}{3} \), and so the solution is \( y(x) = 3x^2 + 2x - \frac{7}{3} - \frac{8}{3}e^{3(x-1)}. \)

8. (a) Observe that the Wronskian
\[ W[y_1, y_2](x) = e^x(e^x \sin x + e^x \cos x) - e^x(e^x \sin x) = e^x \cos x \]
is zero when \( x \) is of the form \( (n + \frac{1}{2})\pi \) for some integer \( n \) and nonzero otherwise. Therefore \( y_1 \) and \( y_2 \) cannot be solutions to the same second-order linear homogeneous differential equation.

(b) These are solutions to a second-order linear homogeneous differential equation with constant coefficients whose auxiliary equation is \( (r-1-2i)(r-1+2i) = r^2 - 2r + 5 = 0 \). This differential equation is \( y'' - 2y' + 5y = 0 \).
(c) Observe that the Wronskian

\[ W[y_1, y_2](x) = x^3 \cdot 4x^3 - 3x^2 \cdot x^4 = x^6 \]

is zero when \( x = 0 \) and nonzero otherwise. Therefore \( y_1 \) and \( y_2 \) cannot be solutions to the same second-order linear homogeneous differential equation.

(d) These are solutions to a second-order linear homogeneous differential equation with constant coefficients whose auxiliary equation is \( r(r+4) = r^2 + 4r = 0 \). This differential equation is \( y'' + 4y' = 0 \).