MATH 308
Practice Problems for Examination 1
Fall 2008

1. Is $4x + y^3 - 5y = 0$ an implicit solution to the differential equation $\frac{dy}{dx} = \frac{8}{3y^2 - 5}$?

2. Find the general solution to each of the following differential equations.

   (a) $y'' + 4y' - 5y = 0$

   (b) $y' = 4xy^4$ for $x > 0$

   (c) $xy' + 3y = x$ for $x > 0$

   (d) $y'' - 2y + 5 = 0$

   (e) $y'' - y = e^x - x - 1$

3. Construct a second-order linear differential equation that has both $y_1(x) = 2xe^{4x} + \sin x$ and $y_2(x) = 7e^{4x} + \sin x$ as solutions.

4. Given that $f(x) = x^{-1}$ is a solution to the differential equation $x^2y'' + 4xy' + 2y = 0$, find another linearly independent solution.

5. Compute the Wronskian for each of the following pairs of functions.

   (a) $y_1(x) = 1, \quad y_2(x) = \sin x$

   (b) $y_1(x) = e^{3x}, \quad y_2(x) = x$

   (c) $y_1(x) = 3, \quad y_1(x) = -2$

   (d) $y_1(x) = e^x \sin x, \quad y_2(x) = e^x \cos x$

6. Find a particular solution to the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = \cos x.$$

7. Solve the following initial value problems.

   (a) $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$

   (b) $y' = 2xy, \quad y(0) = 3$

   (c) $y'' - 3y' = 18x, \quad y(1) = 0, \quad y'(1) = 0$

8. Which of the following pairs of functions are solutions to the same second-order linear homogeneous differential equation? Justify your answer.

   (a) $y_1(x) = e^x, \quad y_2(x) = e^x \sin x$

   (b) $y_1(x) = e^x \cos 2x, \quad y_1(x) = e^x \sin 2x$

   (c) $y_1(x) = x^3, \quad y_2(x) = x^4$

   (d) $y_1(x) = 1, \quad y_2(x) = e^{-4x}$

Solutions

1. Implicitly differentiating the equation $4x + y^3 - 5y = 0$ gives $4 + 3y^2y' - 5y' = 0$, or $y'(3y^2 - 5) = -4$. Since $-4 \neq 8$ such a y is not a solution to the given differential equation.

2. (a) This is a linear second-order homogeneous equation with constant coefficients. The auxiliary equation is $r^2 + 4r - 5 = (r - 1)(r + 5) = 0$, which has roots 1 and -5. So the general solution is $y(x) = c_1e^x + c_2e^{-5x}$ where c_1 and c_2 are arbitrary constants.

   (b) Here we can separate the variables:

$$\int \frac{dy}{y^4} = \int 4x \, dx.$$

   Thus $-\frac{1}{3}y^{-3} = 2x^2 + C$ and so $y = (-6x^2 + D)^{-1/3}$ where D is an arbitrary constant.

(c) Put the equation in the standard form $y' + \frac{3}{x}y = 1$ and compute the integrating factor (recall that $x > 0$)

$$\mu(x) = \exp\left[\int \frac{3}{x}dx\right] = e^{3\ln x} = x^3.$$

Then

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\left[\int \mu(x)\cdot 1\,dx + C\right] = \frac{1}{x^3}\left[\int x^3\,dx + C\right] \\ &= \frac{1}{x^3}\left(\frac{1}{4}x^4 + C\right) = \frac{1}{4}x + \frac{C}{x^3} \end{aligned}$$

where C is an arbitrary constant.

(d) The corresponding homogeneous equation is $y'' - 2y = 0$, which has auxiliary equation $r^2 - 2 = 0$ with roots $\pm\sqrt{2}$, so that its general solution is $y(x) = c_1e^{\sqrt{2}x} + c_2e^{-\sqrt{2}x}$ where c_1 and c_2 are arbitary constants. As a particular solution to the given equation $y''-2y = -5$, try $y_p(x) = A$ where A is a constant. Then $y_p''-2y_p = -2A = -5$ and hence $A = \frac{5}{2}$. Thus the general solution to the given equation is

$$y(x) = \frac{5}{2} + c_1e^{\sqrt{2}x} + c_2e^{-\sqrt{2}x}$$

where c_1 and c_2 are arbitary constants.

(e) The corresponding homogeneous equation is $y''-y = 0$, which has auxiliary equation $r^2-1 = 0$ with roots ± 1, so that its general solution is $y(x) = c_1e^x+c_2e^{-x}$ where c_1 and c_2 are arbitary constants. Since $y(x) = e^x$ is a solution to the homogeneous equation, let us try $y_p(x) = Axe^x + Bx + C$ as a particular solution to the given equation. We have $y_p'(x) = Axe^x + Ae^x + B$ and $y_p''(x) = Axe^x + 2Ae^x$ so that

$$y_p'' - y_p = Axe^x + 2Ae^x - (Axe^x + Bx + C) = 2Ae^x - Bx - C,$$

and this is equal to $e^x - x - 1$ when $A = \frac{1}{2}$, $B = 1$, and $C = 1$. Thus the general solution to the given equation is

$$y(x) = \frac{1}{2}xe^x + x + 1 + c_1e^x + c_2e^{-x}$$

where c_1 and c_2 are arbitary constants.

3. Notice that a second-order linear homogeneous differential equation with constant coefficients whose auxiliary equation is $(r-4)^2 = r^2 - 8r + 16 = 0$ has the particular solutions $w_1(x) = 2xe^{4x}$ and $w_2(x) = 7e^{4x}$. This differential equation is $y'' - 8y' + 16y = 0$. Now if we compute $L[y] := y'' - 8y' + 16y$ for the function $y(x) = \sin x$, then $L[y] = 15\sin x - 8\cos x$. Since L is a linear operator, we see that the given functions y_1 and y_2 are both solutions to the equation $y'' - 8y' + 16y = 15\sin x - 8\cos x$.

4. Put the equation in the standard form $y'' + \frac{4}{x}y' + \frac{2}{x^2}y = 0$ and use the reduction of order formula:

$$\begin{aligned} y(x) = f(x)\int \frac{e^{-\int 4x^{-1}\,dx}}{[f(x)]^2}\,dx \; &= x^{-1}\int \frac{e^{-4\ln|x|}}{x^{-2}}\,dx \\ = x^{-1}\int x^{-2}\,dx &= x^{-1}(-x^{-1}) = -x^{-2}. \end{aligned}$$

5. (a) $W[y_1, y_2](x) = 1 \cdot \cos x - 0 \cdot \sin x = \cos x$.

   (b) $W[y_1, y_2](x) = e^{3x} \cdot 1 - 3e^{3x} \cdot x = (1 - 3x)e^{3x}$.

   (c) $W[y_1, y_2](x) = 3 \cdot 0 - 0 \cdot (-2) = 0$.

   (d) $W[y_1, y_2](x) = e^x \sin x(e^x \cos x - e^x \sin x) - (e^x \sin x + e^x \cos x)e^x \cos x = -e^{2x}$.

6. Using the method of undetermined coefficients, we look for a particular solution of the form $y_p(x) = A\sin x + B\cos x$. Substituting into the differential equation yields

$$\begin{aligned} y_p'' - y_p' + y_p &= (-A\sin x - B\cos x) - (A\cos x - B\sin x) + (A\sin x + B\cos x) \\ &= B\sin x - A\cos x = \cos x \end{aligned}$$

   so that $A = -1$ and $B = 0$. Thus $y_p(x) = -\sin x$ is a particular solution.

7. (a) This is a linear second-order homogeneous equation with constant coefficients. The auxiliary equation is $r^2 + r - 2 = (r-1)(r+2) = 0$, which has roots 1 and -2. So the general solution is $y(x) = c_1 e^x + c_2 e^{-2x}$ where c_1 and c_2 are arbitrary constants. To solve the initial value problem we compute $y'(x) = c_1 e^x - 2c_2 e^{-2x}$ and require that $1 = y(0) = c_1 + c_2$ and $0 = y'(0) = c_1 - 2c_2$, which together yield $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$. Thus the solution is $y(x) = \frac{2}{3}e^x + \frac{1}{3}e^{-2x}$.

(b) Here we can separate the variables:

$$\int \frac{dy}{y} = \int 2x\,dx.$$

Thus $\ln|y| = x^2 + C$ and so the general solution to the differential equation is $y = De^{x^2}$ where D is an arbitrary constant. For the initial value problem, we require that $3 = y(0) = D$, and so the solution is $y(x) = 3e^{x^2}$.

(c) The corresponding homogeneous equation is $y'' - 3y' = 0$, which has auxiliary equation $r^2 - 3r = r(r-3) = 0$ with roots 0 and 3, so that its general solution is $y(x) = c_1 + c_2e^{3x}$ where c_1 and c_2 are arbitary constants. Using the method of undetermined coefficients, let us look for a particular solution to the given equation of the form $y_p(x) = Ax^2 + Bx$ (we use this instead of $Ax + B$ because constant functions are solutions to the homogeneous equation). We have $y_p'(x) = 2Ax+B$ and $y_p''(x) = 2A$ so that

$$y_p'' - 3y_p' = 2A - 3(2Ax + B) = -6Ax + (2A - 3B) = -18x.$$

Therefore $A = 3$ and $B = 2$ and a particular solution is $y_p(x) = 3x^2 + 2x$. Thus the general solution to the given equation is

$$y(x) = 3x^2 + 2x + c_1 + c_2e^{3x}$$

where c_1 and c_2 are arbitary constants. For the initial value problem we compute $y'(x) = 6x + 2 + 3c_2e^{3x}$ and require that $0 = y(1) = 3 + 2 + c_1 + c_2e^3$ and $0 = y'(1) = 6 + 2 + 3c_2e^3$. Then $c_2 = -\frac{8}{3}e^{-3}$ and $c_1 = -\frac{7}{3}$, and so the solution is $y(x) = 3x^2 + 2x - \frac{7}{3} - \frac{8}{3}e^{3(x-1)}$.

8. (a) Observe that the Wronskian

$$W[y_1, y_2](x) = e^x(e^x \sin x + e^x \cos x) - e^x(e^x \sin x) = e^x \cos x$$

is zero when x is of the form $(n + \frac{1}{2})\pi$ for some integer n and nonzero otherwise. Therefore y_1 and y_2 cannot be solutions to the same second-order linear homogeneous differential equation.

(b) These are solutions to a second-order linear homogeneous differential equation with constant coefficients whose auxiliary equation is $(r-1-2i)(r-1+2i) = r^2-2r+5 = 0$. This differential equation is $y'' - 2y' + 5y = 0$.

(c) Observe that the Wronskian

$$W[y_1, y_2](x) = x^3 \cdot 4x^3 - 3x^2 \cdot x^4 = x^6$$

is zero when $x = 0$ and nonzero otherwise. Therefore y_1 and y_2 cannot be solutions to the same second-order linear homogeneous differential equation.

(d) These are solutions to a second-order linear homogeneous differential equation with constant coefficients whose auxiliary equation is $r(r+4) = r^2+4r = 0$. This differential equation is $y'' + 4y' = 0$.