MATH 308
Practice Problems for Examination 2
Fall 2008

1. Using variation of parameters find the general solution to the differential equation $y'' + 2y' + y = x^{-4}e^{-x}$.

2. Determine the Laplace transform of each of the following functions.

   (a) $f(t) = \begin{cases} t, & 0 < t < 2 \\ e^t, & t > 2 \end{cases}$

   (b) $f(t) = te^{-t}\sin 2t$

   (c) $f(t) = 2 + t^5 - e^{3t}\sin 2t$

   (d) $f(t) = e^t u(t-1)u(2-t)$ where u is the unit step function

   (e) $f(t) = t^2 g(t)$ where g is a function such that $\mathscr{L}\{g\}(s) = (s^2+1)^{-4}$

   (f) f is a periodic function with period 3 and $f(t) = \begin{cases} 1, & 0 < t < 2 \\ 0, & 2 < t < 3 \end{cases}$

3. Determine the inverse Laplace transform of each of the following functions.

   (a) $F(s) = \dfrac{s}{s^2+2s+10}$

   (b) $F(s) = \dfrac{e^{-s}}{s^2-4} + \dfrac{3}{s^2}$

   (c) $F(s) = \dfrac{1}{(s^2+4)(s^2+1)}$

   (d) $F(s) = G''(s-2)$ where G is a function such that $\mathscr{L}^{-1}\{G\}(t) = e^{-t^2}$

4. Find the transfer function and impulse response function for the system described by the initial value problem

$$y'' - 6y' + 10y = g(t), \quad y(0) = 1, \; y'(0) = -1.$$

and give a formula for the solution.

5. Solve the initial value problem

$$y' + y = g(t), \quad y(0) = 2$$

where $g(t) = \begin{cases} 4, & 0 < t < 6 \\ 2, & t > 6. \end{cases}$

6. Let f be a function such that $\mathscr{L}\{f\}(s) = \dfrac{s^2}{s^2+4}$. Determine the Laplace transform of the function $h(t) = t^2(f * f)(t)$.

7. Solve the initial value problem

$$y'' - y' = 2te^t, \quad y(0) = 0, \; y'(0) = 2.$$

Solutions

1. The corresponding homogeneous equation $y'' + 2y' + y = 0$ has auxiliary equation $r^2 + 2r + 1 = (r+1)^2 = 0$ and hence has fundamental solutions $y_1(x) = e^{-x}$ and $y_2(x) = xe^{-x}$. We compute

$$W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = e^{-x}(e^{-x} - xe^{-x}) - (-e^{-x})(xe^{-x}) = e^{-2x}$$

and then

$$v_1(x) = \int \frac{-y_2(x)x^{-4}e^{-x}}{W[y_1, y_2](x)} dx = \int -x^{-3}\, dx = \frac{1}{2}x^{-2},$$
$$v_2(x) = \int \frac{y_1(x)x^{-4}e^{-x}}{W[y_1, y_2](x)} dx = \int x^{-4}\, dx = -\frac{1}{3}x^{-3}.$$

Thus a particular solution to the given equation is

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x) = \frac{1}{2}x^{-2}e^{-x} - \frac{1}{3}x^{-3}(xe^{-x}) = \frac{1}{6}x^{-2}e^{-x}.$$

2. (a) Write $f(t) = t + (e^t - t)u(t-2)$. Since

$$\mathscr{L}\{(e^t - t)u(t-2)\}(s) = e^{-2s}\mathscr{L}\{e^{t+2} - (t+2)\} = e^{-2s}\left(e^2\frac{1}{s-1} - \frac{1}{s^2} - \frac{2}{s}\right)$$

we have

$$\mathscr{L}\{f\}(s) = \frac{1}{s} + e^{-2s+2}\frac{1}{s-1} + e^{-2s}\left(\frac{1}{s^2} - \frac{2}{s}\right).$$

(b) We have

$$\begin{aligned}\mathscr{L}\{te^{-t}\sin 2t\}(s) &= -\frac{d}{ds}\big(\mathscr{L}\{e^{-t}\sin 2t\}(s)\big) = -\frac{d}{ds}\big(\mathscr{L}\{\sin 2t\}(s+1)\big)\\ &= -\frac{d}{ds}\left(\frac{2}{(s+1)^2+4}\right) = -\frac{d}{ds}\left(\frac{2}{s^2+2s+5}\right)\\ &= \frac{4s+4}{(s^2+2s+5)^2}.\end{aligned}$$

(c) We have

$$\begin{aligned}\mathscr{L}\{2 + t^5 - e^{3t}\sin 2t\}(s) &= \mathscr{L}\{2\}(s) + \mathscr{L}\{t^5\}(s) - \mathscr{L}\{e^{3t}\sin 2t\}(s)\\ &= \frac{2}{s} + \frac{5!}{s^6} - \frac{2}{(s-3)^2+4}.\end{aligned}$$

(d) We can express f as

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ e^t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

and so

$$\begin{aligned}\mathscr{L}\{f\}(s) &= \int_0^\infty e^{-st} f(t)\,dt = \int_1^2 e^{-st}e^t\,dt = \int_1^2 e^{(-s+1)t}\,dt \\ &= \frac{1}{-s+1}e^{(-s+1)t}\bigg|_{t=1}^{t=2} = \frac{1}{-s+1}\big(e^{-2s+2} - e^{-s+1}\big).\end{aligned}$$

One could alternatively write $f(t) = e^t u(t-1) - e^t u(t-2)$ and compute the transform using the appropriate formula.

(e) We have

$$\mathscr{L}\{f\}(s) = \frac{d^2}{ds^2}\big(\mathscr{L}\{g(t)\}(s)\big) = \frac{d^2}{ds^2}(s^2+1)^{-4} = \frac{8(9s^2-1)}{(s^2+1)^6}.$$

(f) Setting

$$f_3(t) = \begin{cases} 1, & 0 < t < 2 \\ 0, & t > 2 \end{cases}$$

we have

$$\mathscr{L}\{f_3(t)\}(s) = \int_0^\infty e^{-st} f_3(t)\,dt = \int_0^2 e^{-st}\,dt = -\frac{1}{s}e^{-st}\bigg|_{t=0}^{t=2} = -\frac{1}{s}(e^{-2s} - 1)$$

and so $\mathscr{L}\{f\}(s) = \dfrac{\mathscr{L}\{f_3(t)\}(s)}{1-e^{3s}} = \dfrac{1-e^{-2s}}{s(1-e^{-3s})}$.

3. (a) Writing $F(s) = \dfrac{s}{(s+1)^2+9} = \dfrac{s+1}{(s+1)^2+9} - \dfrac{1}{3}\cdot\dfrac{3}{(s+1)^2+9}$ we see that

$$\mathscr{L}^{-1}\{F\}(t) = e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t.$$

(b) Using partial fractions we have

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2-4}\right\}(t) = \mathscr{L}^{-1}\left\{\frac{1}{4}\cdot\frac{1}{s-2} - \frac{1}{4}\cdot\frac{1}{s+2}\right\} = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$$

and so $\mathscr{L}^{-1}\{F\}(t) = \frac{1}{4}\big(e^{2(t-1)} - e^{-2(t-1)}\big)u(t-1) + 3t$.

(c) Using partial fractions we have

$$\frac{1}{(s^2+4)(s^2+1)} = -\frac{1}{6} \cdot \frac{2}{s^2+4} + \frac{1}{3} \cdot \frac{1}{s^2+1}$$

and so $\mathscr{L}^{-1}\{F\}(t) = -\frac{1}{6}\sin 2t + \frac{1}{3}\sin t$.

(d) $\mathscr{L}^{-1}\{F(s)\}(t) = e^{2t}\mathscr{L}^{-1}\{G''(s)\}(t) = e^{2t}t^2\mathscr{L}^{-1}\{G(s)\}(t) = t^2e^{2t-t^2}$.

4. The transfer function is

$$H(s) = \frac{1}{s^2-6s+10} = \frac{1}{(s-3)^2+1}$$

and so the impulse response function is $h(t) = e^{3t}\sin t$. The corresponding homogeneous differential equation has auxiliary equation $r^2 - 6r + 10 = 0$ with roots $3 \pm i$, yielding the general solution $y(t) = c_1e^{3t}\cos t + c_2e^{3t}\sin t$, with $c_1 = 1$ and $c_2 = -3$ for the given initial conditions. Thus the solution to the given initial value problem is

$$\begin{aligned} y(t) &= (h * g)(t) + e^{3t}\cos t - 3e^{3t}\sin t \\ &= \int_0^t e^{3(t-v)}\sin(t-v)g(v)\,dv + e^{3t}\cos t - 3e^{3t}\sin t. \end{aligned}$$

5. Writing $Y(s) = \mathscr{L}\{y\}(s)$, we take Laplace transforms to get

$$sY(s) - 2 + Y(s) = \mathscr{L}\{4 - 2u(t-6)\} = \frac{4}{s} - \frac{2e^{-6s}}{s}$$

and so $Y(s) = \dfrac{2}{s+1} + \dfrac{4}{s(s+1)} - \dfrac{2e^{-6s}}{s(s+1)}$. Using partial fractions we have

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

so that

$$Y(s) = \frac{4}{s} - \frac{4}{s+1} - \frac{2e^{-6s}}{s} - \frac{2e^{-6s}}{s+1}$$

and hence $y(t) = 2e^{-t} + 4 - 4e^{-t} - 2(1 + e^{-(t-6)})u(t-6)$.

6. $\mathscr{L}\{h\}(s) = \dfrac{d^2}{ds^2}\mathscr{L}\{f * f\}(s) = \dfrac{d^2}{ds^2}\left(\dfrac{s^2}{s^2+4}\right)^2 = \dfrac{d^2}{ds^2}\left(\dfrac{s^4}{(s^2+4)^2}\right) = \dfrac{48s^2(4-s^2)}{(s^2+4)^4}$.

7. Writing $Y(s) = \mathscr{L}\{y\}(s)$, we take Laplace transforms to get

$$s^2Y(s) - 2 - sY(s) = -2\frac{d}{ds}\left(\frac{1}{s-1}\right) = \frac{2}{(s-1)^2}$$

and so $Y(s) = \dfrac{2}{s(s-1)} + \dfrac{2}{s(s-1)^3}$. Using partial fractions we have

$$\frac{2}{s(s-1)} = -\frac{2}{s} + \frac{2}{s-1}$$

and

$$\frac{2}{s(s-1)^3} = -\frac{2}{s} + \frac{2}{s-1} - \frac{2}{(s-1)^2} + \frac{2}{(s-1)^3}$$

and therefore $y(t) = 2 + 2e^t - 2 + 2e^t - 2te^t + t^2e^t = 4e^t - 2te^t + t^2e^t$.