MATH 308
Solutions to Chapter 9 review problems

1. We have

$$|\mathbf{A} - r\mathbf{I}| = \begin{vmatrix} 6-r & -3 \\ 2 & 1-r \end{vmatrix} = r^2 - 7r + 12 = (r-3)(r-4).$$

To find an eigenvector for the eigenvalue $r = 3$ we solve $(\mathbf{A} - 3\mathbf{I})\mathbf{u} = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \mathbf{u} = \mathbf{0}$ and take $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For the eigenvalue $r = 4$ we solve $(\mathbf{A} - 4\mathbf{I})\mathbf{u} = \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \mathbf{u} = \mathbf{0}$ and take $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. So the general solution is

$$\mathbf{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

4. We have

$$|\mathbf{A} - r\mathbf{I}| = \begin{vmatrix} 1-r & 1 & 0 \\ 0 & 1-r & 0 \\ 0 & 0 & 2-r \end{vmatrix} = -(r-1)^2(r-2).$$

To find an eigenvector for the eigenvalue $r = 2$ we solve $(\mathbf{A} - 2\mathbf{I})\mathbf{u} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{u} = \mathbf{0}$ by row reducing the matrix to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and taking $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. For the eigenvalue $r = 1$ we solve $(\mathbf{A} - \mathbf{I})\mathbf{u} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{u} = \mathbf{0}$ and take $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We need to find a generalized eigenvector $\mathbf{u}_3$ for $r = 1$ such that $\mathbf{u}_2$ and $\mathbf{u}_3$ are linearly independent. For this we observe that the solution to $(\mathbf{A} - \mathbf{I})^2\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{u} = \mathbf{0}$ is $\mathbf{u} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ for $s, t \in \mathbb{R}$, and so we can take $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The solution to the system corresponding to this generalized eigenvector is

$$\mathbf{x}(t) = e^t(\mathbf{u}_3 + t(\mathbf{A} - \mathbf{I})\mathbf{u}_3) = e^t \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}.$$

So the general solution is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}.$$

5. We have $|\mathbf{A}-r\mathbf{I}| = \begin{vmatrix} 1-r & -1 \\ 2 & 4-r \end{vmatrix} = r^2-5r+6 = (r-3)(r-2)$. To find an eigenvector for the eigenvalue $r=3$ we solve $(\mathbf{A}-2\mathbf{I})\mathbf{u} = \left[\begin{smallmatrix} -2 & -1 \\ 2 & 1 \end{smallmatrix}\right]\mathbf{u} = \mathbf{0}$ and take $\mathbf{u}_1 = \left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]$. For the eigenvalue $r=2$ we solve $(\mathbf{A}-2\mathbf{I})\mathbf{u} = \left[\begin{smallmatrix} -1 & -1 \\ 2 & 2 \end{smallmatrix}\right]\mathbf{u} = \mathbf{0}$ and take $\mathbf{u}_2 = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$. So a fundamental matrix for the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ is

$$\begin{bmatrix} e^{3t} & e^{2t} \\ -2e^{3t} & -e^{2t} \end{bmatrix}.$$

8. We have

$$|\mathbf{A}-r\mathbf{I}| = \begin{vmatrix} -4-r & 2 \\ 2 & -1-r \end{vmatrix} = r^2+5r = r(r+5).$$

To find an eigenvector for the eigenvalue $r=0$ we solve $\mathbf{A}\mathbf{u} = \left[\begin{smallmatrix} -4 & 2 \\ 2 & -1 \end{smallmatrix}\right]\mathbf{u} = \mathbf{0}$ and take $\mathbf{u}_1 = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$. For the eigenvalue $r=-5$ we solve $(\mathbf{A}-5\mathbf{I})\mathbf{u} = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 4 \end{smallmatrix}\right]\mathbf{u} = \mathbf{0}$ and take $\mathbf{u}_2 = \left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right]$. So the general solution to the homogeneous system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ is

$$\mathbf{x}_{\mathrm{h}}(t) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2e^{-5t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

To find a particular solution to the given system, let us use the method of undetermined coefficients. As a trial solution we take $\mathbf{x}_{\mathrm{p}}(t) = e^{4t}\mathbf{a}$ where $\mathbf{a} = \left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right]$ is to be determined. Substituting into the system, we obtain

$$4e^{4t}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} e^{4t}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} e^{4t} \\ 3e^{4t} \end{bmatrix} = e^{4t}\begin{bmatrix} -4a_1+2a_2+1 \\ 2a_1-a_2+3 \end{bmatrix}.$$

Equating vector entries on either side and simplifying, we get

$$\begin{aligned} 8a_1 - 2a_2 &= 1 \\ -2a_1 + 5a_2 &= 3. \end{aligned}$$

To solve this system of equations we row reduce $\left[\begin{array}{cc|c} 8 & -2 & 1 \\ -2 & 5 & 3 \end{array}\right]$ to $\left[\begin{array}{cc|c} 1 & 0 & \frac{11}{36} \\ 0 & 1 & \frac{13}{18} \end{array}\right]$. So $a_1 = \frac{11}{36}$ and $a_2 = \frac{13}{18}$ and the general solution to the given system is

$$\mathbf{x}(t) = \mathbf{x}_{\mathrm{h}}(t) + \mathbf{x}_{\mathrm{p}}(t) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2e^{-5t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + e^{4t}\begin{bmatrix} 11/36 \\ 13/18 \end{bmatrix}.$$

16. In this case we notice that $\mathbf{A}^3 = \mathbf{0}$, and so we can compute $e^{\mathbf{A}t}$ using the definition since the series will terminate after three terms:

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2}\mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t & 4t + t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix}.$$

Alternatively we can find three linearly independent generalized eigenvectors for the eigenvalue $r = 0$ by solving in succession $\mathbf{A}\mathbf{u} = \mathbf{0}$, $\mathbf{A}^2\mathbf{u} = \mathbf{0}$, and $\mathbf{A}^3\mathbf{u} = \mathbf{0}$ and selecting a suitable vector at each stage. We then write down the corresponding fundamental matrix $\mathbf{X}(t)$ and compute $e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$.