1.3.0. (a) True under the assumption that $A \cap B \neq \emptyset$ (otherwise $\sup(A \cap B)$ is undefined). Indeed let $A$ and $B$ be nonempty bounded subsets of $\mathbb{R}$ such that $A \cap B \neq \emptyset$. Note that $\sup(A \cap B)$ exists since $A \cap B$, being a subset of the bounded set $A$, is itself bounded. Since $A \cap B \subseteq A$, every upper bound for $A$ is also an upper bound for $A \cap B$. In particular, $\sup(A)$ is an upper bound for $A \cap B$ by the definition of $\sup(A)$. Thus $\sup(A \cap B) \leq \sup(A)$ by the definition of $\sup(A \cap B)$.

(c) True. Let $A$ and $B$ be nonempty bounded subsets of $\mathbb{R}$. Let $x \in A + B$. Then we can write $x = a + b$ for some $a \in A$ and some $b \in B$, and so $x = a + b \leq \sup(A) + \sup(B)$. This shows that $\sup(A) + \sup(B)$ is a upper bound for $A + B$, and so $\sup(A + B) \leq \sup(A) + \sup(B)$ by the definition of $\sup(A + B)$.

Now suppose that $\sup(A + B) < \sup(A) + \sup(B)$. Set $\epsilon = \sup(A) + \sup(B) - \sup(A + B)$. By the approximation property for suprema, we can find an $a \in A$ and a $b \in B$ such that $a > \sup(A) - \epsilon/2$ and $b > \sup(B) - \epsilon/2$. Then

$$a + b > \sup(A) + \sup(B) - \epsilon = \sup(A + B),$$

contradicting the fact that $\sup(A + B)$ is an upper bound for $A + B$. Therefore we must have $\sup(A + B) \geq \sup(A) + \sup(B)$ and hence $\sup(A + B) = \sup(A) + \sup(B)$.

1.3.2. (a) $E = \{-3, 1\}$, and so $\sup(E) = 1$.

(e) $\sup(E) = \frac{3}{2}$.

1.3.6. (a) Let $E$ be a subset of $\mathbb{R}$ with an infimum and let $\epsilon > 0$. Then $\sup(-E)$ exists and is equal to $-\inf E$ by the reflection principle. By the approximation property for suprema, there is an $a \in E$ such that $\sup(-E) - \epsilon < -a \leq \sup(-E)$. Then

$$\inf E + \epsilon = -\sup(-E) + \epsilon > a \geq -\sup(-E) = \inf E.$$ 

(b) Let $E$ be a subset of $\mathbb{R}$ which is nonempty and bounded below. Then $-E$ is nonempty, and it is also bounded above, for if we take a lower bound $M$ for $E$ then $-M$ is an upper bound for $-E$. Hence $-E$ has a supremum. By the reflection principle, $E$ has an infimum.
1.3.7. (a) Let $E$ be a subset of $\mathbb{R}$ and let $x$ be an upper bound for $E$ such that $x \in E$. Then $\sup E \leq x$ since $x$ is an upper bound and $\sup E$ is less than or equal to every upper bound for $E$. On the other hand, $x \leq \sup E$ since $\sup E$ is an upper bound for $E$. Thus $x = \sup E$.

(b) Let $E$ be a subset of $\mathbb{R}$ and let $x$ be a lower bound for $E$ such that $x \in E$. Then $x = \inf E$. To see this, we note that $-x$ is an upper bound for $-E$, and hence $-x = \sup(-E)$ by part (a), in which case the reflection principle yields $\inf E = -\sup(-E) = x$.

(c) Consider the set $E = (0, 1)$. Then $\sup E = 1$ and $\inf E = 0$, but neither of these numbers is an element of $E$.

1.3.8. Let $A$ and $B$ be nonempty subsets of $\mathbb{R}$ such that $\sup(A \cup B)$ exists. Since $A \cup B$ contains both $A$ and $B$, $\sup(A \cup B)$ is an upper bound for both $A$ and $B$. Thus $\sup A$ and $\sup B$ both exist, and we have $\sup(A \cup B) \geq \sup A$ and $\sup(A \cup B) \geq \sup B$. Now we treat two cases. Suppose first that $\sup A \geq \sup B$. Then $\sup A$ is an upper bound for both $A$ and $B$, and hence for $A \cup B$. Consequently $\sup(A \cup B) \leq \sup A$, and so $\sup(A \cup B) = \sup A$. In the case $\sup B > \sup A$ we obtain by a similar argument $\sup(A \cup B) = \sup B$.

1.4.2. (b) Let $a, b \geq 0$. In the base case $n = 1$ we trivially have $(a + b)^1 = a + b = a^1 + b^1$. Suppose then that the statement is true for a given $n \geq 1$. Then

$$(a + b)^{n+1} = (a + b)(a + b)^n \geq (a + b)(a^n + b^n) \]
$$

$$= a^{n+1} + ba^n + ab^n + b^{n+1} \geq a^{n+1} + b^{n+1}.$$

Thus the statement holds for all $n$ by induction.

(d) Let $n \in \mathbb{N}$. Using the binomial formula $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$ in the case $a = b = 1$, we obtain

$$\sum_{k=1}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} - \binom{n}{0} = 2^n - 1.$$

It remains to show that $\sum_{k=0}^{n-1} 2^k = 2^n - 1$. We do this by induction. The base case $n = 1$ is clear. Suppose that the formula holds for a given $n \geq 1$. Then

$$\sum_{k=0}^{n} 2^k = 2^n + \sum_{k=0}^{n-1} 2^k = 2 \cdot 2^n - 1 = 2^{n+1} - 1,$$

and so the formula holds for all $n$. 
1.4.4. (a) In the base case \( n = 1 \) the formula holds since \( 1(1 + 1)/2 = 1 \). Suppose that the formula holds for a given \( n \geq 1 \). Then

\[
\sum_{k=1}^{n+1} k = n + 1 + \sum_{k=1}^{n} k = n + 1 + \frac{n(n+1)}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n + 1)(n + 2)}{2}.
\]

Thus the statement holds for all \( n \) by induction.

(c) In the case \( n = 1 \) we have \((a - 1)/a = 1 - 1/a\) and so the formula holds. Suppose that the formula holds for a given \( n \geq 1 \). Then

\[
\sum_{k=1}^{n+1} \frac{a - 1}{a^k} = \frac{a - 1}{a^{n+1}} + \sum_{k=1}^{n} \frac{a - 1}{a^k} = \frac{a - 1}{a^{n+1}} + 1 - \frac{1}{a^n} = \frac{a - 1 - a}{a^{n+1}} = 1 - \frac{1}{a^{n+1}}.
\]

Thus the formula holds for all \( n \) by induction.