The exam consists of 5 questions. The point value for a question is written next to the question number. There is a total of 100 points. No aids are permitted.

1. [20] (a) State the completeness axiom for \( \mathbb{R} \).

(b) Give an example of a nonempty bounded subset of \( \mathbb{R} \) which contains its infimum but does not contain its supremum.
(c) Let $A$ and $B$ be nonempty bounded subsets of $\mathbb{R}$. Define

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$ 

Prove that $\sup(A + B) = \sup A + \sup B$. 
2. [25] (a) State the definition of the limit of a sequence in $\mathbb{R}$.

(b) Give an example of a bounded sequence in $\mathbb{R}$ that does not converge.

(c) Give an example of a sequence $\{x_n\}_{n=1}^\infty$ in $\mathbb{R}$ which is not bounded above and has a subsequence $\{x_{n_k}\}_{k=1}^\infty$ that converges to 1 as $k \to \infty$.

(d) Compute $\lim_{n \to \infty} \frac{n^4 + 3n - 1}{3n^4 + n^2}$.
(e) Let \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) be sequences in \( \mathbb{R} \) such that \( \lim_{n \to \infty} x_n = 2 \) and \( \lim_{n \to \infty} y_n = 1 \). Prove directly from the definition of a limit that

\[
\lim_{n \to \infty} (x_n + 2y_n) = 4.
\]
3. [20] (a) State what it means for a set to be countable.

(b) Let $m \in \mathbb{N}$, and let $A$ be the set of all open intervals in $\mathbb{R}$ of the form $(m, n)$ for some integer $n > m$. Show that $A$ is countable.

(c) Let $B$ be the set of all open intervals in $\mathbb{R}$ of the form $(m, n)$ for some $m, n \in \mathbb{N}$ with $m < n$. Show that $B$ is countable.
4. [15] (a) Give an example of a function $f : X \to Y$ and a set $E \subseteq X$ such that
$f^{-1}(f(E)) \neq E$.

(b) Let $E$ be a nonempty bounded subset of $\mathbb{R}$ that does not contain its supremum. Show that there exists an injective function $f : \mathbb{N} \to E$. 
5. [20] (a) State the well-ordering principle.

(b) Prove that \( n < 2^n \) for all \( n \in \mathbb{N} \).

(c) Let \( \{x_n\}_{n=1}^{\infty} \) be a bounded sequence in \( \mathbb{R} \). Prove directly from the definition of a limit that \( x_n/2^n \to 0 \) as \( n \to \infty \).