

MATH 409.200/501
Spring 2009
Exam #1 Solutions

1. (a) Every nonempty subset of \mathbb{R} which is bounded above has a supremum.

(b) An example is the interval $[0, 1)$.

(c) First we show that $\sup(A + B) \leq \sup A + \sup B$. If $a \in A$ and $b \in B$ then $a + b \leq \sup A + \sup B$ since $\sup A$ is an upper bound for A and $\sup B$ is an upper bound for B . Therefore $\sup A + \sup B$ is an upper bound for $A + B$. By the definition of $\sup(A + B)$, this yields $\sup(A + B) \leq \sup A + \sup B$.

Now we show that $\sup(A + B) \geq \sup A + \sup B$. Let $\varepsilon > 0$. By the approximation property for suprema, there are an $a \in A$ and a $b \in B$ such that $\sup A \leq a + \varepsilon/2$ and $\sup B \leq b + \varepsilon/2$. Then

$$\sup(A + B) \geq a + b \geq \sup A - \frac{\varepsilon}{2} + \sup B - \frac{\varepsilon}{2} = \sup A + \sup B + \varepsilon.$$

Since this is true for every $\varepsilon > 0$, we conclude that $\sup(A + B) \geq \sup A + \sup B$. Thus $\sup(A + B) = \sup A + \sup B$.

2. (a) A sequence $\{x_n\}_{n=1}^{\infty}$ in \mathbb{R} has limit L as $n \rightarrow \infty$ if for every $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that if $n \geq N$ then $|x_n - L| < \varepsilon$.

(b) An example is $\{(-1)^n\}_{n=1}^{\infty}$.

(c) Set x_n equal to 1 if n is even n if n is odd. Then $\{x_n\}_{n=1}^{\infty}$ is not bounded above by the Archimedean principle and $\{x_{2n}\}_{n=1}^{\infty}$ is a subsequence which converges to 1 as $n \rightarrow \infty$.

(d)
$$\lim_{n \rightarrow \infty} \frac{n^4 + 3n - 1}{3n^4 + n^2} = \lim_{n \rightarrow \infty} \frac{1 + 3/n^3 - 1/n^4}{3 + 1/n^2} = \frac{1}{3}.$$

(e) Let $\varepsilon > 0$. Take an $N_1 \in \mathbb{N}$ such that $|x_n - 2| < \varepsilon/2$ for all $n \geq N_1$ and an $N_2 \in \mathbb{N}$ such that $|y_n - 1| < \varepsilon/4$ for all $n \geq N_2$. Then for all $n \geq \max\{N_1, N_2\}$ we have

$$|x_n + 2y_n - 4| = |x_n - 2 + 2(y_n - 1)| \leq |x_n - 2| + 2|y_n - 1| < \frac{\varepsilon}{2} + 2 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

Therefore $\lim_{n \rightarrow \infty} (x_n + 2y_n) = 4$.

3. (a) A set A is countable if there exists a bijective function $f : \mathbb{N} \rightarrow A$.
- (b) Define a function $f : \mathbb{N} \rightarrow A$ by $f(n) = (m, m + n)$ for all $n \in \mathbb{N}$. Then f is a bijection, and so A is countable.
- (c) For each $m \in \mathbb{N}$ write A_m for the set of all open intervals in \mathbb{R} of the form (m, n) for some integer $n > m$. Then A_m is countable by part (b). Since $B = \bigcup_{m=1}^{\infty} A_m$ and a countable union of countable sets is countable, we conclude that B is countable.
4. (a) Define $f : \{0, 1\} \rightarrow \{0\}$ by $f(0) = 0$ and $f(1) = 0$. Set $E = \{0\}$. Then $f^{-1}(f(E)) = \{0, 1\} \neq E$.
- (b) Set $a = \sup E$. By nonemptiness we can pick an $x_1 \in E$. By the approximation property of suprema, there is an $x_2 \in E$ such that $x_1 < x_2 \leq a$, and $x_2 \neq a$ since $a \notin E$ by assumption. Again by the approximation property of suprema, there is an $x_3 \in E$ such that $x_2 < x_3 \leq a$, and $x_3 \neq a$ since $a \notin E$. Continue recursively applying the approximation property of suprema to find elements $x_n \in E$ such that $x_{n-1} < x_n < a$ for every $n > 1$. Now define $f : \mathbb{N} \rightarrow E$ by $f(n) = x_n$. Then f is an injection since the sequence $\{x_n\}_{n=1}^{\infty}$ is increasing.
5. (a) Every nonempty subset of \mathbb{N} has a least element.
- (b) We proceed by induction. The case $n = 1$ is obvious. Suppose that the assertion is true for some $n \in \mathbb{N}$. Then

$$n + 1 \leq n + n = 2n < 2 \cdot 2^n = 2^{n+1}.$$

Thus $n < 2^n$ for all $n \in \mathbb{N}$.

(c) Let $\varepsilon > 0$. Take an $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Take an $N \in \mathbb{N}$ such that $N > M/\varepsilon$. Then for all $n \geq N$ we have, using part (b),

$$\left| \frac{x_n}{2} \right| = |x_n| \cdot \frac{1}{2^n} < M \cdot \frac{1}{n} \leq M \cdot \frac{1}{N} < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

We conclude that $x_n/2^n \rightarrow 0$ as $n \rightarrow \infty$.