1. (a) Every nonempty subset of \( \mathbb{R} \) which is bounded above has a supremum.

(b) An example is the interval \([0, 1)\).

(c) First we show that \( \sup(A + B) \leq \sup A + \sup B \). If \( a \in A \) and \( b \in B \) then \( a + b \leq \sup A + \sup B \) since \( \sup A \) is an upper bound for \( A \) and \( \sup B \) is an upper bound for \( B \). Therefore \( \sup A + \sup B \) is an upper bound for \( A + B \). By the definition of \( \sup(A + B) \), this yields \( \sup(A + B) \leq \sup A + \sup B \).

Now we show that \( \sup(A + B) \geq \sup A + \sup B \). Let \( \varepsilon > 0 \). By the approximation property for suprema, there are an \( a \in A \) and \( b \in B \) such that \( \sup A \leq a + \varepsilon/2 \) and \( \sup B \leq b + \varepsilon/2 \). Then

\[
\sup(A + B) \geq a + b \geq \sup A - \frac{\varepsilon}{2} + \sup B - \frac{\varepsilon}{2} = \sup A + \sup B - \varepsilon.
\]

Since this is true for every \( \varepsilon > 0 \), we conclude that \( \sup(A + B) \geq \sup A + \sup B \). Thus \( \sup(A + B) = \sup A + \sup B \).

2. (a) A sequence \( \{x_n\}_{n=1}^{\infty} \) in \( \mathbb{R} \) has limit \( L \) as \( n \to \infty \) if for every \( \varepsilon > 0 \) there is an \( n \in \mathbb{N} \) such that if \( n \geq N \) then \( |x_n - L| < \varepsilon \).

(b) An example is \( \{(−1)^n\}_{n=1}^{\infty} \).

(c) Set \( x_n \) equal to 1 if \( n \) is even \( n \) if \( n \) is odd. Then \( \{x_n\}_{n=1}^{\infty} \) is not bounded above by the Archimedean principle and \( \{x_{2n}\}_{n=1}^{\infty} \) is a subsequence which converges to 1 as \( n \to \infty \).

(d) \[
\lim_{n \to \infty} \frac{n^4 + 3n - 1}{3n^4 + n^2} = \lim_{n \to \infty} \frac{1 + 3/n^3 - 1/n^4}{3 + 1/n^2} = \frac{1}{3}.
\]

(e) Let \( \varepsilon > 0 \). Take an \( N_1 \in \mathbb{N} \) such that \( |x_n - 2| < \varepsilon/2 \) for all \( n \geq N_1 \) and an \( N_2 \in \mathbb{N} \) such that \( |y_n - 1| < \varepsilon/4 \) for all \( n \geq N_2 \). Then for all \( n \geq \max\{N_1, N_2\} \) we have

\[
|x_n + 2y_n - 4| = |x_n - 2 + 2(y_n - 1)| \leq |x_n - 2| + 2|y_n - 1| < \frac{\varepsilon}{2} + 2 \cdot \frac{\varepsilon}{4} = \varepsilon.
\]

Therefore \( \lim_{n \to \infty}(x_n + 2y_n) = 4 \).
3. (a) A set $A$ is countable if there exists a bijective function $f : \mathbb{N} \to A$.

(b) Define a function $f : \mathbb{N} \to A$ by $f(n) = (m, m + n)$ for all $n \in \mathbb{N}$. Then $f$ is a bijection, and so $A$ is countable.

(c) For each $m \in \mathbb{N}$ write $A_m$ for the set of all open intervals in $\mathbb{R}$ of the form $(m, n)$ for some integer $n > m$. Then $A_m$ is countable by part (b). Since $B = \bigcup_{m=1}^{\infty} A_m$ and a countable union of countable sets is countable, we conclude that $B$ is countable.

4. (a) Define $f : \{0, 1\} \to \{0\}$ by $f(0) = 0$ and $f(1) = 0$. Set $E = \{0\}$. Then $f^{-1}(f(E)) = \{0, 1\} \neq E$.

(b) Set $a = \sup E$. By nonemptiness we can pick an $x_1 \in E$. By the approximation property of suprema, there is an $x_2 \in E$ such that $x_1 < x_2 \leq a$, and $x_2 \neq a$ since $a \notin E$ by assumption. Again by the approximation property of suprema, there is an $x_3 \in E$ such that $x_2 < x_3 \leq a$, and $x_2 \neq a$ since $a \notin E$. Continue recursively applying the approximation property of suprema to find elements $x_n \in E$ such that $x_{n-1} < x_n < a$ for every $n > 1$. Now define $f : \mathbb{N} \to E$ by $f(n) = x_n$. Then $f$ is an injection since the sequence $\{x_n\}_{n=1}^{\infty}$ is increasing.

5. (a) Every nonempty subset of $\mathbb{N}$ has a least element.

(b) We proceed by induction. The case $n = 1$ is obvious. Suppose that the assertion is true for some $n \in \mathbb{N}$. Then

$$n + 1 \leq n + n = 2n < 2 \cdot 2^n = 2^{n+1}.$$

Thus $n < 2^n$ for all $n \in \mathbb{N}$.

(c) Let $\varepsilon > 0$. Take an $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Take an $N \in \mathbb{N}$ such that $N > M/\varepsilon$. Then for all $n \geq N$ we have, using part (b),

$$\left| \frac{x_n}{2} \right| = \left| x_n \right| \cdot \frac{1}{2^n} < M \cdot \frac{1}{n} \leq M \cdot \frac{1}{N} < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

We conclude that $x_n/2^n \to 0$ as $n \to \infty$. 