MATH 409.200/501
Spring 2009
Exam #2 Solutions

1. (b) Since a sequence is Cauchy if and only if it converges, it suffices to show that $\{x_n + (-1)^n\}_{n=1}^{\infty}$ fails to be Cauchy. Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy there is an $N \in \mathbb{N}$ such that if $n, m \geq N$ then $|x_n - x_m| < 1$. Then for all $n \geq N$ we have

$$\begin{aligned} |(x_n + (-1)^n) - (x_{n+1} - (-1)^{n+1})| &\geq |(-1)^n - (-1)^{n+1}| - |x_n - x_{n+1}| \\ &= 2 - |x_n - x_{n+1}| \\ &> 2 - 1 = 1. \end{aligned}$$

It follows that $\{x_n\}_{n=1}^{\infty}$ fails to be Cauchy.

(c) First we show by induction that $\{x_n\}_{n=1}^{\infty}$ is bounded above by 3 and below by 0. This is given for $n = 1$. Suppose that it is true for some $n \in \mathbb{N}$. Then

$$x_{n+1} = \sqrt{6 + x_n} \leq \sqrt{6 + 3} = \sqrt{9} = 3$$

and $x_{n+1} = \sqrt{6 + x_n} \geq \sqrt{6} \geq 0$. Thus $0 \leq x_n \leq 3$ for all $n \in \mathbb{N}$.

Next we show that $\{x_n\}_{n=1}^{\infty}$ is increasing. Let $n \in \mathbb{N}$. Then $x_{n+1} \geq x_n$ is equivalent to $\sqrt{6 + x_n} \geq x_n$, which in turn is equivalent to $6 + x_n \geq x_n^2$ or $(x_n - 3)(x_n + 2) \leq 0$, which is true because we showed above that $0 \leq x_n \leq 3$.

It follows by the Monotone Convergence Theorem that $\{x_n\}_{n=1}^{\infty}$ converges to some number L as $n \to \infty$. Then

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{6 + x_n} = \sqrt{6 + \lim_{n\to\infty} x_n} = \sqrt{6 + L}$$

so that $L^2 - L - 6 = 0$, that is, $(L - 3)(L + 2) = 0$. Thus $L = 3$ or $L = -2$. Since $0 \leq x_n \leq 3$ for all $n \in \mathbb{N}$, we must have $L = 3$.

2. (b) Let $\varepsilon > 0$. Set $\delta = \sqrt{\varepsilon}$. Then for all $x \in \mathbb{R}$ satisfying $0 < |x| < \delta$ we have

$$\left| x^2 \cos\left(\frac{x^3 + 5}{x^2 + 2}\right) - 0 \right| \leq |x|^2 \left| \cos\left(\frac{x^3 + 5}{x^2 + 2}\right) \right| \leq |x|^2 \leq \delta^2 = \varepsilon.$$

Thus $\lim_{x\to 0} x^2 \cos\left(\frac{x^3 + 5}{x^2 + 2}\right) = 0$.

(c) $\lim_{x\to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x + 2)}{x - 1} = \lim_{x\to 1} x + 2 = 3.$

3. (b) Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^6 + x^4 + 1 - 2\cos x^3$. Since $\cos x$ is continuous and polynomials are continuous, f is continuous. Note that $f(0) = 1+1+1-2 = -1$ and $f(1) = 1+1+1-2\cos 1 \geq 3-2 = 1$. Thus by the Intermediate Value Theorem there is an $x \in (0,1)$ such that $f(x) = 0$, that is, $x^6+x^4+1 = 2\cos x^3$.

4. (b) Take for example $f(x) = \sin(1/x)$.

   (c) Let $\varepsilon > 0$. Since f and g are bounded on E we can find an $M > 0$ such that $|f(x)| \leq M$ and $|g(x)| \leq M$ for all $x \in E$. By the uniform continuity of f and g there is a $\delta > 0$ such that if $x, y \in E$ and $|x-y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{2M}$ and $|g(x) - g(y)| < \frac{\varepsilon}{2M}$. Then for $x, y \in E$ with $|x-y| < \delta$ we have

$$
\begin{aligned}
|fg(x) - fg(y)| &= |f(x)(g(x) - g(y)) - (f(x) - g(x))g(y)| \\
&\leq |f(x)(g(x) - g(y))| + |(f(x) - g(x))g(y)| \\
&\leq |f(x)||g(x) - g(y)| + |f(x) - g(x)||g(y)| \\
&< M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2M} \cdot M \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

   Thus fg is uniformly continuous on E.

   (d) Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ and $g(x) = x$ for all $x \in \mathbb{R}$.

5. (a) For $x \neq a$ we have

$$
\begin{aligned}
\frac{g(x) - g(a)}{x-a} = \frac{xf(x) - af(a)}{x-a} &= \frac{(x-a)f(x) + a(f(x) - f(a))}{x-a} \\
&= f(x) + a\frac{f(x) - f(a)}{x-a}
\end{aligned}
$$

   and so

$$
\lim_{x\to a} \frac{g(x) - g(a)}{x-a} = \lim_{x\to a} f(a) + a\left(\lim_{x\to a} \frac{f(x) - f(a)}{x-a}\right) = f(a) + af'(a).
$$

   Thus g is differentiable at a.

   (b) By the Chain Rule, $(g \circ f)'(0) = g'(f(0))f'(0) = g'(1)f'(0) = 3(-1) = -3$.