Write \( \tilde{g} \) for the extension of \( g \in C(A) \) to \( X \) obtained by setting \( \tilde{g}(x) = 0 \) for all \( x \in A^c \). To verify that \( \tilde{g} \) is continuous, let \( K \) be a closed subset of \( C \) and let us show that \( \tilde{g}^{-1}(K) \) is closed. Since \( g \) is continuous, \( g^{-1}(K) \) is a closed subset of \( A \). It follows by the definition of the relative topology that there is an open set \( U \subseteq X \) such that \( (g^{-1}(K))^c \cap A = U \cap A \), and \( g^{-1}(K) = A \cap (\tilde{U} \cap A^c) = A \cap U^c \), which is closed in \( X \). Now if \( 0 \not\in K \) then \( \tilde{g}^{-1}(K) \) is equal to \( g^{-1}(K) \) and hence is closed in \( X \), while if \( 0 \in K \) then \( \tilde{g}^{-1}(K) = g^{-1}(K) \cup (A^o)^c \), which again is closed in \( X \). Therefore \( \tilde{g} \) is continuous.

4.15. a. Set \( A = \{ x \in X : f(x) \neq g(x) \} \). Since \( Y \) is Hausdorff, for each \( x \in A \) there are disjoint open sets \( U_{f,x}, V_{g,x} \subseteq Y \) such that \( f(x) \in U_{f,x} \) and \( g(x) \in V_{g,x} \). Then \( x \in f^{-1}(U_{g,x}) \cap g^{-1}(V_{g,x}) \subseteq A \). Therefore \( A = \bigcup_{x \in A} (f^{-1}(U_{g,x}) \cap g^{-1}(V_{g,x})) \), which is open since each of the sets \( f^{-1}(U_{g,x}) \cap g^{-1}(V_{g,x}) \) is open by the continuity of \( f \) and \( g \). Hence \( A^c = \{ x \in X : f(x) = g(x) \} \) is closed.

b. Suppose that \( f = g \) on a dense set \( D \subseteq X \). Then by part (a) we have \( f = g \) on the closure of \( D \), which is equal to \( X \).

4.16. Let \( f \in \mathbb{C}^X \). For \( n, m \in \mathbb{N} \) set \( U_{f,m,n} = \{ g \in \mathbb{C}^X : \sup_{x \in \mathbb{N}} |g(x) - f(x)| < m^{-1} \} \). Our goal is to show that these opens sets form a neighborhood base for \( f \). So let \( U \) be an open neighbourhood of \( f \). By Propositions 4.2 and 4.4 and the definition of the topology, there are \( f_1, \ldots, f_\ell \in \mathbb{C}^X \), compact sets \( K_1, \ldots, K_\ell \subseteq X \), and \( m_1, \ldots, m_\ell \in \mathbb{N} \) such that the set

\[
V = \bigcap_{i=1}^\ell \left\{ g \in \mathbb{C}^X : \sup_{x \in K_i} |g(x) - f_i(x)| < \frac{1}{m_i} \right\}
\]

contains \( f \) and is contained in \( U \). For each \( i = 1, \ldots, \ell \), \( \{ U_n : n \in \mathbb{N} \} \) is an open cover of \( K_i \), and thus, since \( U_1 \subseteq U_2 \subseteq \ldots \), there exists an \( n \) such that \( K_i \subseteq U_n \) for all \( i = 1, \ldots, \ell \). Take an \( \varepsilon > 0 \) such that \( \sup_{x \in K_i} |f(x) - f_i(x)| < m_i^{-1} - \varepsilon \) for each \( i = 1, \ldots, \ell \). Now take an \( m \in \mathbb{N} \) which is larger than both \( 2 \cdot \max\{m_1, \ldots, m_\ell\} \) and \( 1/\varepsilon \). Then for \( g \in U_{f,m,n}, i \in \{1, \ldots, \ell\} \), and \( x \in K_i \) we have

\[
|g(x) - f_i(x)| \leq |g(x) - f(x)| + |f(x) - f_i(x)| < \frac{1}{m} + \frac{1}{m_i} - \varepsilon < \frac{1}{m_i},
\]

so that \( U_{f,m,n} \subseteq V \subseteq U \). We conclude that the sets \( U_{f,m,n} \) form a neighborhood base for \( f \).

4.70. a. We can write \( h(\mathbb{J}) \) as \( \bigcap_{f \in \mathbb{J}} f^{-1}(\{0\}) \), which, since \( \{0\} \) is closed and each \( f \in \mathbb{J} \) is continuous, is an intersection of closed sets and hence is closed.
b. The set $k(E)$ is clearly a vector subspace, and if $f \in k(E)$ and $g \in C(X, \mathbb{R})$ then for $x \in E$ we have $(fg)(x) = f(x)g(x) = 0$ if $g(x) = 0$ so that $fg \in k(E)$, showing that $k(E)$ is an ideal. Now if $f$ is a function in $C(X, \mathbb{R})$ such that $f(x) \neq 0$ for some $x \in E$ then $\{g \in C(X, \mathbb{R}) : \|g - f\| < |f(x)|\}$ is a neighborhood of $f$ which does not intersect $E$. Thus $E^c$ is open and hence $E$ is closed.

c. Obviously $E \subseteq h(k(E))$. By part (a) the set $h(k(E))$ is closed, and so $\overline{E} \subseteq h(k(E))$. Now let $x \notin \overline{E}$. By Urysohn’s lemma there is an $f \in C(X, \mathbb{R})$ such that $f|_E = 0$ and $f(x) = 1$. Then $f \in k(E)$, and so $x \notin h(k(E))$. We conclude that $h(k(E)) = \overline{E}$.

d. Obviously $\mathcal{J} \subseteq k(h(\mathcal{J}))$. By part (b) the set $k(h(\mathcal{J}))$ is closed and hence $\overline{\mathcal{J}} \subseteq k(h(\mathcal{J}))$. Observe that $\mathcal{J}$ is a subalgebra of $C_0(h(\mathcal{J})^c, \mathbb{R})$, since for every $f \in \mathcal{J}$ and $\varepsilon > 0$ we can find, using the continuity of $f$ and the fact that $h(\mathcal{J})$ is compact by part (a), a neighborhood $V$ of $h(\mathcal{J})$ such that $|f(x)| < \varepsilon$ for all $x$ in the set $V^c$, which is compact in $X$ and hence also compact in $h(\mathcal{J})^c$. Now if $x$ and $y$ are distinct points in $h(\mathcal{J})^c$ then we can find an $f \in \mathcal{J}$ such that $f(x) \neq 0$ and, by Urysohn’s lemma, an $g \in C_0(h(\mathcal{J})^c, \mathbb{R})$ such that $g(x) = 1$ and $g(y) = 0$, in which case $fg$ is an element of $\mathcal{J}$ whose values are different on $x$ and $y$. We can thus apply Proposition 4.36 to the one-point compactification of $h(\mathcal{J})^c$ and appeal to the Stone-Weierstrass theorem to conclude that $\overline{\mathcal{J}} = C_0(h(\mathcal{J})^c, \mathbb{R})$. (One can also take an approach that avoids the Stone-Weierstrass theorem in favor of the argument in the solution to 4.71(b) below.)

e. The map $E \mapsto k(E)$ from the closed subsets of $X$ to the closed ideals of $C(X, \mathbb{R})$ is injective since $h(k(E)) = E$ by part (c). It is surjective since given a closed ideal $\mathcal{J}$ in $C(X, \mathbb{R})$ we have $k(h(\mathcal{J})) = \mathcal{J}$ by part (d).

4.71. a. Set $\mathcal{J} = \{f \in C(X, \mathbb{R}) : \varphi(f) = 0\}$. Suppose there is a proper linear subspace $\mathcal{J}$ of $C(X, \mathbb{R})$ which contains $\mathcal{J}$ as a proper subset. Take a $g \in \mathcal{J} \setminus \mathcal{J}$ and an $h \in C(X, \mathbb{R}) \setminus \mathcal{J}$. Then $\varphi(g) \neq 0$, and we can define $\tilde{g} = (\varphi(h) / \varphi(g))g$, which is an element of $\mathcal{J} \setminus \mathcal{J}$ since $\mathcal{J}$ and $\mathcal{J}$ are subspaces. But then $\varphi(h - \tilde{g}) = 0$ so that $h - \tilde{g} \in \mathcal{J}$ and hence $h \in \mathcal{J} + \tilde{g} \subseteq \mathcal{J}$, a contradiction.

We conclude that $\mathcal{J}$ is a maximal proper ideal.

b. Suppose to the contrary that for every $x \in X$ there is an $f_x \in \mathcal{J}$ such that $f_x(x) \neq 0$. Then by continuity we can find for each $x \in X$ an open neighborhood $U_x$ of $x$ such that $|f_x(x)| > 0$ for all $x \in U_x$. By compactness there is a finite set $F \subseteq X$ such that $\bigcup_{x \in F} U_x = X$. Then $g = \sum_{x \in F} f_x^2$ is a function in $\mathcal{J}$ which is strictly positive and hence invertible. It follows that $1 = g^{-1}g \in \mathcal{J}$, and since every $f \in C(X, \mathbb{R})$ can be written as $f \cdot 1$ this implies that $\mathcal{J} = C(X, \mathbb{R})$, a contradiction.

c. The map $x \mapsto \hat{x}$ from $X$ to $M$ is injective, for if $x$ and $y$ are distinct elements of $X$ then Urysohn’s lemma yields an $f \in C(X, \mathbb{R})$ such that $f(x) = 1$ and $f(y) = 0$, whence $\hat{x}(f) = f(x) \neq f(y) = \hat{y}(f)$. Now suppose we are given a $\varphi \in M$. Since $\ker \varphi$ is an ideal, by part (b) there exists an $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \ker \varphi$. Then $\ker \varphi \subseteq \ker \hat{x}_0$ and so $\ker \varphi = \ker \hat{x}_0$ since $\ker \hat{x}_0$ is a proper ideal and, by part (a), $\ker \varphi$ is a maximal proper ideal. It follows that $\hat{x}_0 = \varphi$, showing that the map $x \mapsto \hat{x}$ is surjective.
c. By Proposition 4.28, we need only show that the map $x \mapsto \hat{x}$ from $X$ to $M$ is continuous. But if $\{x_\alpha\}_\alpha$ is a net in $X$ converging to some $x \in X$ then for every $f \in C(X)$ we have $\hat{x}_\alpha(f) = f(x_\alpha) \to f(x) = \hat{x}(f)$ by continuity.

4.71. In $\ell^2(A)$, the coordinate vectors $e_\alpha$ for $\alpha \in A$ given by $e_\alpha(\alpha') = 1$ if $\alpha' = \alpha$ and $e_\alpha(\alpha') = 0$ otherwise form a set which is evidently orthonormal and complete by the definition of $\ell^2(A)$ and hence is an orthonormal basis. We similarly have the coordinate vector basis $\{e_\beta\}_{\beta \in B}$ for $\ell^2(B)$. Thus if $\text{card}(A) = \text{card}(B)$ we can take a bijection $\theta : A \to B$ and define a bijective linear map $U : \ell^2(B) \to \ell^2(A)$ by $U((x_\beta)_\beta) = (x_{\theta(\alpha)})_\alpha$, which is isometric by Parseval's identity and hence is a unitary isomorphism.

Conversely, if we are given a unitary isomorphism $U : \ell^2(B) \to \ell^2(A)$ then the image of $\{e_\beta\}_{\beta \in B}$ is an orthonormal basis for $\ell^2(A)$ with cardinality $\text{card}(B)$. Since $\{e_\alpha\}_{\alpha \in A}$ is an orthonormal basis for $\ell^2(A)$ with cardinality $\text{card}(B)$ and any two orthonormal bases of a given Hilbert space have the same cardinality (we proved this for Hilbert spaces admitting a countable orthonormal basis, but the same argument works in general), we conclude that $\text{card}(A) = \text{card}(B)$.

7.11. Let $0 < \alpha < \mu(A)$. To show the existence of a Borel set $B \subseteq A$ with $\mu(B) = \alpha$ we may assume that $A$ is compact by replacing it with a compact subset of measure greater than $\alpha$ using inner regularity. Let $\mathcal{C}$ be the collection of all relatively open subsets $B \subseteq A$ with $\mu(B) \leq \alpha$, directed by inclusion. Note that every linearly ordered subcollection $\{U_i\}_{i \in I}$ of $\mathcal{C}$ has an upper bound, namely $U = \bigcup_{i=1}^\infty U_i$. Indeed $U$ is a relatively open subset of $A$, and if it were the case that $\mu(U) > \alpha$ then by inner regularity we could find a compact set $K \subseteq U$ with $\mu(K) > \alpha$, and the open cover $\{U_i\}_{i \in I}$ of $K$ would admit a finite subcover, in which case the largest member of this subcover would contain $K$ and hence have measure great than $\alpha$, a contradiction. Therefore by Zorn’s lemma $\mathcal{C}$ has a maximal element $U$.

Suppose that $\mu(U) < \alpha$. For every $x \in A \setminus U$ there is, by outer regularity and the hypothesis $\mu(\{x\}) = 0$, an open set $V_x$ containing $x$ such that $\mu(V_x) < \alpha - \mu(U)$. Since $A \setminus U$ is compact, there is a finite set $F \subseteq A \setminus U$ such that $\{V_x\}_{x \in F}$ covers $A \setminus U$. Since $\sum_{x \in F} \mu(V_x \setminus U) \geq \mu(A \setminus U) > 0$ there is a particular $x \in F$ such that $\mu(V_x \setminus U) > 0$. Now $\mu(V_x \cup U) \leq \mu(V_x) + \mu(U) < \alpha$ so that $V_x \cup U \in \mathcal{C}$, while $\mu(V_x \cup U) = \mu(U) + \mu(V_x \setminus U) > \mu(U)$ so that $U$ is a proper subset of $V_x \cup U$, contradicting maximality. We conclude that $\mu(U) = \alpha$. 

3