

MATH 689—TOPICS IN ERGODIC THEORY

Problems

Fall 2009

1. Give a proof of the mean ergodic theorem using the spectral theorem for unitary operators.
2. Let G be a countable discrete group and let $G \curvearrowright (\{0, 1\}^G, \mu)$ be the Bernoulli action where μ is the product of measures which give weight $\frac{1}{2}$ to both 0 and 1 at every coordinate. Determine the associated unitary representation of G on $L^2(\{0, 1\}^G, \mu)$.
3. Using the multiple recurrence theorem, show that for every $\eta > 0$ and $k \in \mathbb{N}$ there are an $M \in \mathbb{N}$ and a $\delta > 0$ such that for every measure-preserving \mathbb{Z} -system (X, μ, T) and every set $B \subset X$ with $\mu(B) \geq \eta$ there is an $m \in \{1, \dots, M\}$ for which $\mu(\bigcap_{l=0}^{k-1} T^{lm} B) \geq \delta$.
4. Let G be a countably infinite amenable discrete group. Show that there are infinitely many left invariant means on $\ell_\infty(G)$.
5. Give examples of unitary representations π and ρ of \mathbb{Z} such that $\pi \otimes \rho$ is ergodic but neither π nor ρ is weakly mixing.
6. Show that a unitary representation π of a countable discrete group G is weakly mixing if and only if $\pi \otimes \rho$ is weakly mixing for every unitary representation ρ of G .
7. Show that if (X, μ, T) and (Y, ν, S) are measure-preserving \mathbb{Z} -systems and (X, μ, T) is weakly mixing then the extension $X \times Y \rightarrow Y$ is weakly mixing.
8. For bounded sets in the Hilbert module $L^2(X|Y)$ we can define conditional precompactness either in terms of approximation by finitely generated module zonotopes or in terms of approximation by expansions over finite orthonormal sets. Show that these two definitions are equivalent.
9. Let G be a countable discrete group with property (T). Prove that G is finitely generated.
10. Show that a countable discrete group G is amenable if and only if every continuous action of G on a compact Hausdorff space has an invariant Borel probability measure.
11. Show that a subgroup of an amenable countable discrete group is amenable.
12. Let G be an amenable countable discrete group. Show that G has property (T) if and only if it is finite.
13. Let G be a countable discrete group and H a finite-index subgroup of G . Show that G has property (T) if and only if H has property (T).