

NOTES FOR MATH 689—TOPICS IN DYNAMICS AND ANALYSIS

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1. INTRODUCTION

Dynamics might be described as the analytic or asymptotic study of symmetry. As the term dynamics suggests, asymptotics may refer to the long-term behaviour of a system evolving in time, such as the universe. In some models, as in statistical mechanics, dynamics may represent a mechanism for mapping out space rather than time evolution. This is the physical motivation for the development of dynamical entropy, which provides a means for calculating the entropy of a statistical mechanical system of infinite spatial extent (this spatial unboundedness is paradoxically necessary to model phase transitions mathematically).

In a general abstract sense, a dynamical system consists of the action of a group on a space of a certain type by automorphisms, i.e., a homomorphism of a group G into the group $\text{Aut}(X)$ of automorphisms of a space X . If G is equipped with a topology then one might ask for this homomorphism to be continuous with respect to some topology on $\text{Aut}(X)$. In this course we will concentrate on topological and measure-preserving systems, and our groups will be discrete and countable.

Definition 1.1. By a *topological (dynamical) system* we mean a pair (X, G) where X is a compact Hausdorff space and G is a countable group acting on X by homeomorphisms.

We will say that a topological system (X, G) is *metrizable* if X is metrizable.

Definition 1.2. By a *measure-preserving (dynamical) system* we mean a quadruple (X, \mathcal{X}, μ, G) where (X, \mathcal{X}, μ) is a standard probability space and G is a countable group acting on (X, \mathcal{X}, μ) by μ -preserving bimeasurable transformations.

We will also speak of *G-systems*. On occasion we might need to use a symbol T for the homomorphism representing the action, in which case we will write T_s for the image of an element s in G . Usually however we will simply write sx for the image of a point $x \in X$ under T_s and not bother introducing an extra symbol in accord with the notation in the above definitions. We can also describe a system or action as a map $G \times X \rightarrow X$ where $(s, x) \mapsto sx$ and $s(tx) = (st)x$ for all $s, t \in G$ and $x \in X$. In the topological case this map is continuous, while in the measure-preserving case it is measurable.

A topological \mathbb{Z} -system can be alternatively described as a pair (X, T) where T is a homeomorphism from X to itself. The action of \mathbb{Z} in this case is generated by taking powers of T . Similarly a measure-preserving \mathbb{Z} -system can be described as a quadruple (X, \mathcal{X}, μ, T) where T is a μ -preserving bimeasurable transformation of the standard probability space (X, \mathcal{X}, μ) . Integer actions are the domain of classical topological and measurable dynamics and all of the basic concepts stem from this setting. However new types of phenomena (e.g., rigidity) can arise once one moves away from the case $G = \mathbb{Z}$. In fact the integers are quite special in the sense of being the only nontrivial group which is both free and amenable (we will examine these notions later), which is important in enabling certain kinds of constructions.

The main goal of dynamics is to study the asymptotic behaviour of a system as we move outward towards infinity in the group. For a \mathbb{Z} -system with generator T this means looking at the way the iterates T^n transform the space as $n \rightarrow \pm\infty$. One basic question is to what degree a system exhibits recurrence. One can also ask to what extent the dynamics mixes up the space, leading to notions like entropy and weak mixing. Our principle aim in this course is to examine recurrence and mixing properties from a systematic perspective based in combinatorial independence and involving ideas rooted in the geometric theory of Banach spaces. Taking a perspective which is dual to the usual picture of an action on a space X , we can study topological and measurable dynamics in terms of symmetry in Banach spaces or operator algebras by looking at how the dynamics interacts with the linear-geometric structure of a suitable space of functions over X . We will investigate certain Banach space phenomena in the presence of symmetry and relate these back to what is happening at the level of the space X in order to gain new insight into various dynamical properties. These properties will typically involve the notion of independence, which originates in probability theory but also appears in combinatorial form in Banach space theory. Combinatorial independence manifests itself dynamically in many ways and has long played an important role in the study of topological systems, although it has not until now received a unified and systematic treatment, which is what we aim to provide in this course.

A recurring theme in both the topological and measurable settings will be the appearance of dichotomies which separate tame systems from those which exhibit chaotic or random behavior. For example we will see how Rosenthal's theorem characterizing those Banach spaces which contain isomorphs of ℓ_1 translates dynamically into a tame/mixing dichotomy that can be formulated in terms of combinatorial independence. Rosenthal's

theorem is a prototypical example of a result in Banach space theory which yields the existence of a subspace possessing one of two properties at different extremes. This type of dichotomy is essentially local in nature, even though infinite-dimensional subspaces are involved. In the dynamical context where a considerable degree of symmetry is involved, it is possible sometimes to leverage local dichotomous behaviour into a global structure theorem. This idea underlies Furstenberg's ergodic-theoretic approach to Szemerédi theorem on the existence of arithmetic progressions in subsets of natural numbers of positive upper density, which we will examine towards the end of the course.

2. WEAK MIXING VS. COMPACTNESS FOR UNITARY OPERATORS

Let \mathcal{H} be a separable infinite-dimensional Hilbert space and U a unitary operator on \mathcal{H} . For a vector $\xi \in \mathcal{H}$ write $Z(\xi)$ for the cyclic subspace generated by ξ , i.e., the closed linear span of the orbit $\{U^n \xi : n \in \mathbb{Z}\}$, and σ_ξ for the measure on \mathbb{T} whose n th Fourier coefficient is $\langle U^n \xi, \xi \rangle$. Recall that the spectral theorem asserts that there exists a sequence $\{\xi_n\}_{n=1}^\infty$ in \mathcal{H} such that $\mathcal{H} = \bigoplus_{n=1}^\infty Z(\xi_n)$ and $\sigma_{\xi_1} \gg \sigma_{\xi_2} \gg \dots$, and that if $\{\xi'_n\}_{n=1}^\infty$ is another sequence in \mathcal{H} with the same properties then $\sigma_{\xi_n} \sim \sigma_{\xi'_n}$ for all n . The *spectral type* σ_U of U is the measure (class) of σ_{ξ_1} . We can also record the information captured by the spectral theorem by using σ_U along with the multiplicity function $M_U : \mathbb{T} \rightarrow \mathbb{N} \cup \{\infty\}$. Thus, up to unitary equivalence, a unitary on a separable Hilbert space is a direct sum of operators acting as multiplication by z on $L^2(\mathbb{T}, \mu)$ for some Borel probability measure μ on \mathbb{T} .

The measure σ_U has an atomic part σ_d and a continuous part σ_c . These correspond to orthogonal U -invariant closed subspaces \mathcal{H}_{wm} and $\mathcal{H}_{\text{cpct}}$ with $\mathcal{H} = \mathcal{H}_{\text{wm}} \oplus \mathcal{H}_{\text{cpct}}$. The vectors ξ in $\mathcal{H}_{\text{cpct}}$ are characterized by the relative compactness of their orbit $\{U^n \xi : n \in \mathbb{Z}\}$ (such vectors are called *compact* vectors). The vectors ξ in \mathcal{H}_{wm} on the other hand satisfy the weak mixing condition $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} |\langle U^i \xi, \xi \rangle| = 0$.

Given a Hilbert space \mathcal{H} , its conjugate $\overline{\mathcal{H}}$ is the Hilbert space which is the same as \mathcal{H} as an additive group but with the values of the inner product conjugate to those of \mathcal{H} (for multiplication by a scalar c this means that $c\xi$ in $\overline{\mathcal{H}}$ must correspond to $\bar{c}\xi$ in \mathcal{H}). Given a unitary operator U on \mathcal{H} we write \overline{U} for the unitary operator on $\overline{\mathcal{H}}$ which agrees with U under the identification of $\overline{\mathcal{H}}$ and \mathcal{H} as sets.

By using the canonical identification of $\mathcal{H} \otimes \overline{\mathcal{H}}$ with the space of Hilbert-Schmidt operators on \mathcal{H} , one can show that the unitary operator $U \otimes \overline{U}$ on $\mathcal{H} \otimes \overline{\mathcal{H}}$ has a nonzero invariant vector if and only if U has an eigenvector.

Theorem 2.1. *For a unitary operator U on a separable Hilbert space \mathcal{H} , the following are equivalent:*

- (1) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} |\langle U^i \xi, \zeta \rangle| = 0$ for all $\xi, \zeta \in \mathcal{H}$,
- (2) there are positive integers $n_1 < n_2 < n_3 < \dots$ such that $\lim_{k \rightarrow \infty} |\langle U^{n_k} \xi, \zeta \rangle| = 0$ for all $\xi, \zeta \in \mathcal{H}$,
- (3) the only compact vector in \mathcal{H} is the zero vector,
- (4) U has no eigenvectors,
- (5) $U \otimes \overline{U}$ has no nonzero invariant vectors.

Theorem 2.1 extends, with a couple of modifications, to unitary representations of arbitrary groups. In this case the averaging in condition (1) should be done via the mean on weakly almost periodic functions, while condition (4) should assert the nonexistence of nontrivial finite-dimensional invariant subspaces.

References: [4, 1].

3. BASIC PROPERTIES AND EXAMPLES OF TOPOLOGICAL SYSTEMS

Many properties of dynamical systems can be categorized as being either of a *recurrence/transitivity* variety (e.g., minimality, transitivity) or of a *mixing* variety (e.g., mixing, weak mixing, positive entropy). These two basic categories are not completely unrelated: weak mixing is defined as transitivity on a product system.

Minimality is the appropriate notion of irreducibility for a topological system and is defined as follows. The system (Y, G) is a *subsystem* of the topological system (X, G) if Y is a closed subset of X and the action on Y is the restriction of the action on X .

Definition 3.1. A topological system (X, G) is said to be *minimal* if it has no proper subsystems.

Proposition 3.2. *A topological system (X, G) is minimal if and only if for every $x \in X$ the orbit $\{sx : s \in G\}$ is dense.*

Every topological system has a minimal subsystem by Zorn's lemma. However there is no sense in which a general topological system can be decomposed into minimal components (this is different from the case of measure-preserving systems). Consider for example the left shift action $(s, t) \mapsto st$ of a countable discrete group G on itself compactified with a fixed point at ∞ . This system has a unique minimal subsystem (the fixed point). In particular it cannot be written as a union of minimal systems. It does however have a dense orbit.

Definition 3.3. A topological system (X, G) is said to be *transitive* if for all nonempty open sets $U, V \subseteq X$ there exists an $s \in G$ such that $sU \cap V \neq \emptyset$. The system is *point transitive* if it has a dense orbit. It is *weakly mixing* if the product system $(X \times X, G)$ is transitive, and *mixing* if for all nonempty open sets $U, V \subseteq X \times X$ there is a finite set $F \subseteq G$ such that $sU \cap V \neq \emptyset$ for all $s \in G \setminus F$.

Thus (X, G) is weak mixing if for all open sets $U_1, U_2, U_3, U_4 \subseteq X$ there is an $s \in G$ such that $sU_1 \cap U_3 \neq \emptyset$ and $sU_2 \cap U_4 \neq \emptyset$, and mixing if for all open sets $U_1, U_2, U_3, U_4 \subseteq X$ there is a finite set $F \subseteq G$ such that $sU_1 \cap U_3 \neq \emptyset$ and $sU_2 \cap U_4 \neq \emptyset$ for all $s \in G \setminus F$.

Proposition 3.4. *For topological systems (X, G) with X metrizable, transitivity and point transitivity are equivalent.*

Definition 3.5. A topological system (X, G) is said to be *equicontinuous* if for every neighbourhood U of the diagonal $\Delta = \{(x, x) : x \in X\}$ there is a neighbourhood V of Δ such that $sV \subseteq U$ for every $s \in G$. It is said to be *distal* if for all distinct $x, y \in X$ there is a neighbourhood U of Δ such that $\{(sx, sy) : s \in G\} \cap U = \emptyset$.

If X is metrizable then (X, G) is equicontinuous if and only if it is isometric, i.e., there is a compatible metric on X which is G -invariant.

Every topological system has a largest equicontinuous factor and a largest distal factor.

It is easy to give examples of minimal systems (X, G) with X finite. This is less trivial however if X is infinite, especially if one wants to arrange for some other properties like positive entropy to hold at the same time. An irrational rotation of the circle is a basic and important example of a minimal \mathbb{Z} -system. This system arises as a compactification of \mathbb{Z} as a group and is equicontinuous.

All transitive systems arise as a compactification of G as a discrete topological space, and there is a universal such system with respect to taking factors.

Definition 3.6. Let (X, G) and (Y, G) be topological G -systems and $\pi : X \rightarrow Y$ a surjective continuous map such that $s\pi(x) = \pi(sx)$ for all $s \in G$ and $x \in X$. We refer to π as a *factor map* or *extension*, (X, G) as an *extension* of (Y, G) , and (Y, G) as a *factor* of (X, G) .

The shift action $(s, t) \mapsto st$ of a countable discrete group G on itself extends to a continuous action of G on the Stone-Ćech compactification βG . By the universal property of the Stone-Ćech compactification, every point transitive system is a factor of $(\beta G, G)$.

Example 3.7. The free group F_2 acts continuously on its Gromov boundary as follows. Let a, b be the generators of F_2 . The Gromov boundary ∂F_2 is the set of infinite reduced words in a, b, a^{-1}, b^{-1} topologized by viewing it as a closed subset of $\{a, b, a^{-1}, b^{-1}\}^{\mathbb{N}}$ with the product topology. It is a Cantor set. We let F_2 act on X by multiplication on the left, reducing as necessary. This system is minimal and not distal. Is it weakly mixing?

Example 3.8. Let G be a countable discrete group and Y a compact Hausdorff space. Then G acts continuously on Y^G with the product topology by shifting the index, i.e., $s(y_t)_{t \in G} = (y_{st})_{t \in G}$ for all $s \in G$ and $(y_t)_{t \in G} \in Y^G$. This system is mixing and in particular transitive, although it is point transitive only when X is separable.

Example 3.9. Let X be the group $\{0, 1\}^{\mathbb{N}}$ where addition is modulo 2 with carry over to the right. We define a homeomorphism T of $\{0, 1\}^{\mathbb{N}}$ by $Tx = x + (1, 0, 0, \dots)$. The system (X, T) is called the *dyadic adding machine* or *dyadic odometer*. It is minimal and equicontinuous.

Example 3.10. Let θ be an irrational number and define the homeomorphism T of \mathbb{T}^2 by $T(z, w) = (e^{2\pi i \theta} z, zw)$. The system (\mathbb{T}^2, T) is minimal and distal but not equicontinuous. Defining $\pi : \mathbb{T}^2 \rightarrow \mathbb{T}$ by $\pi(z, w) = z$ yields a factor map of (\mathbb{T}^2, T) onto its largest equicontinuous factor.

References: [4, 18, 23].

4. COMBINATORIAL INDEPENDENCE AND ℓ_1

Definition 4.1. A collection $\{(A_{i,1}, \dots, A_{i,k}) : i \in I\}$ of k -tuples of subsets of a given set is said to be *independent* if $\bigcap_{i \in J} A_{i, \sigma(i)} = \emptyset$ for every finite set $J \subseteq I$ and $\sigma \in \{1, \dots, k\}^J$.

Definition 4.2. Let (X, G) be a topological system. For a tuple $\mathbf{A} = (A_1, A_2, \dots, A_k)$ of subsets of X , we say that a set $J \subseteq G$ is an *independence set* for \mathbf{A} if for every nonempty

finite subset $I \subseteq J$ and function $\sigma : I \rightarrow \{1, 2, \dots, k\}$ we have $\bigcap_{s \in J} s^{-1}A_{\sigma(s)} \neq \emptyset$, where $s^{-1}A$ for a set $A \subseteq X$ refers to the inverse image $\{x \in X : sx \in A\}$.

To realize ℓ_1 isomorphically in a Banach space V , it is enough to produce a bounded sequence of functions whose values on the dual are separated in a uniform way by an independent collection of pairs in B_{V^*} , as the following proposition demonstrates.

Proposition 4.3. *Let V be a Banach space. Let D_0, D_1 be closed disks in the complex plane with respective centres z_0, z_1 and common radius r such that $r \leq |z_0 - z_1|/6$. Let $\{f_i\}_{i=1}^\infty$ a bounded sequence of vectors in V such that the collection $\{(A_{i,0}, A_{i,1})\}_{i=1}^\infty$ is independent, where $A_{i,j} = \{\omega \in B_{V^*} : \omega(f_i) \in D_j\}$. Then $\{f_i\}_{i=1}^\infty$ is $2r^{-1}C$ -equivalent to the standard basis of ℓ_1 , where $C = \sup_{i \in \mathbb{N}} \|f_i\|$.*

Proof. We may assume by multiplying the vectors f_i by $|z_1 - z_0|/(z_1 - z_0)$ that $z_1 - z_0$ is real and positive. It suffices to show that for any complex scalars c_1, \dots, c_n we have $\|\sum_{i=1}^n c_i f_i\| \geq 2r^{-1} \sum_{i=1}^n |c_i|$. Writing $c_i = a_i + ib_i$ we may assume that $\sum_{i=1}^n |a_i| \geq \sum_{i=1}^n |b_i|$.

Consider $\sigma \in \{0, 1\}^n$ such that $\sigma(i)$ is 0 or 1 depending on whether $a_i < 0$ or $a_i \geq 0$. By independence there exist $s \in \bigcap_{i=1}^n A_{i,\sigma(i)}$ and $t \in \bigcap_{i=1}^n A_{i,1-\sigma(i)}$. Since $\operatorname{re}(f_i(s) - f_i(t)) \geq \operatorname{dist}(D_0, D_1) \geq 4r$ when $\sigma(i) = 1$ and $\operatorname{re}(f_i(t) - f_i(s)) \geq \operatorname{dist}(D_0, D_1) \geq 4r$ when $\sigma(i) = 0$, we have

$$\operatorname{re}\left(\sum_{i=1}^n a_i f_i((s-t)/2)\right) \geq 2r \sum_{i=1}^n |a_i|.$$

Note also that

$$\operatorname{im}\left(\sum_{i=1}^n b_i f_i((s-t)/2)\right) \leq \sum_{i=1}^n |b_i| |\operatorname{im}(f_i((s-t)/2))| \leq r \sum_{i=1}^n |b_i| \leq r \sum_{i=1}^n |a_i|.$$

Therefore

$$\begin{aligned} \left\| \sum_{i=1}^n c_i f_i \right\| &\geq \operatorname{re}\left(\sum_{i=1}^n c_i f_i((s-t)/2)\right) \\ &= \operatorname{re}\left(\sum_{i=1}^n a_i f_i((s-t)/2)\right) - \operatorname{im}\left(\sum_{i=1}^n b_i f_i((s-t)/2)\right) \\ &\geq r \sum_{i=1}^n |a_i| \geq \frac{r}{2} \sum_{i=1}^n |c_i|. \end{aligned}$$

□

Proposition 4.3 forms part of the proof of the complex version of Rosenthal's ℓ_1 theorem [19] which was established by Dor in [3]. The theorem states that a bounded sequence of vectors in a Banach space has a subsequence which is either weakly Cauchy or equivalent to the standard basis of ℓ_1 . We will discuss it in Section 12.

What can we say in the converse direction? Given a set S of vectors in a Banach space V which is C -equivalent to the standard basis of ℓ_1 , can we find a collection of pairs of sets in B_{V^*} which separate in a uniform way the values of the vectors in S or at least in some subset of S whose relative size depends on C ? When S is infinite then it follows from

Rosenthal's ℓ_1 theorem that we can find an infinite subset of S over which equivalence with the standard ℓ_1 basis occurs due to independence. In the topological study of entropy and weak mixing, however, what we need is a quantitative finite-dimensional result that we can apply asymptotically as the number of dimensions grows large.

Given a finite collection $\mathcal{A} = \{(A_{i,1}, \dots, A_{i,k}) : i \in I\}$ of k -tuples of subsets of a given set, we can define its combinatorial entropy $h_c(\mathcal{A})$ as the number of nonempty intersections of the form $\bigcap_{i \in I} A_{i,\sigma(i)}$ where $\sigma \in \{1, \dots, k\}^I$. If we know something about the entropy $h_c(\mathcal{A})$, can we say something about the possible size of an index subset $J \subset I$ for which $\{(A_{i,1}, \dots, A_{i,k}) : i \in J\}$ is independent? In the case that $k = 2$ and $h_c(\mathcal{A})$ is exponentially large with respect to $|I|$ there will be an index subset of proportional size over which independence occurs, as expressed by the following crude form of the Sauer-Shelah lemma [20, 21].

Lemma 4.4. *For every $b > 0$ there is a $c > 0$ such that, for all $n \in \mathbb{N}$, if $S \subseteq \{0, 1\}^{\{1,2,\dots,n\}}$ satisfies $|S| \geq e^{bn}$ then there is an $I \subseteq \{1, 2, \dots, n\}$ with $|I| \geq cn$ and $S|_I = \{0, 1\}^I$.*

A similar result holds for general k (see lemma 7.9) and is a consequence of Karpovskiy and Milman's generalization of the Sauer-Shelah lemma [8].

This suggests a combinatorial definition of entropy for a tuple (A_1, \dots, A_k) of subsets of a compact Hausdorff space X with the action of a group G as the growth of the number of nonempty intersections of the form $\bigcap_{s \in F} sA_{\sigma(s)}$ where $\sigma \in \{1, \dots, k\}^F$ and F is a finite subset of G . The problem is to determine exactly how the growth is to be measured, i.e., in what way do we take a limit over finite subsets? There is a satisfactory way of doing this if G is amenable, which means that G has good averaging properties. In our topological framework it is more natural to define entropy in a not so strictly combinatorial way using open covers, as we will see in Section 6. First however we will need to study amenability, to which we turn in the next section.

References: [19, 3, 6].

5. AMENABLE GROUPS

Let G be a countable discrete group. A *mean* on $\ell^\infty(G)$ is a positive unital linear functional $\sigma : \ell^\infty(G) \rightarrow \mathbb{C}$. Consider the action α of G on $\ell^\infty(G)$ given by $\alpha_s(f)(t) = f(s^{-1}t)$ for all $f \in \ell^\infty(G)$ and $s \in G$. The mean σ is *left invariant* if $\sigma(\alpha_s(f)) = \sigma(f)$ for all $f \in \ell^\infty(G)$ and $s \in G$.

Definition 5.1. We say that G is *amenable* if $\ell^\infty(G)$ admits a left invariant mean.

As a C^* -algebra, $\ell^\infty(G)$ is isomorphic to $C(\beta G)$, and so by the Riesz representation theorem every left invariant mean on $\ell^\infty(G)$ corresponds to a regular Borel probability measure on βG . Thus G is amenable precisely when there exists a G -invariant regular Borel probability measure on βG . By the universality of βG among transitive systems, this means that G is amenable precisely when every topological G -system admits a G -invariant regular Borel probability measure.

Properties of amenable groups:

The following Følner characterization of amenability is crucial for many applications. Given a finite $K \subseteq G$ and $\delta > 0$, we say that a nonempty finite set $F \subseteq G$ is (K, δ) -invariant if

$$|\{s \in F : Ks \subseteq F\}| \geq (1 - \delta)|F|.$$

Theorem 5.2. *The group G is amenable if and only if for every finite set $K \subseteq G$ and $\varepsilon > 0$ there exists a nonempty finite set $F \subseteq G$ such that $|\{s \in F : Ks \subseteq F\}| \geq (1 - \delta)|F|$. This is equivalent to the existence of a sequence $\{F_n\}_n$ of nonempty finite subsets of G such that $\lim_{n \rightarrow \infty} |sF_n \Delta F_n|/|F_n| = 0$ for all $s \in G$.*

The sequence $\{F_n\}_n$ is called a *Følner sequence*.

Namioka gave a relatively short proof of the above theorem which produces from an invariant mean approximately invariant functions in $\ell_1(G)$ (whose dual is $\ell^\infty(G)$) and applies an averaging argument to extract a Følner sequence from level sets of step approximations to these functions.

To define topological entropy in Section 6 we will need a subadditivity result (Theorem 5.6 below) which requires the quasitiling machinery for amenable groups developed by Ornstein and Weiss [16]. The key result of Ornstein and Weiss (Theorem 5.5 below) uses the Følner characterization of amenability.

Definition 5.3. A collection $\{A_i\}_{i \in I}$ of nonempty finite sets is ε -disjoint if there exist pairwise disjoint sets $A'_i \subseteq A_i$ with $|A'_i|/|A_i| \geq 1 - \varepsilon$ for all $i \in I$.

Definition 5.4. Let G be a group. We say that a collection $\{T_i\}$ of subsets of G ε -quasitiles G if there exist a finite set $K \subseteq G$ and a $\delta > 0$ such that for every (K, δ) -invariant finite set $A \subseteq G$ there is an ε -disjoint collection $\{T_i c_{ij}\}_{i,j}$ of translates of the T_i satisfying

$$|A \cap \bigcup_{i,j} T_i c_{ij}|/|A| \geq 1 - \varepsilon.$$

Theorem 5.5. *Let G be a countable amenable group. Let $\varepsilon > 0$. Then there exists an $n \in \mathbb{N}$ depending only on ε such that if $e \in T_1 \subseteq T_2 \subseteq \dots \subseteq T_n$ are subsets of G such that for $i = 1, \dots, n-1$ we have $|\partial_{T_i} T_{i+1}| \leq \eta_i |T_{i+1}|$ for some sufficiently small η_i depending on i and T_i , then the collection $\{T_i : i = 1, \dots, n\}$ ε -quasitiles G .*

Theorem 5.6. *If φ is a real-valued function which is defined on the set of finite subsets of G and satisfies*

- (1) $0 \leq \varphi(A) < +\infty$ and $\varphi(\emptyset) = 0$,
- (2) $\varphi(A) \leq \varphi(B)$ for all $A \subseteq B$,
- (3) $\varphi(As) = \varphi(A)$ for all finite $A \subseteq G$ and $s \in G$,
- (4) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ if $A \cap B = \emptyset$,

then $\frac{1}{|F|}\varphi(F)$ converges to some limit b as the set F becomes more and more invariant in the sense that for every $\varepsilon > 0$ there exist a finite set $K \subseteq G$ and a $\delta > 0$ such that $|\frac{1}{|F|}\varphi(F) - b| < \varepsilon$ for all (K, δ) -invariant sets $F \subseteq G$.

References: [13, 16, 7].

6. TOPOLOGICAL ENTROPY

Topological entropy was originally formulated for \mathbb{Z} -actions but the natural general setting is that of actions of amenable groups. We will first treat the classical definition for \mathbb{Z} -actions (i.e., single homeomorphisms) and then extend the definition to actions of amenable groups using Theorem 5.6.

Let X be a compact metric space and $T : X \rightarrow X$ a homeomorphism. The topological entropy $h_{\text{top}}(T, \mathcal{U})$ of an open cover \mathcal{U} of X with respect to T is defined as

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln N(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \dots \vee T^{-n+1}\mathcal{U}),$$

where $N(\cdot)$ denotes the minimal cardinality of a subcover. The topological entropy $h_{\text{top}}(T)$ of T is the supremum of $h_{\text{top}}(T, \mathcal{U})$ over all open covers \mathcal{U} of X .

We can equivalently define entropy in terms of spanning and separated sets. Let d be a compatible metric on X . Given an $n \in \mathbb{N}$ and $\varepsilon > 0$, a set $E \subset X$ is said to be (n, ε) -spanning if for every $x \in X$ there is a $y \in E$ such that $d(T^i x, T^i y) < \varepsilon$ for every $i = 0, \dots, n-1$, and (n, ε) -separated if for all distinct $x, y \in E \subset X$ we have $d(T^i x, T^i y) > \varepsilon$ for some $i = 0, \dots, n-1$. We write $\text{spn}(n, \varepsilon)$ for the smallest cardinality of an (n, ε) -spanning set and $\text{sep}(n, \varepsilon)$ for the largest cardinality of an (n, ε) -separated set. Then

$$h_{\text{top}}(T) = \sup_{\varepsilon > 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{spn}(n, \varepsilon) = \sup_{\varepsilon > 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{sep}(n, \varepsilon).$$

For every $n \in \mathbb{N}$ we can define a metric d_n on X by

$$d_n(x, y) = \max_{i=0, \dots, n-1} d(T^i x, T^i y)$$

so that at the n th stage in the above formulation of entropy we can alternatively speak of ε -spanning and ε -separated sets with respect to d_n .

Example 6.1. The shift on $\{1, \dots, d\}^{\mathbb{Z}}$ has entropy $\log d$.

A rotation on the circle is an example of a system with zero entropy. In fact:

Proposition 6.2. *Every homeomorphism of the circle has zero entropy.*

For the remainder of this section (X, G) will be a topological system with G a countable amenable group. We define the topological entropy $h_{\text{top}}(X)$ of (X, G) by first using Theorem 5.6 to define the entropy of a finite open cover \mathcal{U} of X as the limit of

$$\frac{1}{|F|} \ln N(\{s\mathcal{U} : s \in F\})$$

as F becomes more and more invariant, and then defining $h_{\text{top}}(X)$ to be the supremum of these quantities over all finite open covers. (this was originally introduced in [14] without the subadditivity result).

In the measure-preserving setting, the notion of conditional entropy can be used to show that the function φ to which Theorem 5.6 is applied to define entropy satisfies the stronger subadditive inequality

$$\varphi(A \cup B) + \varphi(A \cap B) \leq \varphi(A) + \varphi(B)$$

in which case it is possible to give a simpler proof of the theorem that does not rely on quasitilings (see Section 2.2 of [14]).

Example 6.3. The shift on $\{1, \dots, d\}^G$ has entropy $\log d$.

Example 6.3 illustrates why amenability is the natural context for defining entropy. Consider the points in $\{1, \dots, d\}^G$ as representing states of a statistical-mechanical system. Each element of G is a site occupied by a particle which can be in one of the d possible states $1, \dots, d$. We wish to determine the mean entropy by taking a limit of the entropy density (the logarithmic average of the number of possible states) over finite subsets of G along a suitable sequence. The appropriate type of finite sets over which to take a limit are Følner sets, which play the role of boxes in the case $G = \mathbb{Z}^d$ and provide accurate local models for the group structure. The structure of the group can effectively be factored out by containing it within approximately invariant finite sets, and on finite sets we have a formula for entropy which we can use to take an infinite-volume limit. In theory we want to average over the whole group but this quantity is ill-defined. In the amenable case we can approximately do this by capturing the local structure in finite sets.

Intuitively positive entropy indicates a certain degree of indeterminism in the system, as quantified by exponential growth in the number of possible states up to a small observational error. One of our goals is to show that positive entropy implies local randomness along a subset of powers of T of positive density. We can do this by means of combinatorial arguments inspired by problems in Banach space theory. We will also describe entropy production at the function level and for this we will need to linearize the measurement of exponential growth so as to be able to invoke tools from the geometric theory of Banach spaces.

We will now show using the Sauer-Shelah lemma how positive entropy produces an isomorph of ℓ_1 along a positive density subset of the orbit of some function in $C(X)$. We will reprove this later using a more powerful combinatorial lemma that yields the ℓ_1 isomorphism by producing independence inside of X .

The following is Lemma 2.3 in [5].

Lemma 6.4. *For all $b > 0$ and $\delta > 0$ there exist $c > 0$ and $\varepsilon > 0$ such that, for all sufficiently large n , if E is a subset of the unit ball of $(\ell_\infty^n)_\mathbb{R}$ which is δ -separated and $|E| \geq e^{bn}$, then there are a $t \in [-1, 1]$ and a set $J \subseteq \{1, 2, \dots, n\}$ for which*

- (1) $|J| \geq cn$, and
- (2) for every $\sigma \in \{0, 1\}^J$ there is a $v \in E$ such that for all $j \in J$

$$\begin{aligned} v(j) &\geq t + \varepsilon && \text{if } \sigma(j) = 1, \text{ and} \\ v(j) &\leq t - \varepsilon && \text{if } \sigma(j) = 0. \end{aligned}$$

Let $f \in C(X)$. We say that $J \subset G$ is an ℓ_1 -isomorphism set if $\{sf : s \in J\}$ is equivalent to the standard basis of ℓ_1^J , i.e., if the map which sends the standard basis element of ℓ_1^J at $s \in J$ to sf extends to an isomorphism from ℓ_1^J to the closed linear span of $\{sf : s \in J\}$.

Proposition 6.5. *Suppose that $h_{\text{top}}(X) > 0$. Then there is an $f \in C_\mathbb{R}(X)$ such that for every tempered Følner sequence $\{F_n\}_n$ in G there is an ℓ_1 -isomorphism set J for f such that $\lim_{n \rightarrow \infty} |F_n \cap J|/|F_n| > 0$. In particular, if we are dealing with a \mathbb{Z} -system (X, T) then the function f has an ℓ_1 -isomorphism set of positive density.*

References: [23, 18, 4, 9].

7. ENTROPY PAIRS AND IE-PAIRS

Definition 7.1. We call a pair $(x_1, x_2) \in X \times X \setminus \Delta_2(X)$ an *entropy pair* if whenever U_1 and U_2 are closed disjoint subsets of X with $x_1 \in \text{int}(U_1)$ and $x_2 \in \text{int}(U_2)$, the open cover $\{U_1^c, U_2^c\}$ has positive topological entropy. More generally, we call a tuple $\mathbf{x} = (x_1, \dots, x_k) \in X^k \setminus \Delta_k(X)$ an *entropy tuple* if whenever U_1, \dots, U_l are closed pairwise disjoint neighbourhoods of the distinct points in the list x_1, \dots, x_k , the open cover $\{U_1^c, \dots, U_l^c\}$ has positive topological entropy.

Definition 7.2. We call a pair $(x_1, x_2) \in X \times X$ an *IE-pair* if for every product neighbourhood $U_1 \times U_2$ of (x_1, x_2) the pair (U_1, U_2) has an independence set of positive density. More generally, we call a tuple $\mathbf{x} = (x_1, \dots, x_k) \in X^k$ an *IE-tuple* if for every product neighbourhood $U_1 \times \dots \times U_k$ of \mathbf{x} the tuple (U_1, \dots, U_k) has an independence set of positive density. We denote the set of IE-tuples of length k by $\text{IE}_k(X, G)$.

Definition 7.3. For a finite tuple $\mathbf{A} = (A_1, \dots, A_k)$ of subsets of X , Theorem 5.6 also applies to the function $\varphi_{\mathbf{A}}$ given by

$$\varphi_{\mathbf{A}}(F) = \max\{|F \cap J| : J \text{ is an independence set for } \mathbf{A}\},$$

and we define the *independence density* $I(\mathbf{A})$ of \mathbf{A} as the limit of $\frac{1}{|F|}\varphi_{\mathbf{A}}(F)$ as F becomes more and more invariant.

A sequence $\{F_n\}_n$ of nonempty finite subsets of G is said to be *tempered* if for some $c > 0$ we have $|\bigcup_{k=1}^{n-1} F_k^{-1}F_n| \leq c|F_n|$ for all $n \in \mathbb{N}$.

Proposition 7.4. *Every Følner sequence has a tempered subsequence.*

Lindenstrauss established the following pointwise ergodic theorem [12].

Theorem 7.5. *Let (X, \mathcal{X}, μ, G) be a measure-preserving system with G amenable and let $\{F_n\}_n$ be a tempered Følner sequence. Then for every $f \in L^1(X, \mu)$ there is a G -invariant $\bar{f} \in L^1(X, \mu)$ (which is equal to $\int f(x) d\mu(x)$ if the system is ergodic) such that*

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{s \in F_n} f(sx) = \bar{f}(x)$$

for almost all $x \in X$.

For a tuple $\mathbf{A} = (A_1, \dots, A_k)$ of subsets of X we write $\mathcal{P}_{\mathbf{A}}$ for the collection of all independence sets for \mathbf{A} . We can view $\mathcal{P}_{\mathbf{A}}$ as a subset of $\{0, 1\}^G$ by identifying subsets of G with elements in $\{0, 1\}^G$ via indicator functions.

The following lemma permits us to convert finitary density statements to infinitary ones. The proof makes use of Lindenstrauss's pointwise ergodic theorem.

Proposition 7.6. *Let (X, G) be a metrizable system. Let $\mathbf{A} = (A_1, \dots, A_k)$ be a tuple of subsets of X . Let $c > 0$. Then the following are equivalent:*

- (1) $I(\mathbf{A}) \geq c$,
- (2) for every $\varepsilon > 0$ there exist a finite set $K \subseteq G$ and a $\delta > 0$ such that for every $F \in M(K, \delta)$ there is an independence set J for \mathbf{A} with $|J \cap F| \geq (c - \varepsilon)|F|$.

- (3) for every finite set $K \subseteq G$ and $\varepsilon > 0$ there exist an $F \in M(K, \varepsilon)$ and an independence set J for \mathbf{A} such that $|J \cap F| \geq (c - \varepsilon)|F|$.
- (4) for every tempered Følner sequence $\{F_n\}_{n \in \mathbb{N}}$ of G there is an independence set J for \mathbf{A} such that $\lim_{n \rightarrow \infty} \frac{|F_n \cap J|}{|F_n|} \geq c$.
- (5) there are a tempered Følner sequence $\{F_n\}_{n \in \mathbb{N}}$ of G and an independence set J for \mathbf{A} such that $\lim_{n \rightarrow \infty} \frac{|F_n \cap J|}{|F_n|} \geq c$.

Proof. The equivalences (1) \Leftrightarrow (2) \Leftrightarrow (3) follow from Theorem 5.6, and the implications (4) \Rightarrow (5) \Rightarrow (3) are trivial. Suppose that (3) holds and let us establish (4). We regard $\mathcal{P}_{\mathbf{A}}$ as a subset of $\{0, 1\}^G$ by taking indicator functions. It is straightforward to show that there is a G -invariant Borel probability measure μ on $\mathcal{P}_{\mathbf{A}} \subseteq \{0, 1\}^G$ with $\mu([e] \cap \mathcal{P}_{\mathbf{A}}) \geq c$, where $\{0, 1\}^G$ is equipped with the shift given by $sx(t) = x(ts)$ for all $x \in \{0, 1\}^G$ and $s, t \in G$. Replacing μ by a suitable ergodic G -invariant Borel probability measure in the ergodic decomposition of μ , we may assume that μ is ergodic. Let $\{F_n\}_{n \in \mathbb{N}}$ be a tempered Følner sequence for G . The pointwise ergodic theorem [12, Theorem 1.2] asserts that $\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{s \in F_n} f(sx) = \int f d\mu$ μ -a.e. for every $f \in L^1(\mu)$. Setting f to be the characteristic function of $[e] \cap \mathcal{P}_{\mathbf{A}}$ and taking J to be some x satisfying the above equation, we get (4). \square

In the case $G = \mathbb{Z}$, we see from the above theorem that a tuple $\mathbf{A} = (A_1, \dots, A_k)$ has an independence set of positive density if and only if there exists a $d > 0$ such that for every $M > 0$ there is an interval I in \mathbb{Z} with $|I| \geq M$ and an independence set J for \mathbf{A} included in I for which $|J| \geq d|I|$.

The following result provides the combinatorial key for establishing the precise relationship between entropy tuples and IE-tuples.

Lemma 7.7. *Let $k \geq 2$ and let $b > 0$. Then there exists a $c > 0$ depending only on k and b such that for every finite set Z and $S \subseteq \{0, 1, \dots, k\}^Z$ with $F_S \geq k^{b|Z|}$ there exists a $W \subseteq Z$ with $|W| \geq c|Z|$ and $S|_W \supseteq \{1, \dots, k\}^W$.*

Lemma 7.8. *Let $k \geq 2$. Let U_1, \dots, U_k be pairwise disjoint subsets of X and set $\mathcal{U} = \{U_1^c, \dots, U_k^c\}$. Then $\mathbf{U} := (U_1, \dots, U_k)$ has positive independence density if and only if $h_c(T, \mathcal{U}) > 0$.*

The following is a consequence of Karpovskiy and Milman's generalization of the Sauer-Shelah lemma [20, 21, 8] and can also be deduced from Lemma 7.7.

Lemma 7.9. *Given $k \geq 2$ and $\lambda > 1$ there is a $c > 0$ such that, for all $n \in \mathbb{N}$, if $S \subseteq \{1, 2, \dots, k\}^{\{1, 2, \dots, n\}}$ satisfies $|S| \geq ((k-1)\lambda)^n$ then there is an $I \subseteq \{1, 2, \dots, n\}$ with $|I| \geq cn$ and $S|_I = \{1, 2, \dots, k\}^I$.*

The case $|Z| = 1$ of the following lemma appeared in [17].

Lemma 7.10. *Let Z be a finite set such that $Z \cap \{1, 2, 3\} = \emptyset$. There exists a constant $c > 0$ depending only on $|Z|$ such that, for all $n \in \mathbb{N}$, if $S \subseteq (Z \cup \{1, 2\})^{\{1, 2, \dots, n\}}$ is such that $\Gamma_n|_S : S \rightarrow (Z \cup \{3\})^{\{1, 2, \dots, n\}}$ is bijective, where $\Gamma_n : (Z \cup \{1, 2\})^{\{1, 2, \dots, n\}} \rightarrow (Z \cup \{3\})^{\{1, 2, \dots, n\}}$ converts the coordinate values 1 and 2 to 3, then there is some $I \subseteq \{1, 2, \dots, n\}$ with $|I| \geq cn$ and either $S|_I \supseteq (Z \cup \{1\})^I$ or $S|_I \supseteq (Z \cup \{2\})^I$.*

8. POSITIVE ENTROPY AND LI-YORKE CHAOS

The notion of Li-Yorke chaos was introduced in [2] and is motivated by the study of interval dynamics in [11].

Let (X, T) be a metrizable topological \mathbb{Z} -system and let d be a compatible metric on X . We say that a pair $(x_1, x_2) \in X \times X$ is a *Li-Yorke pair* (with modulus δ) if

$$\limsup_{n \rightarrow \infty} \rho(T^n x_1, T^n x_2) = \delta > 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \rho(T^n x_1, T^n x_2) = 0.$$

We say that a set $Z \subseteq X$ is *scrambled* if all nondiagonal pairs of points in Z are Li-Yorke. The system (X, T) is said to be *Li-Yorke chaotic* if X contains an uncountable scrambled set.

For a subset \mathcal{P} of Ω_2 , we say that a finite subset $J \subset G$ has *positive density with respect to \mathcal{P}* if there exists a $K \subseteq G$ with positive density such that $(K - K) \cap (J - J) = \{0\}$ and $K + J \in \mathcal{P}$. We say that a subset $J \subseteq G$ has *positive density with respect to \mathcal{P}* if every finite subset of J has positive density with respect to \mathcal{P} .

Lemma 8.1. *Let \mathcal{P} be a hereditary closed shift-invariant subset of $\{0, 1\}^{\mathbb{Z}}$ with positive density. Then there exists a $J \subseteq \mathbb{Z}_{\geq 0}$ with positive density which also has positive density with respect to \mathcal{P} .*

Theorem 8.2. *Let $k \geq 2$ and let $\mathbf{x} = (x_1, \dots, x_k)$ be an IE-tuple in $X^k \setminus \Delta_k(X)$. For each $1 \leq j \leq k$, let A_j be a neighbourhood of x_j . Then there exist a $\delta > 0$ and a Cantor set $Z_j \subseteq A_j$ for each $j = 1, \dots, k$ such that the following hold:*

- (1) *every nonempty finite tuple of points in $Z := \bigcup_j Z_j$ is an IE-tuple;*
- (2) *for all $m \in \mathbb{N}$, distinct $y_1, \dots, y_m \in Z$, and $y'_1, \dots, y'_m \in Z$ we have*

$$\liminf_{n \rightarrow \infty} \max_{1 \leq i \leq m} \rho(T^n y_i, y'_i) = 0.$$

Corollary 8.3. *If $h_{\text{top}}(T) > 0$ then (X, T) is Li-Yorke chaotic.*

Corollary 8.4. *The set $\text{IE}_2(X, T)$ of IE-pairs does not have isolated points.*

9. EMBEDDING ℓ_1^k INTO M_d

We write M_d for the C^* -algebra of $d \times d$ matrices over the complex numbers and Tr for the unnormalized trace on M_d , i.e., $\text{Tr}((a_{ij})_{ij}) = \sum_{i=1}^d a_{ii}$. Recall that all of our Banach spaces are over the complex numbers unless otherwise indicated.

We will need the generalized Horn inequality. Associated to a matrix $a \in M_d$ are its s -numbers $s_1(a) \geq s_2(a) \geq \dots \geq s_d(a)$ which are the eigenvalues of $|a| = (a^*a)^{1/2}$ with multiplicity.

Theorem 9.1. *For $a_1, \dots, a_q \in M_d$ we have*

$$\text{Tr}(|a_1 \cdots a_q|) = \sum_{i=1}^k s_i(a_1) \cdots s_i(a_q).$$

The type 2 constant $T_2(V)$ of a Banach space V is defined as the infimum of the positive numbers T such that, for all $m \in \mathbb{N}$ and $v_1, \dots, v_m \in V$,

$$\frac{1}{2^m} \sum_{\varepsilon_1, \dots, \varepsilon_m = \pm 1} \left\| \sum_{i=1}^m \varepsilon_i v_i \right\|^2 \leq T^2 \sum_{i=1}^m \|v_i\|^2.$$

Notice that $T_2(\ell_1^n) \geq \sqrt{n}$, as can be seen by considering the standard basis v_1, \dots, v_n .

Lemma 9.2. *We have $T_2(M_d) \geq c\sqrt{\log d}$ for $d \geq 2$ where $c > 0$ is a universal constant.*

Theorem 9.3. *Let E be a n -dimensional subspace of M_d which is λ -isomorphic to ℓ_1^n . Then*

$$n \leq a\lambda^2 \log d$$

where $a > 0$ is a universal constant.

References: [22, 15, 5].

10. THE TOPOLOGICAL PINSKER ALGEBRA

Let (X, G) be a topological system. This gives rise to an action α of G on $C(X)$ by $*$ -automorphisms defined by $\alpha_s(f)(x) = f(s^{-1}x)$ for all $f \in C(X)$, $x \in X$, and $s \in G$. This is an example of a C^* -dynamical system.

For the remainder of this section G will be assumed to be amenable.

We denote by $\text{Fin}(C(X))$ the collection of finite subsets of $C(X)$. As before the C^* -algebra of $d \times d$ matrices over \mathbb{C} is written M_d . For each $\Omega \subseteq \mathcal{P}_{\text{fin}}(X)$ and $\delta > 0$ we denote by $\text{CMA}(\Omega, \delta)$ (*contractive matricial approximation*) the collection of triples (φ, ψ, d) where d is a positive integer and $\varphi : C(X) \rightarrow M_d$ and $\psi : M_d \rightarrow C(X)$ are contractive linear maps such that

$$\|(\psi \circ \varphi)(f) - f\| < \delta$$

for all $f \in \Omega$. We then set

$$\text{rm}(\Omega, \delta) = \inf\{d : (\varphi, \psi, d) \in \text{CMA}(\Omega, \delta)\}.$$

Considering our action α from above we now define

$$\begin{aligned} \text{hm}(\alpha, \Omega, \delta) &= \lim_{(K, \delta)} \sup_{F \in \mathcal{M}(K, \delta)} \frac{1}{n} \log \text{rm}(\{\alpha_s(\Omega) : s \in F\}, \delta) \\ \text{hm}(\alpha, \Omega) &= \sup_{\delta > 0} \text{hm}(\alpha, \Omega, \delta) \\ \text{hm}(\alpha, \Omega) &= \sup_{\Omega \in \text{Fin}(C(X))} \text{hm}(\alpha, \Omega). \end{aligned}$$

Lemma 10.1. *Let $\Omega = \{f_1, \dots, f_n\}$ be a finite subset of $C(X)$ and suppose that the linear map $\gamma : \ell_1^n \rightarrow X$ sending the i th standard basis element of ℓ_1^n to x_i for each $i = 1, \dots, n$ is an isomorphism. Let $\delta > 0$ be such that $\delta < \|\gamma^{-1}\|^{-1}$. Then*

$$\log \text{rm}(\Omega, \delta) \geq na\|\gamma\|^{-2}(\|\gamma^{-1}\|^{-1} - \delta)^2$$

where $a > 0$ is a universal constant.

Proof. Let $(\varphi, \psi, d) \in \text{CMA}(\Omega, \delta)$. For any linear combination $\sum c_i x_i$ of the elements x_1, \dots, x_n we have

$$\begin{aligned} \left\| \sum c_i x_i \right\| &\leq \left\| \sum c_i x_i - (\psi \circ \varphi) \left(\sum c_i x_i \right) \right\| + \left\| (\psi \circ \varphi) \left(\sum c_i x_i \right) \right\| \\ &\leq \delta \sum |c_i| + \left\| \varphi \left(\sum c_i x_i \right) \right\| \\ &\leq \delta \|\gamma^{-1}\| \left\| \sum c_i x_i \right\| + \left\| \varphi \left(\sum c_i x_i \right) \right\| \end{aligned}$$

and so $\left\| \varphi \left(\sum c_i x_i \right) \right\| \geq (1 - \delta \|\gamma^{-1}\|) \left\| \sum c_i x_i \right\|$. Since φ is contractive, it follows that the composition $\varphi \circ \gamma$ is a $\|\gamma\|(\|\gamma^{-1}\|^{-1} - \delta)^{-1}$ -isomorphism onto its image in M^d . The desired conclusion now follows from Theorem 9.3. \square

Lemma 10.2. *For every $\delta > 0$ there exist $c > 0$ and $\varepsilon > 0$ such that, for every compact Hausdorff space Y and finite subset Θ of the unit ball of $C(Y)$ of sufficiently large cardinality, if the linear map $\gamma : \ell_1^\Theta \rightarrow C(Y)$ sending the standard basis of ℓ_1^Θ to Θ is an isomorphism with $\|\gamma^{-1}\|^{-1} \geq \delta$, then there exist closed disks $B_1, B_2 \subseteq \mathbb{C}$ of diameter at most $\varepsilon/6$ with $\text{dist}(B_1, B_2) \geq \varepsilon$ and an $I \subseteq \Theta$ with $|I| \geq c|\Theta|$ such that the collection $\{(f^{-1}(B_1), f^{-1}(B_2)) : f \in I\}$ is independent.*

Theorem 10.3. *Let $f \in C(X)$. Then the following are equivalent:*

- (1) $f \notin \mathfrak{P}_{X,G}$,
- (2) $\text{hm}(\alpha, \{f\}) > 0$,
- (3) *there is an IE-pair $(x, y) \in X \times X$ with $f(x) \neq f(y)$,*
- (4) $h_{d_f}(X) > 0$,
- (5) *for every tempered Følner sequence $\{F_n\}_n$ in G there is an ℓ_1 -isomorphism set J for f such that $\lim_{n \rightarrow \infty} |F_n \cap J|/|F_n| > 0$.*

Example 10.4. The Stone-Cech compactification $\beta\mathbb{Z}$ provides an example of a compact Hausdorff space which admits a G -action with infinite topological entropy but no homeomorphism with finite nonzero topological entropy.

References: [9, 10].

11. SEQUENCE ENTROPY AND NULLNESS

12. ROSENTHAL'S ℓ_1 THEOREM AND TAMENESS

13. THE RADON-NIKODÝM PROPERTY AND HEREDITARY NONSENSITIVITY

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