Abstract: Given a Borel probability measure $\mu$ on $\mathbb{R}^d$, a number $r \in (0, +\infty)$ and a natural number $n \in \mathbb{N}$, the $n$th quantization error of order $r$ for $\mu$, is defined by

$$
e_{n,r} = \inf \{ \left( \int d(x, \alpha)^r d\mu(x) \right)^{\frac{1}{r}} : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \},$$

where $d(x, \alpha)$ denotes the distance from the point $x$ to the set $\alpha$ with respect to a given norm $\| \cdot \|$ on $\mathbb{R}^d$. Note that if $\int \| x \|^r d\mu(x) < \infty$ then there is some set $\alpha$ for which the infimum is achieved. The upper and lower quantization dimensions for $\mu$ of order $r$ are defined by

$$
\overline{D}_r(\mu) := \limsup_{n \to \infty} \frac{\log n}{-\log e_{n,r}}; \quad \underline{D}_r(\mu) = \liminf_{n \to \infty} \frac{\log n}{-\log e_{n,r}}.
$$

If $\overline{D}_r(\mu)$ and $\underline{D}_r(\mu)$ coincide, we call the common value the quantization dimension of $\mu$ of order $r$ and is denoted by $D_r(\mu)$. One sees that the quantization dimension is actually a function $r \mapsto D_r$ which measures the asymptotic rate at which $e_{n,r}$ goes to zero. If $D_r$ exists, then asymptotically

$$\log e_{n,r} \sim \log \left( \frac{1}{n} \right)^{1/D_r}.$$

Let $P = [p_{ij}]_{1 \leq i,j \leq N}$ be an $N \times N$ irreducible row stochastic matrix and $X \subset \mathbb{R}^d$ be a compact set such that $X = \text{cl}(\text{int}X)$. To each $p_{ij}$ if $p_{ij} > 0$, let us associate a contractive similitude $S_{ij}$ mapping $X$ into $X$ with the similarity ratio $s_{ij}$ ($0 < s_{ij} < 1$). Then the collection $\{X, S_{ij}, p_{ij} > 0, 1 \leq i,j \leq N\}$ is called a recurrent iterated function system (RIFS) of similarity mappings. Let us now consider the ergodic Markov measure $\nu$ on the coding space, and take its image measure $\mu := \nu \circ \pi^{-1}$ on the recurrent self-similar set via the coding map $\pi$. I will talk about the quantization dimension function for the probability measure $\mu$, and the relationship between the quantization dimension function and the temperature function of the thermodynamic formalism that arises in multifractal analysis.