

# Entropy in Measurable Dynamics

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The triple  $(G, X, \mu)$  is a *dynamical system*.

Two systems  $(G, X_1, \mu_1)$  and  $(G, X_2, \mu_2)$  are *isomorphic* if there exists a measure-space isomorphism  $\phi : X_1 \rightarrow X_2$  with  $\phi(gx) = g\phi(x)$  for a.e.  $x \in X_1$  and for all  $g \in G$ .

**Main Problem:** Classify systems up to isomorphism.

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- $(G, K^G, \kappa^G)$  is the *Bernoulli shift* over  $G$  with *base space*  $(K, \kappa)$ .

# von Neumann's question

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**von Neumann's question:** Is the full 2-shift over  $\mathbb{Z}$  isomorphic to the full 3-shift over  $\mathbb{Z}$ ?

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$I(t)$  for  $0 \leq t \leq 1$  should satisfy:

- 1  $I(t) \geq 0$ .
- 2  $I(t)$  is continuous.
- 3  $I(ts) = I(t) + I(s)$ .

So  $I(t) = -\log_b(t)$  for some  $b > 1$ .

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Let  $T : X \rightarrow X$  be measure-preserving. The *entropy rate* of  $\phi$  w.r.t  $T$  is:

$$h(T, \phi) = \lim_{n \rightarrow \infty} \frac{1}{2n+1} H \left( \bigvee_{i=-n}^n \phi \circ T^i \right).$$

# Coding

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$\phi$  is a **generator** if  $\Phi$  is an isomorphism from  $(G, X, \mu)$  to  $(G, A^G, \Phi_*\mu)$ .

# Kolmogorov's entropy

## Theorem (Kolmogorov, 1958)

*Let  $T : X \rightarrow X$  be an automorphism of  $(X, \mu)$ . If  $\phi$  and  $\psi$  are finite-entropy generators for  $(\mathbb{Z}, X, \mu) = (\langle T \rangle, X, \mu)$  then  $h(T, \phi) = h(T, \psi)$ .*

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## Theorem (Sinai, 1959)

If  $\phi$  is any finite-entropy observable then  $h(T, \phi) \leq h(\mathbb{Z}, X, \mu)$ . Hence we may define the entropy of  $(\mathbb{Z}, X, \mu)$  to be  $\sup_{\phi} h(T, \phi)$ .

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### Theorem (Kolmogorov, 1958)

*If  $(\mathbb{Z}, K^{\mathbb{Z}}, \kappa^{\mathbb{Z}})$  is isomorphic to  $(\mathbb{Z}, L^{\mathbb{Z}}, \lambda^{\mathbb{Z}})$  then  $H(K, \kappa) = H(L, \lambda)$ . So the full 2-shift is not isomorphic to the full 3-shift.*

# Questions

- Does the converse hold?
  
- What if  $\mathbb{Z}$  is replaced with some other group  $G$ ?

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A group  $G$  is *Ornstein* if whenever  $(K, \kappa), (L, \lambda)$  are two standard probability spaces with  $H(\kappa) = H(\lambda)$  then  $(G, K^G, \kappa^G)$  is isomorphic to  $(G, L^G, \lambda^G)$ .

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- Is every countably infinite group Ornstein?

# Classification

Theorem (Ornstein, 1970)

*Bernoulli shifts over  $\mathbb{Z}$  are completely classified by their entropy.*

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## Theorem

*If  $G$  is infinite and amenable then Bernoulli shifts over  $G$  are completely classified by their entropy (which equals their base measure entropy).*

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What if  $G$  is nonamenable?

# Factor maps

## Definition

Let  $(G, X, \mu)$ ,  $(G, Y, \nu)$  be two systems and  $\phi : X \rightarrow Y$  a measurable map with  $\phi_*\mu = \nu$ ,  $\phi(gx) = g\phi(x)$  for a.e.  $x \in X$  and all  $g \in G$ . Then  $\phi$  is a *factor map* from  $(G, X, \mu)$  to  $(G, Y, \nu)$ .

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- The full  $n$ -shift over  $G$  has entropy  $\log(n)$ .

$\implies$  the full 2-shift over  $G$  cannot factor onto the full 4-shift over  $G$ .

# The Ornstein-Weiss Example

## Theorem (Ornstein-Weiss, 1987)

*If  $\mathbb{F} = \langle a, b \rangle$  is the rank 2 free group then the full 2-shift over  $\mathbb{F}$  factors onto the full 4-shift over  $\mathbb{F}$ .*

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# More Counterexamples

Theorem (Karen Ball, 2005)

*If  $G$  has infinitely many ends then the 2-shift over  $G$  factors onto every Bernoulli shift over  $G$ .*

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### Theorem

*If  $G$  contains a nonabelian free subgroup then every nontrivial Bernoulli shift over  $G$  factors onto every other Bernoulli shift over  $G$ .*

# New Results

## Theorem

*If  $G$  is a sofic group (e.g., a linear group) then Kolmogorov's direction holds. I.e., if  $(G, K^G, \kappa^G)$  is isomorphic to  $(G, L^G, \lambda^G)$  then  $H(K, \kappa) = H(L, \lambda)$ .*

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**The idea:** For  $n > 0$ , count the number of sequences  $(a_1, a_2, \dots, a_n)$  with elements  $a_i \in A$  that **approximate** the above sequence.

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Let  $u$  be the uniform measure on  $\{1, \dots, n\}$ .  $\psi_*^W u$  is a measure on  $A^W$  that encodes the **local statistics** of the sequence  $(\psi(1), \dots, \psi(n)) \in A^n$ .

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## Theorem

$$h(T, \phi) = \inf_{W \subset \mathbb{Z}} \inf_{\epsilon > 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \left\{ \psi : \{1, \dots, n\} \rightarrow A : d_W(\phi, \psi) < \epsilon \right\} \right|.$$

# Sofic Groups

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For  $W \subset G$ , let  $\mathcal{G}(W) \subset \{1, \dots, m\}$  be the set of all  $p$  such that

$$\begin{aligned}\sigma(fg)p &= \sigma(f)\sigma(g)p \quad \forall f, g \in W \text{ with } fg \in W, \\ \sigma(f)p &\neq \sigma(g)p \iff f \neq g \in W.\end{aligned}$$

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$\sigma$  is a  $(W, \epsilon)$ -*approximation* to  $G$  if  $|\mathcal{G}(W)| \geq (1 - \epsilon)m$ .

# Sofic Groups

A sequence  $\Sigma = \{\sigma_i\}_{i=1}^{\infty}$  of maps  $\sigma_i : G \rightarrow \text{Sym}(m_i)$  is a *sofic approximation* if  $\sigma_i$  is an  $(W_i, \epsilon_i)$ -approximation with  $\epsilon_i \rightarrow 0$  and  $W_i \rightarrow G$  (i.e.,  $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} W_i = G$ ).

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**The idea:** Count the number of observables  $\psi : \{1, \dots, m_i\} \rightarrow A$  so that  $(G, [m_i], u_i, \psi)$  **approximates**  $(G, X, \mu, \phi)$ .

# Approximating

If  $W \subset G$  is finite, let  $\phi^W: X \rightarrow A^W$  be the map  $\phi^W(x) := \left( \phi(wx) \right)_{w \in W}$ .

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Let  $d_W(\phi, \psi)$  be the  $l^1$ -distance between  $\phi_*^W \mu$  and  $\psi_*^W u$ .

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$$h(\Sigma, \phi) := \inf_{W \subset G} \inf_{\epsilon > 0} \limsup_{i \rightarrow \infty} \frac{\log |\{\psi : \{1, \dots, m_i\} \rightarrow A : d_W(\phi, \psi) \leq \epsilon\}|}{m_i}.$$

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Two observables  $\phi : X \rightarrow A$ ,  $\psi : X \rightarrow B$  are **equivalent** if the partitions  $\{\phi^{-1}(a) : a \in A\}$ ,  $\{\psi^{-1}(b) : b \in B\}$  agree up to measure zero.

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$\phi$  is a *simple splitting* of  $\psi$  if there exists  $f \in G$  and an observable  $\omega$  refined by  $\psi$  such that

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## Proposition

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## Applications: von Neumann algebras

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### Theorem (Connes, 1976)

*If  $G$  is infinite and amenable and the action  $G \curvearrowright (X, \mu)$  is free and ergodic then  $L^\infty(X, \mu) \rtimes G$  is hyperfinite. In particular, all such algebras are isomorphic.*

# Rigidity

## Definition

$(G_1, X_1, \mu_1)$  and  $(G_2, X_2, \mu_2)$  are *von Neumann equivalent* (vNE) if  $L^\infty(X_1, \mu_1) \rtimes G_1 \cong L^\infty(X_2, \mu_2) \rtimes G_2$ .

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## Corollary

*If, in addition,  $G$  is sofic and Ornstein then Bernoulli shifts over  $G$  are classified up to vNE by base measure entropy. E.g., this occurs when  $G = \mathrm{PSL}_n(\mathbb{Z})$  for  $n > 2$ .*

# Applications: orbit equivalence

## Definition

$(G_1, X_1, \mu_1)$  is **orbit equivalent** (OE) to  $(G_2, X_2, \mu_2)$  if there exists a measure-space isomorphism  $\phi : X_1 \rightarrow X_2$  such that  $\phi(G_1 x) = G_2 \phi(x)$  for a.e.  $x \in X_1$ .

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## Theorem (Dye 1959, Connes-Feldman-Weiss 1981)

*If  $G_1$  and  $G_2$  are amenable and infinite and their respective actions are ergodic and free then  $(G_1, X_1, \mu_1)$  is OE to  $(G_2, X_2, \mu_2)$ .*

## OE rigidity

### Theorem (Kida, 2008)

*Let  $G$  be the mapping class group of a genus  $g$  surface with  $n$  holes. Assume  $3g + n - 4 > 0$  and  $(g, n) \notin \{(1, 2), (2, 0)\}$ . If  $(G, X, \mu)$  is free and ergodic then it is strongly orbitally rigid. I.e., if  $(G_2, X_2, \mu_2)$  is free, ergodic and OE to  $(G, X, \mu)$  then it is isomorphic to  $(G, X, \mu)$ .*

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Let  $\mathbb{F} = \langle s_1, \dots, s_r \rangle$ . Let  $\mathbb{F}$  act on  $(X, \mu)$ .

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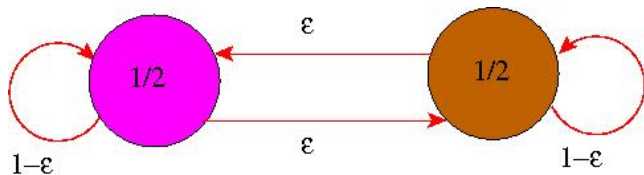
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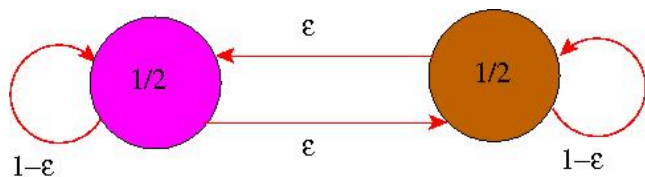
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$$h_*(\phi) = f(\phi).$$

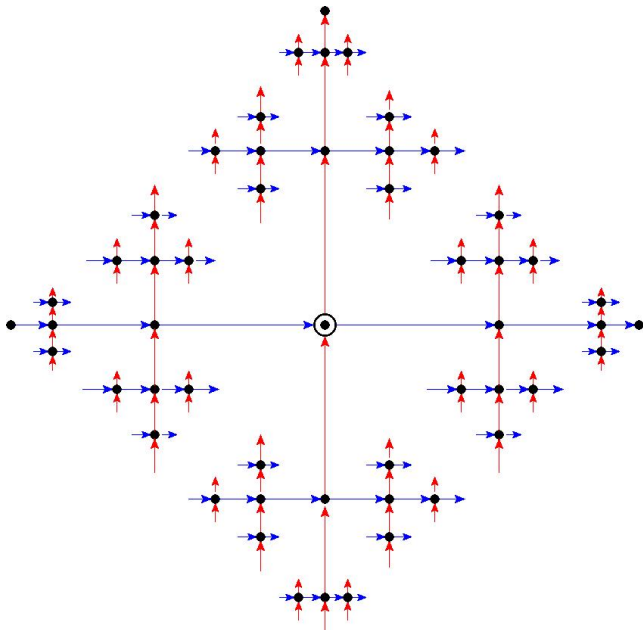
# A Markov chain example



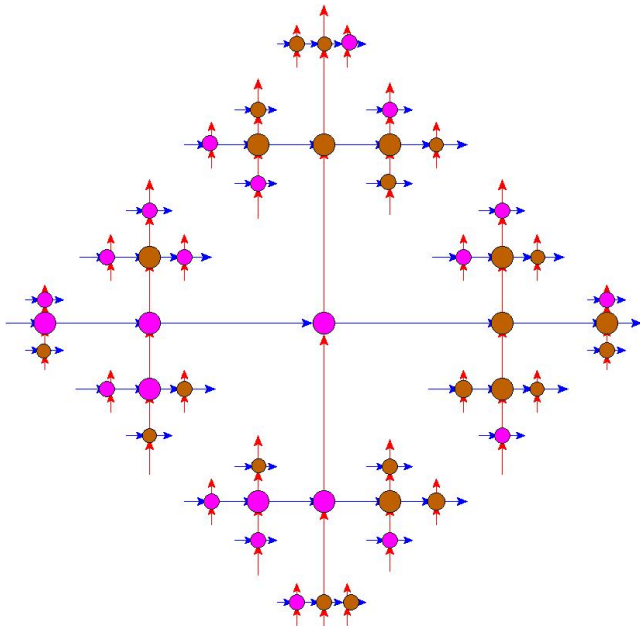
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# The Cayley graph



# The Ising model



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## Systems of algebraic origin

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### Theorem

*If  $\mathcal{G}$  is totally disconnected and  $\mathbb{F}$  acts by automorphisms on  $\mathcal{G}$  with closed normal subgroup  $\mathcal{N}$  then*

$$f(\mathbb{F}, \mathcal{G}, \text{Haar}_{\mathcal{G}}) = f(\mathbb{F}, \mathcal{N}, \text{Haar}_{\mathcal{N}}) + f(\mathbb{F}, \mathcal{G}/\mathcal{N}, \text{Haar}_{\mathcal{G}/\mathcal{N}}).$$

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- Extend the  $f$ -invariant to more general groups.