Entropy in Measurable Dynamics

Lewis Bowen

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Notation

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The triple \((G, X, \mu)\) is a \textit{dynamical system}. 

Main Problem: Classify systems up to isomorphism.
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Two systems \((G, X_1, \mu_1)\) and \((G, X_2, \mu_2)\) are \textit{isomorphic} if there exists a measure-space isomorphism \(\phi : X_1 \to X_2\) with \(\phi(gx) = g\phi(x)\) for a.e. \(x \in X_1\) and for all \(g \in G\).

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Bernoulli shifts

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- $(G, K^G, \kappa^G)$ is the Bernoulli shift over $G$ with base space $(K, \kappa)$. 
von Neumann’s question

If \(|K| = n\) and \(\kappa\) is the uniform probability measure on \(K\), then \((G, K^G, \kappa^G)\) is the full \(n\)-shift over \(G\).

von Neumann’s question: Is the full 2-shift over \(\mathbb{Z}\) isomorphic to the full 3-shift over \(\mathbb{Z}\)?
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Ideas from Information Theory

Let \( x \in X \) be a point unknown to us. Let \( E \subset X \).

\[ I(E) = I(\mu(E)) \]

\( I(t) \) for \( 0 \leq t \leq 1 \) should satisfy:

1. \( I(t) \geq 0 \)
2. \( I(t) \) is continuous.
3. \( I(ts) = I(t) + I(s) \).

So \( I(t) = -\log_b(t) \) for some \( b > 1 \).
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Let \( x \in X \) be a point unknown to us. Let \( E \subset X \).

**Goal:** quantify the “amount of information” we gain by being told that \( x \in E \).
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Entropy

An *observable* is a measurable map $\phi : X \rightarrow A$ into a finite (or countable) set $A$. 

The **Shannon entropy** of $\phi$ is the average amount of information one gains by learning the value of $\phi$. I.e.,

$$H(\phi) = -\sum_{a \in A} \mu(\phi^{-1}(a)) \log(\mu(\phi^{-1}(a))).$$

If $\phi : X \rightarrow A$ and $\psi : X \rightarrow B$ are two observables then their join is defined by

$$\phi \lor \psi(x) := (\phi(x), \psi(x)) \in A \times B.$$

Let $T : X \rightarrow X$ be measure-preserving. The **entropy rate** of $\phi$ w.r.t. $T$ is:

$$h(T, \phi) = \lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} H(n \lor \bigcup_{i=-n}^{n-1} \phi \circ T^{-i}).$$
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\(\phi\) is a generator if \(\Phi\) is an isomorphism from \((G, X, \mu)\) to \((G, A^G, \Phi^*\mu)\).
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Kolmogorov’s entropy

Theorem (Kolmogorov, 1958)

Let $T : X \to X$ be an automorphism of $(X, \mu)$. If $\phi$ and $\psi$ are finite-entropy generators for $(\mathbb{Z}, X, \mu) = (\langle T \rangle, X, \mu)$ then $h(T, \phi) = h(T, \psi)$.

So $h((\mathbb{Z}, X, \mu) := h(T, \phi)$ is the entropy of the action.

Theorem (Sinai, 1959)

If $\phi$ is any finite-entropy observable then $h(T, \phi) \leq h((\mathbb{Z}, X, \mu)$. Hence we may define the entropy of $(\mathbb{Z}, X, \mu)$ to be $\sup \phi h(T, \phi)$. 
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Bernoulli shifts

For a probability space \((K, \kappa)\), define the *base entropy* by

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H(K, \kappa) := - \sum_{k \in K} \kappa(k) \log \left( \kappa(k) \right).
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A calculation reveals:

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h(\mathbb{Z}, K^\mathbb{Z}, \kappa^\mathbb{Z}) = H(K, \kappa).
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**Theorem (Kolmogorov, 1958)**

If \((\mathbb{Z}, K^\mathbb{Z}, \kappa^\mathbb{Z})\) is isomorphic to \((\mathbb{Z}, L^\mathbb{Z}, \lambda^\mathbb{Z})\) then \(H(K, \kappa) = H(L, \lambda)\). So the full 2-shift is not isomorphic to the full 3-shift.
Questions

- Does the converse hold?

- What if $\mathbb{Z}$ is replaced with some other group $G$?
The Converse

A group \(G\) is Ornstein if whenever \((K, \kappa)\) and \((L, \lambda)\) are two standard probability spaces with \(H(\kappa) = H(\lambda)\) then \((G, K, G, \kappa, G)\) is isomorphic to \((G, L, G, \lambda, G)\). No finite group is Ornstein [Ornstein, 1970]. Infinite amenable groups are Ornstein [Ornstein-Weiss, 1987]. If \(G\) contains an Ornstein subgroup \(H\) then \(G\) is Ornstein [Stepin, 1975]. Is every countably infinite group Ornstein?
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Definition

A group $G$ is **Ornstein** if whenever $(K, \kappa)$, $(L, \lambda)$ are two standard probability spaces with $H(\kappa) = H(\lambda)$ then $(G, K^G, \kappa^G)$ is isomorphic to $(G, L^G, \lambda^G)$.

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- Is every countably infinite group Ornstein?
Classification

Theorem (Ornstein, 1970)

*Bernoulli shifts over $\mathbb{Z}$ are completely classified by their entropy.*
Classification


Theorem

If $G$ is infinite and amenable then Bernoulli shifts over $G$ are completely classified by their entropy (which equals their base measure entropy).

What if $G$ is nonamenable?
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What if G is nonamenable?
## Factor maps

### Definition

Let \((G, X, \mu), (G, Y, \nu)\) be two systems and \(\phi : X \to Y\) a measurable map with \(\phi_* \mu = \nu\), \(\phi(gx) = g\phi(x)\) for a.e. \(x \in X\) and all \(g \in G\). Then \(\phi\) is a **factor map** from \((G, X, \mu)\) to \((G, Y, \nu)\).
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- The full \(n\)-shift over \(G\) has entropy \(\log(n)\).

\(\implies\) the full 2-shift over \(G\) cannot factor onto the full 4-shift over \(G\).
The Ornstein-Weiss Example

Theorem (Ornstein-Weiss, 1987)

If $F = \langle a, b \rangle$ is the rank 2 free group then the full 2-shift over $F$ factors onto the full 4-shift over $F$. 

Define $\phi : (\mathbb{Z}/2\mathbb{Z})^F \to (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^F$ by $\phi(x)(g) = (x(g) + x(ga), x(g) + x(gb))$. 

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Theorem (Karen Ball, 2005)

If $G$ has infinitely many ends then the 2-shift over $G$ factors onto every Bernoulli shift over $G$. 
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If $G$ has infinitely many ends then the $2$-shift over $G$ factors onto every Bernoulli shift over $G$.

If $G$ is any nonamenable group then there is some $m > 0$ such that the $2^m$-shift over $G$ factors onto every Bernoulli shift over $G$. 

More Counterexamples

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If $G$ is any nonamenable group then there is some $m > 0$ such that the $2^m$-shift over $G$ factors onto every Bernoulli shift over $G$.

Theorem

If $G$ contains a nonabelian free subgroup then every nontrivial Bernoulli shift over $G$ factors onto every other Bernoulli shift over $G$. 
New Results

**Theorem**

If $G$ is a sofic group (e.g., a linear group) then Kolmogorov’s direction holds. I.e., if $(G, K^G, \kappa^G)$ is isomorphic to $(G, L^G, \lambda^G)$ then $H(K, \kappa) = H(L, \lambda)$. 

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The idea: For $n > 0$, count the number of sequences $(a_1, a_2, \ldots, a_n)$ with elements $a_i \in A$ that approximate the above sequence.
Local statistics

Let $W \subset \mathbb{Z}$ be finite. ($W$ stands for window)
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$$\phi^W(x) := (\phi(T^w x))_{w \in W}.$$
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\( \phi_*^W \mu \) is a measure on \( A^W \) that encodes the local statistics.
Sequences

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Let $u$ be the uniform measure on $\{1, \ldots, n\}$. $\psi^W_* u$ is a measure on $A^W$ that encodes the local statistics of the sequence $(\psi(1), \ldots, \psi(n)) \in A^n$. 
Let $d_W(\phi, \psi)$ be the $l^1$-distance between $\phi_*^W \mu$ and $\psi_*^W u$:
Entropy as a growth rate

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$$d_W(\phi, \psi) := \sum_{\alpha \in A^W} |\phi_*^W \mu(\alpha) - \psi_*^W u(\alpha)|.$$ 

**Theorem**

$$h(T, \phi) = \inf_{W \subset \mathbb{Z}} \inf_{\epsilon > 0} \lim_{n \to \infty} \frac{1}{n} \log \left| \left\{ \psi : \{1, \ldots, n\} \to A : d_W(\phi, \psi) < \epsilon \right\} \right|.$$
Sofic Groups

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For \( W \subset G \), let \( \mathcal{G}(W) \subset \{1, \ldots, m\} \) be the set of all \( p \) such that

\[
\sigma(fg)p = \sigma(f)\sigma(g)p \quad \forall f, g \in W \text{ with } fg \in W,
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\[
\sigma(f)p \neq \sigma(g)p \iff f \neq g \in W.
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\[ \sigma(f)p \neq \sigma(g)p \iff f \neq g \in W. \]

$\sigma$ is a $(W, \epsilon)$-approximation to $G$ if $|\mathcal{G}(W)| \geq (1 - \epsilon)m$. 
Sofic Groups

A sequence $\Sigma = \{\sigma_i\}_{i=1}^{\infty}$ of maps $\sigma_i : G \to \text{Sym}(m_i)$ is a sofic approximation if $\sigma_i$ is an $(W_i, \epsilon_i)$-approximation with $\epsilon_i \to 0$ and $W_i \to G$ (i.e., $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} W_i = G$).
Sofic Groups

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Sofic Groups

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- Is every countable group sofic?
Entropy for Sofic Groups

Let $(G, X, \mu)$ be a system,
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\[ \Sigma = \{\sigma_i\} \]

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\(\phi : X \to A\) be a measurable map into a finite set.

The idea: Count the number of observables \(\psi : \{1, \ldots, m_i\} \to A\) so that \((G, [m_i], u_i, \psi)\) approximates \((G, X, \mu, \phi)\).
Approximating

If \( W \subset G \) is finite, let \( \phi^W : X \rightarrow A^W \) be the map \( \phi^W(x) := \left( \phi(wx) \right)_{w \in W} \).
Approximating

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Let $d_W(\phi, \psi)$ be the $l^1$-distance between $\phi^W_* \mu$ and $\psi^W_* u$. 
Entropy for sofic groups

\[ h(\Sigma, \phi) := \inf_{W \subset G} \inf_{\epsilon > 0} \limsup_{i \to \infty} \log \left| \left\{ \psi : \{1, \ldots, m_i\} \to A : d_W(\phi, \psi) \leq \epsilon \right\} \right| / m_i. \]
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**Theorem**

If \( \phi_1 \) and \( \phi_2 \) are generating then \( h(\Sigma, \phi_1) = h(\Sigma, \phi_2) \). So let \( h(\Sigma, G, X, \mu) \) be this common number.
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If \( G \) is amenable then \( h(\Sigma, G, X, \mu) \) is the classical entropy of \( (G, X, \mu) \).
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**Theorem**

\[ h(\Sigma, G, K^G, \kappa^G) = H(K, \kappa) \]
Proof sketch

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Two observables \( \phi : X \to A, \psi : X \to B \) are equivalent if the partitions \( \{ \phi^{-1}(a) : a \in A \} \), \( \{ \psi^{-1}(b) : b \in B \} \) agree up to measure zero.
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**Definition (Rohlin distance)**

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d(\phi, \psi) := 2H(\phi \vee \psi) - H(\psi) - H(\phi) = H(\phi|\psi) + H(\psi|\phi).
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**Definition**

\( \phi \text{ refines } \psi \text{ if } H(\psi \lor \phi) = H(\phi). \)
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Definition

\( \phi \) refines \( \psi \) if \( H(\psi \lor \phi) = H(\phi) \).

Definition

\( \phi \) and \( \psi \) are combinatorially equivalent if there exists finite subsets \( K, L \subset G \) such that \( \phi^K \) refines \( \psi \) and \( \psi^L \) refines \( \phi \).
Proof sketch

**Theorem**

If $\phi$ is a generator then its combinatorial equivalence class is dense in the space of all generating observables.
Proof sketch

**Theorem**

\[ \text{If } \phi \text{ is a generator then its combinatorial equivalence class is dense in the space of all generating observables.} \]

**Lemma**

\[ h(\Sigma, \phi) \text{ is upper semi-continuous in } \phi. \]
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**Lemma**

$h(\Sigma, \phi)$ is upper semi-continuous in $\phi$.

**Theorem**

If $\phi$ and $\psi$ are combinatorially equivalent then $h(\Sigma, \phi) = h(\Sigma, \psi)$. 
Proof sketch

### Definition

φ is a **simple splitting** of ψ if there exists $f \in G$ and an observable $\omega$ refined by $\psi$ such that

$$\phi = \psi \lor \omega \circ f.$$  

φ is a **splitting** of ψ if it can be obtained from ψ by a sequence of simple splittings.
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Lemma

If $\phi$ and $\psi$ are equivalent then there exists an observable $\omega$ that is a splitting of both $\phi$ and $\psi$. 
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**Proposition**

*If \( \phi \) is a simple splitting of \( \psi \) then \( h(\Sigma, \phi) = h(\Sigma, \psi) \).*
Applications: von Neumann algebras

A system \((G, X, \mu)\) gives rise in a natural way to a \textit{crossed product von Neumann algebra} \(L^\infty(X, \mu) \rtimes G\).
Applications: von Neumann algebras

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If the action is ergodic and free and \(G\) is infinite then \(L^\infty(X, \mu) \rtimes G\) is a \(II_1\) factor.

Major problem: classify these algebras up to isomorphism in terms of the group/action data.
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Major problem: classify these algebras up to isomorphism in terms of the group/action data.

Theorem (Connes, 1976)

If \(G\) is infinite and amenable and the action \(G \curvearrowright (X, \mu)\) is free and ergodic then \(L^\infty(X, \mu) \rtimes G\) is hyperfinite. In particular, all such algebras are isomorphic.
Rigidity

Definition

\((G_1, X_1, \mu_1)\) and \((G_2, X_2, \mu_2)\) are von Neumann equivalent (vNE) if \(L^\infty(X_1, \mu_1) \rtimes G_1 \cong L^\infty(X_2, \mu_2) \rtimes G_2.\)
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**Theorem (Popa, 2006)**

*If* \(G\) *is an ICC property T group then any two von Neumann equivalent Bernoulli shifts over* \(G\) *are isomorphic.*
**Rigidity**

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**Theorem (Popa, 2006)**

If \(G\) is an ICC property \(T\) group then any two von Neumann equivalent Bernoulli shifts over \(G\) are isomorphic.

**Corollary**

If, in addition, \(G\) is sofic and Ornstein then Bernoulli shifts over \(G\) are classified up to vNE by base measure entropy. E.g., this occurs when \(G = PSL_n(\mathbb{Z})\) for \(n > 2\).
Applications: orbit equivalence

Definition

\((G_1, X_1, \mu_1)\) is orbit equivalent (OE) to \((G_2, X_2, \mu_2)\) if there exists a measure-space isomorphism \(\phi : X_1 \to X_2\) such that \(\phi(G_1 x) = G_2 \phi(x)\) for a.e. \(x \in X_1\).
Applications: orbit equivalence

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**Theorem (Dye 1959, Connes-Feldman-Weiss 1981)**

If $G_1$ and $G_2$ are amenable and infinite and their respective actions are ergodic and free then $(G_1, X_1, \mu_1)$ is OE to $(G_2, X_2, \mu_2)$.
OE rigidity

Theorem (Kida, 2008)

Let $G$ be the mapping class group of a genus $g$ surface with $n$ holes. Assume $3g + n - 4 > 0$ and $(g, n) \notin \{(1, 2), (2, 0)\}$. If $(G, X, \mu)$ is free and ergodic then it is strongly orbitally rigid. I.e., if $(G_2, X_2, \mu_2)$ is free, ergodic and OE to $(G, X, \mu)$ then it is isomorphic to $(G, X, \mu)$.

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If $G$ is as above then Bernoulli shifts over $G$ are classified up to OE by base measure entropy.
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Free Groups: a special case

Let $F = \langle s_1, \ldots, s_r \rangle$. Let $F$ act on $(X, \mu)$. 
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Let $\mathbb{F} = \langle s_1, \ldots, s_r \rangle$. Let $\mathbb{F}$ act on $(X, \mu)$.

Given an observable $\phi : X \to A$, define

$$F(\phi) := -(2r - 1)H(\phi) + \sum_{i=1}^{r} H(\phi \vee \phi \circ s_i);$$

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**Theorem**

*If $\phi_1$ and $\phi_2$ are generating then $f(\phi_1) = f(\phi_2)$. So we may define $f(\mathbb{F}, X, \mu) = f(\phi_1)$. Moreover, $f(\mathbb{F}, K^\mathbb{F}, \kappa^\mathbb{F}) = H(K, \kappa)$.***
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For each $n \geq 1$, let $\sigma_n : F = \langle s_1, \ldots, s_r \rangle \to \text{Sym}(n)$ be chosen uniformly at random.
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Define

$$h_*(\phi) := \inf_{W} \inf_{\epsilon > 0} \lim_{n \to \infty} \sup_n \log \mathbb{E} \left[ |\{ \psi : \{1, \ldots, n\} \to A : d_W(\phi, \psi) \leq \epsilon \}| \right].$$
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**Theorem**

$h_*(\phi) = f(\phi)$. 
A Markov chain example
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\begin{align*}
\text{1/2} & \quad \varepsilon \quad \text{1/2} \\
1 - \varepsilon & \quad \varepsilon \quad 1 - \varepsilon
\end{align*}

\begin{align*}
-2 & \quad -1 & \quad 0 & \quad 1 & \quad 2 & \quad 3
\end{align*}
The Cayley graph
The Ising model
Example

Let $\mu_\epsilon$ be the probability measure on \{magenta, brown\}$^F$ determined by this process.
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Then

$$F(\mu_\epsilon, \phi) = -2\epsilon \log(\epsilon) - 2(1 - \epsilon) \log(1 - \epsilon) - \log(2).$$
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Theorem

$$F(\mu_\epsilon, \phi) = h_*(F, \{\text{magenta, brown}\}^F, \mu_\epsilon).$$
Systems of algebraic origin
Let $G$ be a compact separable group and let $T : G \to G$ be a group automorphism fixing a closed normal subgroup $N$.

Theorem (Yuzvinskii, 1965)
$h(T, G, \text{Haar}_G) = h(T, N, \text{Haar}_N) + h(T, G/N, \text{Haar}_{G/N}).$

Theorem
If $G$ is totally disconnected and $F$ acts by automorphisms on $G$ with closed normal subgroup $N$ then
$f(F, G, \text{Haar}_G) = f(F, N, \text{Haar}_N) + f(F, G/N, \text{Haar}_{G/N}).$

Let $G = (\mathbb{Z}/2\mathbb{Z})^F$. Let $N = \{0, 1\}$. By Ornstein-Weiss' example, $G/N \sim G \times G = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^F$. 

\[ f(F, G, \text{Haar}_G) = f(F, N, \text{Haar}_N) + f(F, G/N, \text{Haar}_{G/N}) \log(2) = -\log(2) + \log(4) \log(2) = 2 \log(2). \]
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Let $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^\mathbb{F}$. Let $\mathcal{N} = \{0, 1\}$. By Ornstein-Weiss’ example,

$$\mathcal{G}/\mathcal{N} \cong \mathcal{G} \times \mathcal{G} = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^\mathbb{F}. $$
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Let $G = (\mathbb{Z}/2\mathbb{Z})^F$. Let $N = \{0, 1\}$. By Ornstein-Weiss’ example,

$$G/N \cong G \times G = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^F.$$

$$f(F, G, \text{Haar}_G) = f(F, N, \text{Haar}_N) + f(F, G/N, \text{Haar}_{G/N})$$
Systems of algebraic origin

Let \( \mathcal{G} \) be a compact separable group and let \( T : \mathcal{G} \to \mathcal{G} \) be a group automorphism fixing a closed normal subgroup \( \mathcal{N} \).

**Theorem (Yuzvinskii, 1965)**

\[
\begin{align*}
    h(T, \mathcal{G}, \text{Haar}_\mathcal{G}) &= h(T, \mathcal{N}, \text{Haar}_\mathcal{N}) + h(T, \mathcal{G}/\mathcal{N}, \text{Haar}_{\mathcal{G}/\mathcal{N}}).
\end{align*}
\]

**Theorem**

*If \( \mathcal{G} \) is totally disconnected and \( \mathbb{F} \) acts by automorphisms on \( \mathcal{G} \) with closed normal subgroup \( \mathcal{N} \) then*

\[
\begin{align*}
    f(\mathbb{F}, \mathcal{G}, \text{Haar}_\mathcal{G}) &= f(\mathbb{F}, \mathcal{N}, \text{Haar}_\mathcal{N}) + f(\mathbb{F}, \mathcal{G}/\mathcal{N}, \text{Haar}_{\mathcal{G}/\mathcal{N}}).
\end{align*}
\]

Let \( \mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^F \). Let \( \mathcal{N} = \{0, 1\} \). By Ornstein-Weiss’ example,

\[
\begin{align*}
    \mathcal{G}/\mathcal{N} &\cong \mathcal{G} \times \mathcal{G} = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^F.
\end{align*}
\]

\[
\begin{align*}
    f(\mathbb{F}, \mathcal{G}, \text{Haar}_\mathcal{G}) &= f(\mathbb{F}, \mathcal{N}, \text{Haar}_\mathcal{N}) + f(\mathbb{F}, \mathcal{G}/\mathcal{N}, \text{Haar}_{\mathcal{G}/\mathcal{N}}), \\
    \log(2) &= -\log(2) + \log(4).
\end{align*}
\]
Further Results & Open Questions

- Ornstein theory for free groups: factors of Bernoulli shifts, factors onto Bernoulli shifts, mixing Markov chains, etc.
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- Extend the $f$-invariant to more general groups.