On the classification of non-simple inductive limits of matrix algebras over certain one-dimensional spaces

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The Elliott conjecture

Conjecture (Elliott, 1990)

Let A and B be simple unital separable amenable C-algebras, and suppose that there exists an isomorphism* $\varphi : Ell(A) \to Ell(B)$*. Then there exists an isomorphism* $\phi : A \to B$ *inducing* φ*.*

- Simple, stably finite case:
  $Ell(A) = (K_0(A), K_0(A)^+, [1_A], K_1(A), T(A), r_A)$.

Definition

An approximately interval (AI) algebra is an inductive limit C*-algebra $A = \lim(A_i, \mu_i)$, where $A_i = \bigoplus_{k=1}^{N_i} M_{n_k}(C[0,1])$.

Theorem (Elliott, 1992)

Let A and B be two simple separable unital AI-algebras. Then A is isomorphic to B if and only if $(K_0(A), K_0(A)^+, [1_A], T(A), r_A)$ *is isomorphic to* $(K_0(B), K_0(B)^+, [1_B], T(B), r_B)$.

- Generalized by L.Li to simple inductive limits of building blocks of the form $A_i = \bigoplus_{k=1}^{N_i} M_{n_k}(C(X_k))$, with each X_k=tree.

Theorem (Toms)

There exists a simple, unital, AH -algebra such that for any UHF algebra U we have

1. *A and $A \otimes U$ agree on the Elliott invariant.*
2. *$A \not\cong A \otimes U$.*

Tensoring with U doesn't affect the K_0-group or the traces.
The main tool used to distinguish between these C*-algebras is the Cuntz semigroup.

Definition

1. For $a, b \in (A \otimes \mathbb{K})^+$, say that a is *Cuntz subequivalent* to b (written $a \lesssim_{Cu} b$) if there is a sequence $\{x_n\}$ in $A \otimes \mathbb{K}$ such that
$$\|x_n b x_n^* - a\| \to 0.$$

2. $a \sim_{Cu} b$ if $a \lesssim_{Cu} b$ and $b \lesssim_{Cu} a$. Denote by $[a]_{Cu}$ the Cuntz equivalence class.

3. $Cu(A) = (A \otimes \mathbb{K})^+ / \sim_{Cu}$, equipped with
addition $[a]_{Cu} + [b]_{Cu} = [a \oplus b]_{Cu}$
order relation $[a]_{Cu} \leq [b]_{Cu} \iff a \lesssim_{Cu} b$
becomes an ordered abelian semigroup with zero.

Examples

1. $Cu(\mathbb{K}) \cong \mathbb{N} \cup \{\infty\}$.
2. $Cu(C[0,1]) \cong \{f : [0,1] \to \mathbb{N} \cup \{\infty\} |\ f \text{ lower-semicont. } \}$.

It is a natural generalization of the Murray-von Neumann semigroup of projections $V(A)$. The map

$$V(A) \to Cu(A), [p]_{MvN} \mapsto [p]_{Cu}$$

is injective for the class of stably finite C*-algebras.

Problems to be considered

Problem

Is the augmented invariant $(Ell(A), Cu(A))$ a complete invariant? Compute $Cu(A)$ in terms of other C-invariants.*

Question

Can we use the Cuntz semigroup as an invariant in order to establish isomorphism, not just to distinguish C-algebras?*

Theorem (N. Brown, Toms)

Let A be a simple separable unital AH-algebra with slow dimension growth or a simple unital exact stably finite C-algebra absorbing the Jiang-Su algebra $\mathcal{Z}$. Then*

$$Cu(A) \cong V(A) \sqcup SAff(T(A)),$$

where $SAff(T(A))$ are functions on $T(A)$ which are pointwise suprema of increasing sequences of continuous, affine and strictly positive functions on $T(A)$.

Remark

Simple separable unital AI-algebras are classified by the Cuntz semigroup.

Proposition

Let A be a separable C-algebra. Then the ideals of A are in natural bijective correspondence with the hereditary subsemigroups of the Cuntz semigroup of A which are closed under the operation of taking suprema of increasing sequences.*

Problem

Can we classify non-simple AI algebras using Cuntz semigroup?

The Thomsen semigroup

Definition (Approximate unitary equivalence)

$a, b \in A^+$ are said to be approximately unitary equivalent ($a \sim_{au} b$) if there is a sequence of unitaries $\{u_n\}$ in $A^\sim$ such that

$$\|u_n a u_n^* - b\| \to 0.$$

Definition

$Th(A) = \{a \in (A \otimes \mathbb{K})^+ |\ \|a\| \leq 1\}/ \sim_{au}$, with addition $[a]_{au} + [b]_{au} = [a \oplus b]_{au}$ and metric $D_u([a]_{au}, [b]_{au}) = inf\{\|uau^* - b\|\,|\ u \text{ unitary}\}$ becomes a complete metric abelian semigroup with zero.

Theorem (Thomsen, 1992)

Consider $A = \lim(\bigoplus M_{n_i}(C[0,1]), \mu_i)$ *and* $B = \lim(\bigoplus M_{k_i}(C[0,1]), \nu_i)$ *two separable unital AI-algebras. Let* $\alpha\colon (Th(A), Th(A)^+) \to (Th(B), Th(B)^+)$ *be an isometric semigroup isomorphism. Then there is an isomorphism* $\varphi\colon A \to B$ *inducing* α*.*

Problem

Study the relation between $Th(A)$ *and* $Cu(A)$*.*

Remark

$C_0((0,1]) = C^*(x : x \geq 0, \|x\| \leq 1)$.
Given a C*-algebra A, any positive contraction $a \in A$ defines a *-homomorphism

$$\varphi_a \colon C_0((0,1]) \to A, \quad \varphi_a(id) = a.$$

Definition (Reinterpretation of $Th(A)$)

$Th(A) = \{[\varphi]_{au} \mid \; \varphi \colon C_0((0,1]) \to A \otimes \mathbb{K}\}$.

Relation between $Th(A)$ and $Cu(A)$

To each map $\varphi : C_0(0,1] \to A \otimes K$ there is an associated map $Cu(\varphi) : Cu(C_0(0,1]) \to Cu(A)$.
Moreover, $\varphi \sim_{au} \psi$ implies $Cu(\varphi) = Cu(\psi)$. Therefore we have a well-defined semigroup map
$\Gamma : Th(A) \to Hom(Cu(C_0(0,1]), Cu(A))$.

Theorem (Elliott, C.)

If A has stable rank one ($GL(A)$ dense in A) then the map Γ is an isomorphism of semigroups. $Cu(A)$ is determined by $Hom(Cu(C_0(0,1]), Cu(A))$ as the set of all images of $1_{(0,1]}$ under homomorphisms

$$Cu(A) = \{\alpha(1_{(0,1]}) |\ \alpha : Cu(C_0(0,1]) \to Cu(A)\}.$$

The pseudometric on $Hom(Cu(C_0(0,1]), Cu(A))$

Given Cuntz semigroup morphisms $\alpha, \beta \colon Cu(C_0(0,1]) \to Cu(A)$, define a pseudometric on $Hom(Cu(C_0(0,1]), Cu(A))$ by

$$d_W(\alpha,\beta) := \inf \left\{ r \in \mathbb{R}^+ \,\middle|\, \begin{array}{l} \alpha(1_{(t+r,1]}) \leq \beta(1_{(t,1]}), \\ \beta(1_{(t+r,1]}) \leq \alpha(1_{(t,1]}), \end{array} \text{ for all } t \in \mathbb{R}^+ \right\}, \tag{1}$$

Theorem (Elliott, C.)

If A has stable rank one the pseudometric d_W is a metric and

$$\frac{1}{8}D_u([a]_{au},[b]_{au}) \leq d_W(Cu(\varphi_a), Cu(\varphi_b) \leq D_u([a]_{au},[b]_{au}).$$

Remark

1. Proof of d_W being metric uses cancellation in $Cu(A)$ (Rørdam, Winter).
2. Constant $\frac{1}{8}$ recently improved to $\frac{1}{4}$ by Robert, Santiago.

Question

Is Γ an isometry?

Example

$a, b \in M_n$ positive contractions
$\sigma(a) = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$
$\sigma(b) = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n\}$
$D_u([a],[b]) = \max_i |\lambda_i - \mu_i|$

$$d_W(Cu(\varphi_a), Cu(\varphi_b)) = \inf\left\{ r \geq 0 \;\middle|\; \begin{array}{l} rank(a-t-r)_+ \leq rank(b-t)_+ \\ rank(b-t-r)_+ \leq rank(a-t)_+ \end{array} \right\}, \tag{2}$$

- $D_u([a],[b]) = d_W(Cu(\varphi_a), Cu(\varphi_b))$.
- generalizes to inductive limits

Classification of AI-algebras using the Cuntz semigroup

Theorem (Elliott, C.)

Let $A = \lim(\bigoplus M_{n_i}(C[0,1]), \mu_i)$ and $B = \lim(\bigoplus M_{k_i}(C[0,1]), \nu_i)$ be unital separable AI-algebras. Let $\alpha : Cu(A) \to Cu(B)$ be an isomorphism in the category **Cu**, *satisfying $\alpha[1_A] = [1_B]$. Then there exists an isomorphism $\varphi : A \to B$ giving rise to α.*

Remark

The isomorphism $\alpha : Cu(A) \to Cu(B)$ gives rise to an isometric isomorphism $Th(A) \to Th(B)$, hence to a C*-isomorphism $\varphi\colon A \to B$.

The case of the trees

Problem

Use the Cuntz semigroup to classify C-algebras more general than AI.*

Definition

A tree is a compact one-dimensional CW complex which is contractible.

Similar to the interval case, we have

1. A pseudometric D_u on $Hom(C(X,v),A)$. If A has stable rank one then $Hom(C(X,v),A)$ is complete with respect to D_u.
2. A pseudometric d_W on $Hom(Cu(C(X)),Cu(A))$.

Theorem

Let A be a C-algebra and let (X, v) be a rooted tree. Let $\phi, \psi \colon C(X, v) \to A$ be *-homomorphisms. Then*

$$\frac{1}{2N+2} D_u(\phi, \psi) \leq d(\mathrm{Cu}(\phi), \mathrm{Cu}(\psi)) \leq D_u(\phi, \psi),$$

where $N \in \mathbb{N}$ is a constant depending on the tree X.

Approximate Existence Theorem

Theorem (Elliott, Santiago, C.)

Let A be a unital C-algebra and let X be a tree. Let $\alpha\colon \mathrm{Cu}(C(X)) \to \mathrm{Cu}(A)$ be a morphism in the category* **Cu** *such that $\alpha[1_X] \leq [1_A]$. Then for every $\epsilon > 0$ there is a *-homomorphisms $\varphi\colon C(X) \to A$ such that $d(\alpha, \mathrm{Cu}(\varphi)) < \epsilon$. Moreover, if A has stable rank one, we can lift α exactly, i.e. there is $\varphi\colon C(X) \to A$ such that $\alpha = Cu(\varphi)$.*

Classification

Theorem

Let $A = \lim(\bigoplus M_{n_{k_i}}(C(X_{k_i}), \mu_i)$ *and* $B = \lim(\bigoplus M_{n_{m_i}}(C(X_{m_i}), \nu_i)$ *be unital C*-algebras, where all the spaces* X_j *are trees. Let* $\alpha\colon Cu(A) \to Cu(B)$ *be an isomorphism of semigroups such that* $\alpha[1_A] = [1_B]$*. Then there is an isomorphism of C*-algebras* $\varphi\colon A \to B$ *such that* $Cu(\varphi) = \alpha$*.*