On the classification of non-simple inductive limits of matrix algebras over certain one-dimensional spaces

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The Elliott conjecture

Conjecture (Elliott, 1990)

Let $A$ and $B$ be simple unital separable amenable $C^*$-algebras, and suppose that there exists an isomorphism $\varphi : \Ell(A) \to \Ell(B)$. Then there exists an isomorphism $\phi : A \to B$ inducing $\varphi$.

- Simple, stably finite case:
  
  $\Ell(A) = (K_0(A), K_0(A)^+, [1_A], K_1(A), T(A), r_A)$. 
Definition

An approximately interval (AI) algebra is an inductive limit C*-algebra $A = \lim(A_i, \mu_i)$, where $A_i = \bigoplus_{k=1}^{N_i} M_{n_k}(C[0, 1])$.

Theorem (Elliott, 1992)

Let $A$ and $B$ be two simple separable unital AI-algebras. Then $A$ is isomorphic to $B$ if and only if $(K_0(A), K_0(A)^+, [1_A], T(A), r_A)$ is isomorphic to $(K_0(B), K_0(B)^+, [1_B], T(B), r_B)$.

Generalized by L.Li to simple inductive limits of building blocks of the form $A_i = \bigoplus_{k=1}^{N_i} M_{n_k}(C(X_k))$, with each $X_k = \text{tree}$. 

Theorem (Toms)

There exists a simple, unital, AH-algebra such that for any UHF algebra $U$ we have

1. $A$ and $A \otimes U$ agree on the Elliott invariant.
2. $A \not\cong A \otimes U$.

Tensoring with $U$ doesn’t affect the $K_0$-group or the traces. The main tool used to distinguish between these C*-algebras is the Cuntz semigroup.
Definition

1. For $a, b \in (A \otimes \mathbb{K})^+$, say that $a$ is Cuntz subequivalent to $b$ (written $a \lesssim_{Cu} b$) if there is a sequence $\{x_n\}$ in $A \otimes \mathbb{K}$ such that
   \[ \|x_n b x_n^* - a\| \to 0. \]

2. $a \sim_{Cu} b$ if $a \lesssim_{Cu} b$ and $b \lesssim_{Cu} a$. Denote by $[a]_{Cu}$ the Cuntz equivalence class.

3. $Cu(A) = (A \otimes \mathbb{K})^+/\sim_{Cu}$, equipped with addition $[a]_{Cu} + [b]_{Cu} = [a \oplus b]_{Cu}$
   order relation $[a]_{Cu} \leq [b]_{Cu} \iff a \lesssim_{Cu} b$
   becomes an ordered abelian semigroup with zero.
Examples

1. $Cu(\mathbb{K}) \cong \mathbb{N} \cup \{\infty\}$.
2. $Cu(C[0, 1]) \cong \{f : [0, 1] \to \mathbb{N} \cup \{\infty\} | f \text{ lower-semicont.}\}$.

It is a natural generalization of the Murray-von Neumann semigroup of projections $V(A)$. The map

$$V(A) \to Cu(A), [p]_{MVN} \mapsto [p]_{Cu}$$

is injective for the class of stably finite $C^*$-algebras.
Problems to be considered

Problem
Is the augmented invariant \((\text{Ell}(A), \text{Cu}(A))\) a complete invariant? Compute \(\text{Cu}(A)\) in terms of other \(C^*\)-invariants.

Question
Can we use the Cuntz semigroup as an invariant in order to establish isomorphism, not just to distinguish \(C^*\)-algebras?
Theorem (N. Brown, Toms)

Let $A$ be a simple separable unital AH-algebra with slow dimension growth or a simple unital exact stably finite C*-algebra absorbing the Jiang-Su algebra $\mathcal{Z}$. Then

$$\text{Cu}(A) \cong V(A) \sqcup \text{SAff}(T(A)),$$

where $\text{SAff}(T(A))$ are functions on $T(A)$ which are pointwise suprema of increasing sequences of continuous, affine and strictly positive functions on $T(A)$.

Remark
Simple separable unital AI-algebras are classified by the Cuntz semigroup.
Proposition

Let $A$ be a separable $C^*$-algebra. Then the ideals of $A$ are in natural bijective correspondence with the hereditary subsemigroups of the Cuntz semigroup of $A$ which are closed under the operation of taking suprema of increasing sequences.

Problem

Can we classify non-simple AI algebras using Cuntz semigroup?
Introduction

The Cuntz semigroup

Classification of non-simple AI-algebras

The Thomsen semigroup

Definition (Approximate unitary equivalence)

\(a, b \in A^+\) are said to be approximately unitary equivalent \(\sim_{au}\) if there is a sequence of unitaries \(\{u_n\}\) in \(A^\sim\) such that

\[
\|ua_nu_n^* - b\| \to 0.
\]

Definition

\(Th(A) = \{a \in (A \otimes \mathbb{K})^+ | \|a\| \leq 1\}/\sim_{au}\), with addition

\([a]_{au} + [b]_{au} = [a \oplus b]_{au}\) and metric

\(D_u([a]_{au}, [b]_{au}) = \inf\{\|uau^* - b\| \mid u \text{ unitary}\}\) becomes a complete metric abelian semigroup with zero.
Theorem (Thomsen, 1992)

Consider $A = \lim(\bigoplus M_{n_i}(C[0, 1]), \mu_i)$ and $B = \lim(\bigoplus M_{k_i}(C[0, 1]), \nu_i)$ two separable unital AI-algebras. Let $\alpha: (\text{Th}(A), \text{Th}(A)^+) \rightarrow (\text{Th}(B), \text{Th}(B)^+)$ be an isometric semigroup isomorphism. Then there is an isomorphism $\varphi: A \rightarrow B$ inducing $\alpha$.

Problem

Study the relation between $\text{Th}(A)$ and $\text{Cu}(A)$. 
Remark

$C_0((0, 1]) = C^*(x : x \geq 0, \|x\| \leq 1)$.

Given a C*-algebra $A$, any positive contraction $a \in A$ defines a *-homomorphism

$$\varphi_a : C_0((0, 1]) \to A, \quad \varphi_a(id) = a.$$ 

Definition (Reinterpretation of $Th(A)$)

$Th(A) = \{[\varphi]_{au} | \varphi : C_0((0, 1]) \to A \otimes \mathbb{K}\}.$
Relation between $Th(A)$ and $Cu(A)$

To each map $\varphi : C_0(0, 1] \rightarrow A \otimes K$ there is an associated map $Cu(\varphi) : Cu(C_0(0, 1]) \rightarrow Cu(A)$. Moreover, $\varphi \sim_{au} \psi$ implies $Cu(\varphi) = Cu(\psi)$. Therefore we have a well-defined semigroup map $\Gamma : Th(A) \rightarrow Hom(Cu(C_0(0, 1]), Cu(A))$.

**Theorem (Elliott, C.)**

*If $A$ has stable rank one ($GL(A)$ dense in $A$) then the map $\Gamma$ is an isomorphism of semigroups. $Cu(A)$ is determined by $Hom(Cu(C_0(0, 1]), Cu(A))$ as the set of all images of $1_{(0,1]}$ under homomorphisms*

\[ Cu(A) = \{ \alpha(1_{(0,1]}) | \alpha : Cu(C_0(0, 1]) \rightarrow Cu(A) \}. \]
The pseudometric on $\text{Hom}(\text{Cu}(C_0(0, 1]), \text{Cu}(A))$

Given Cuntz semigroup morphisms $\alpha, \beta : \text{Cu}(C_0(0, 1]) \to \text{Cu}(A)$, define a pseudometric on $\text{Hom}(\text{Cu}(C_0(0, 1]), \text{Cu}(A))$ by

$$d_W(\alpha, \beta) := \inf \left\{ r \in \mathbb{R}^+ \left| \begin{array}{l}
\alpha(1_{(t+r,1]}) \leq \beta(1_{(t,1]}), \\
\beta(1_{(t+r,1]}) \leq \alpha(1_{(t,1]}),
\end{array} \right. \text{ for all } t \in \mathbb{R}^+ \right\},$$

(1)
Theorem (Elliott, C.)

If $A$ has stable rank one the pseudometric $d_W$ is a metric and

$$\frac{1}{8} D_u([a]_{au}, [b]_{au}) \leq d_W(Cu(\varphi_a), Cu(\varphi_b)) \leq D_u([a]_{au}, [b]_{au}).$$

Remark

1. Proof of $d_W$ being metric uses cancellation in $Cu(A)$ (Rørdam, Winter).
2. Constant $\frac{1}{8}$ recently improved to $\frac{1}{4}$ by Robert, Santiago.

Question

Is $\Gamma$ an isometry?
Example

$a, b \in M_n \text{ positive contractions}$

$\sigma(a) = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$

$\sigma(b) = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n\}$

$D_u([a], [b]) = \max_i |\lambda_i - \mu_i|$

\[d_W(Cu(\varphi_a), Cu(\varphi_b)) = \inf \left\{ r \geq 0 \left| \begin{array}{c}
\text{rank}(a - t - r)_+ \leq \text{rank}(b - t)_+ \\
\text{rank}(b - t - r)_+ \leq \text{rank}(a - t)_+
\end{array} \right. \right\}, \tag{2}\]

- $D_u([a], [b]) = d_W(Cu(\varphi_a), Cu(\varphi_b))$.
- generalizes to inductive limits
Classification of AI-algebras using the Cuntz semigroup

Theorem (Elliott, C.)

Let $A = \lim(\bigoplus M_{n_i}(C[0,1]), \mu_i)$ and $B = \lim(\bigoplus M_{k_i}(C[0,1]), \nu_i)$ be unital separable AI-algebras. Let $\alpha : Cu(A) \rightarrow Cu(B)$ be an isomorphism in the category $\text{Cu}$, satisfying $\alpha[1_A] = [1_B]$. Then there exists an isomorphism $\varphi : A \rightarrow B$ giving rise to $\alpha$.

Remark

The isomorphism $\alpha : Cu(A) \rightarrow Cu(B)$ gives rise to an isometric isomorphism $Th(A) \rightarrow Th(B)$, hence to a $C^*$-isomorphism $\varphi : A \rightarrow B$. 
The case of the trees

Problem
Use the Cuntz semigroup to classify C*-algebras more general than AI.

Definition
A tree is a compact one-dimensional CW complex which is contractible.

Similar to the interval case, we have

1. A pseudometric $D_u$ on $\text{Hom}(C(X, \nu), A)$. If $A$ has stable rank one then $\text{Hom}(C(X, \nu), A)$ is complete with respect to $D_u$.
2. A pseudometric $d_W$ on $\text{Hom}(Cu(C(X))), Cu(A))$. 
Theorem

Let $A$ be a $C^*$-algebra and let $(X, v)$ be a rooted tree. Let $\phi, \psi : C(X, v) \to A$ be $*$-homomorphisms. Then

$$\frac{1}{2N + 2} D_u(\phi, \psi) \leq d(\text{Cu}(\phi), \text{Cu}(\psi)) \leq D_u(\phi, \psi),$$

where $N \in \mathbb{N}$ is a constant depending on the tree $X$. 

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Classification of non-simple $C^*$-algebras
Theorem (Elliott, Santiago, C.)

Let $A$ be a unital $C^*$-algebra and let $X$ be a tree. Let
\[ \alpha : \text{Cu}(C(X)) \to \text{Cu}(A) \]
be a morphism in the category $\text{Cu}$ such that $\alpha[1_X] \leq [1_A]$. Then for every $\epsilon > 0$ there is a
*-homomorphisms $\varphi : C(X) \to A$ such that $d(\alpha, \text{Cu}(\varphi)) < \epsilon$.
Moreover, if $A$ has stable rank one, we can lift $\alpha$ exactly, i.e. there is $\varphi : C(X) \to A$ such that $\alpha = \text{Cu}(\varphi)$. 
Classification

Theorem
Let \( A = \lim(\bigoplus M_{n_{k_i}}(C(X_{k_i}), \mu_i)) \) and \( B = \lim(\bigoplus M_{n_{m_i}}(C(X_{m_i}), \nu_i)) \) be unital C*-algebras, where all the spaces \( X_j \) are trees. Let \( \alpha: Cu(A) \to Cu(B) \) be an isomorphism of semigroups such that \( \alpha[1_A] = [1_B] \). Then there is an isomorphism of C*-algebras \( \varphi: A \to B \) such that \( Cu(\varphi) = \alpha \).