Mixing subalgebras of finite von Neumann algebras

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Weak mixing

Theorem (Koopman-von Neumann 1932, etc)

An invertible measure-preserving transformation T *on a probability measure space* $(X, \mathcal{B}, \mu)$ *is weakly mixing if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left|\mu(A \cap T^{-n}B) - \mu(A)\mu(B)\right| = 0, \quad \forall A, B \in \mathcal{B}.$$

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- *for all $\epsilon > 0$, $f_1, \cdots, f_n \in L^2(X, \mu)$ with $\int_X f_i(x)\, d\mu(x) = 0$, there exists an n such that $|\langle U_T^n f_i, f_j\rangle| < \epsilon$.*

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- *If $f \in L^2(X, \mu)$ and $U_T f = \lambda f$ a.e., then $f = const$ a.e.*

Weakly almost periodic functions

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Theorem (Ryll-Nardzewski)

There is a unique G-invariant mean on the set of weakly almost periodic functions.

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Theorem (Bergelson and Rosenblatt,1988)

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$$M\left(|\langle U_g f_1, f_2\rangle|\right) = 0$$

for $f_1, f_2 \in L^2(X, \mu)$ *with* $\int_X f_1(x)\, d\mu(x) = \int_X f_2(x)\, d\mu(x) = 0$.

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- $\{\sigma_g\}$ *has no finite-dimensional subrepresentations other than the trivial one.*

Finite von Neumann algebras

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- Example: $L(G)$ is the von Neumann algebra generated by $\{L_g\}$. $L_g\delta_h = \delta_{gh}$, $\tau(x) = \langle x\delta_e, \delta_e \rangle$.

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- $M_2(\mathbb{C}) \cong \{-1, 1\} \rtimes \mathbb{Z}_2$: $g(1) = -1$ and $g(-1) = 1$.

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- Suppose $M = L^\infty(X) \rtimes G$ and $B = L(G) = \{u_g \otimes l_g | g \in G\}''$:

$$\mathbb{E}_B\left(\sum_{g\in G} f_g g\right) = \sum_{g\in G} \left(\int f_g(x)\, d\mu(x)\right) g.$$

Weakly mixing von Neumann subalgebras

Definition (Jolissaint-Stalder 2009, Robertson-Sinclair-Smith 2003)

We call B a weakly mixing von Neumann subalgebra of M if there exists a sequence of unitary operators $\{u_n\}$ in B such that one of the following equivalent conditions holds.

1. $\lim_{n\to\infty} \|\mathbb{E}_B(xu_ny^*) - \mathbb{E}_B(x)u_n\mathbb{E}_B(y^*)\|_2 = 0, \quad \forall x, y \in M.$

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- $B = \{a\}''$ is weakly mixing in $L(F_2) = \{a, b\}''$: $x_i \in F_2 \setminus \{a^n | n \in \mathbb{Z}\}$.

Theorem (Jolissaint-Stalder, 2009)

The action $G \curvearrowright_{\sigma_g} (X, \mathcal{B}, \mu)$ *is weakly mixing if and only if* $L(G)$ *is weakly mixing in* $L^\infty(X, \mu) \rtimes G$.

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Theorem (Sinclair-Smith-White-Wiggins, 2007)

A masa A of a type II_1 factor M is weakly mixing if and only if A is a singular von Neumann subalgebra of M, i.e., if $u \in M$ is a unitary operator such that $uAu^* = A$ then $u \in A$.

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Corollary

If σ is a measure preserving action of a countable abelian group Γ_0 on a probability space $(X, \mathcal{B}, \mu)$, then the action is weakly mixing if and only if the von Neumann subalgebra $L(\Gamma_0)$ is singular in the cross-product finite von Neumann algebra $L^\infty(X, \mu) \rtimes \Gamma_0$.

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- $\mathcal{N}(B)'' \subset q\mathcal{N}(B)''$.
- **Question:** A von Neumann subalgebra B of a finite von Neumann algebra M is weakly mixing if and only if $q\mathcal{N}(B)'' = B$?

One-sided quasi-normalizers

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- Suppose $H \subset G$ and for $g \in G \setminus H$, the set $\{hgh^{-1} | h \in H\}$ is infinite. Let $\Gamma = \{g | Hg \subset \cup_{i=1}^n g_i H\}$. Let $H_1 = \Gamma \cap \Gamma^{-1}$, and let H_2 be the subgroup of G generated by Γ. Then $q\mathcal{N}(L(H))'' = L(H_1)$, $W^*(q\mathcal{N}^{(1)}(L(H))) = L(H_2)$.

One-sided quasi-normalizer algebras

Corollary

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Let $B_i \subset M_i$ be inclusions of finite von Neumann algebras, $i = 1, 2$. Then $W^* \left(q\mathcal{N}^{(1)} (B_1 \bar{\otimes} B_2)\right) = W^*(q\mathcal{N}^{(1)}(B_1)) \bar{\otimes} W^*(q\mathcal{N}^{(1)}(B_2))$.

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Theorem (Fang-Gao)

Let $e \in B$ be a projection. Then

$$W^*(q\mathcal{N}^{(1)}_{eMe}(eBe)) = eW^*(q\mathcal{N}^{(1)}(B))e.$$

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- $\sum_{1\leq i,j\leq n} \|\mathbb{E}_B(x_i u x_j^*)\|_2^2 = \mathrm{Tr}(zuzu^*)$, where $z = \sum_{i=1}^n x_i e_B x_i^*$.

Sketch of the proof

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3. if $x \in M$ satisfies $Bx \subset \sum_i x_i B$ for finite elements $x_1, \cdots, x_n \in M$, then $x \in B$, i.e., $W^*(q\mathcal{N}^{(1)}(B)) = B$.

Mixing von Neumann subalgebras

Definition

$G \curvearrowright_{\sigma_g} (X, \mathcal{B}, \mu)$ is mixing if the map $g \to \langle U_g f_1, f_2 \rangle \in C_0(G)$ for all $f_1, f_2 \in L^2(X, \mu) \ominus \mathbb{C}$.

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We call B a mixing von Neumann subalgebra of M if

$$\lim_{n \to \infty} \|\mathbb{E}_B(x u_n y)\|_2 = 0, \quad \forall x, y \in M \ominus B,$$

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Theorem (Jolissaint-Stalder)

The action $G \curvearrowright_{\sigma_g} (X, \mathcal{B}, \mu)$ is mixing if and only if $L(G)$ is mixing in $L^\infty(X, \mu) \rtimes G$.

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Theorem

Let B *be a mixing von Neumann subalgebra of* M*. Suppose* A *is a diffuse von Neumann subalgebra of* B *and* $y \in M$*. If* $yAy^* \subseteq B$*, then* $y \in B$*.*

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- Γ_0 is called a malnormal subgroup of Γ if $g\Gamma_0 g^{-1} \cap \Gamma_0 = \{e\}$ for every $g \in \Gamma \setminus \Gamma_0$.

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Remark

Tensor product of weakly mixing von Neumann subalgebras is a weakly mixing von Neumann subalgebra. Tensor product of mixing von Neumann subalgebras is not a mixing von Neumann subalgebra.

Examples of Mixing von Neumann subalgebras

- Let $M = M_1 *_A M_2$ be the amalgamated free product of diffuse finite von Neumann algebras (M_1, τ_1) and (M_2, τ_2) over an atomic finite von Neumann algebra A. Then M_1 is a mixing von Neumann subalgebra of M.

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- There are uncountably many non conjugate mixing abelian von Neumann subalgebras in the free group factors each having Pukansky values $\{1, \infty\}$.

An application

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Theorem

Let $M = L(G \rtimes \Gamma_0)$ *and* $B = L(\Gamma_0)$*. Then* B *is mixing in* M *if and only if for each* $g \in G$*,* $g \neq e$*, the group*

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Theorem (Halmos)

Let X be a compact abelian group, and $T : X \to X$ a continuous automorphism. Then T is mixing if and only if T is ergodic.

An application

Corollary

Let $M = L(G \rtimes \Gamma_0)$ *and* $B = L(\Gamma_0)$. *Suppose* Γ_0 *is a finitely generated, infinite, abelian group or* Γ_0 *is a torsion free group. Then* B *is mixing in* M *if and only if every element* $h \in \Gamma_0$ *of infinite order is ergodic on* $L(G)$.

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Theorem (Kitchens-Schmidt)

Let X be a compact group and let $\Gamma_0 \subset \mathrm{Aut}(X)$ be a finitely generated, infinite, abelian group. Then the action Γ_0 on X is mixing if and only if every element $h \in \Gamma_0$ of infinite order is ergodic on X.