Cocycle Superrigidity for Gaussian Actions

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This talk will cover joint work with Jesse Peterson.

Group Actions

$\Gamma \curvearrowright^{\sigma} (X, \mu)$

- Γ countable discrete group.
- (X, μ) non-atomic standard probability space *i.e.*, $(X, \mu) \cong (\mathbb{R}, g)$, $dg = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)dx$.
- $\sigma : \Gamma \to \mathrm{Aut}(X, \mu)$, μ-preserving.

Example

The **Bernoulli action**: $\Gamma \curvearrowright^{\sigma} \prod_{\Gamma}(\mathbb{R}, g)$, $\sigma_{\gamma}((x_{\gamma'})_{\gamma'}) = (x_{\gamma'})_{\gamma\gamma'}$

Cocycles

$\Gamma \curvearrowright^{\sigma} (X, \mu)$, $\mathcal{G}$, a Polish topological group.

Definition

A **cocycle** is a measurable map $c : \Gamma \times X \to \mathcal{G}$ satisfying the cocycle identity $c(\gamma_1\gamma_2, x) = c(\gamma_1, \sigma_{\gamma_2}(x))c(\gamma_2, x)$, for all $\gamma_1, \gamma_2 \in \Gamma$, a.e. $x \in X$.

Definition

A pair of cocycles c_1, c_2 are **cohomologous** if there exists a measurable map $\xi : X \to \mathcal{G}$ such that $\xi(\sigma_\gamma(x))c_1(\gamma, x)\xi(x)^{-1} = c_2(\gamma, x)$ for a.e. $x \in X$.

Cocycles

Example

To any homomorphism $\rho : \Gamma \to \mathcal{G}$ we can associate a cocycle $\tilde{\rho}$ by $\tilde{\rho}(\gamma, x) = \rho(\gamma)$.

cocycle = homomorphism "twisted" over a space

Definition (S. Popa)

A cocycle c is said to **untwist** if there exists a homomorphism $\rho : \Gamma \to \mathcal{G}$ such that c is cohomologous to $\tilde{\rho}$.

$\mathcal{U}_{\mathrm{fin}}$

Definition

$\mathcal{U}_{\mathrm{fin}}$ is the class of Polish groups which can be realized as a closed subgroup of the unitary group of some II_1 factor.

Example

1. Any countable discrete group Λ.
2. Any compact Polish group *e.g.*, $\mathbb{T}$.

Cocycle Superrigidity

Definition

An action $\Gamma \curvearrowright^{\sigma} (X, \mu)$ is $\mathcal{U}_{\mathrm{fin}}$-**cocycle superrigid** if for any $\mathcal{G} \in \mathcal{U}_{\mathrm{fin}}$, every cocycle $c : \Gamma \times X \to \mathcal{G}$ untwists.

Cocycles of Representations

$\pi : \Gamma \to \mathcal{O}(\mathcal{H})$, an orthogonal representation.

Definition

A **cocycle** is a map $b : \Gamma \to \mathcal{H}$ satisfying the cocycle identity $b(\gamma_1\gamma_2) = \pi(\gamma_1)b(\gamma_2) + b(\gamma_1)$, for all $\gamma_1, \gamma_2 \in \Gamma$.

Definition

A cocycle b is **unbounded** if $\exists \gamma_n$ such that $||b(\gamma_n)|| \to \infty$.

L^2-Betti numbers

Let $\lambda : \Gamma \to \ell^2_{\mathbb{R}}\Gamma$ be the left-regular (orthogonal) representation.

Definition

The **n-th L^2-Betti number**: $\beta_n^{(2)}(\Gamma) = \dim_{L\Gamma} H^n(\Gamma, \lambda \otimes \mathbb{C})$.

Theorem (Bekka-Valette, Peterson-Thom)

If Γ is non-amenable, then $\beta_1^{(2)}(\Gamma) \neq 0$ if and only if λ admits an unbounded cocycle.

Popa's CSR for Bernoulli actions

Theorem (S. Popa)

Let Γ be a countable discrete group which either has Kazhdan's property (T) or is the product of an infinite group and a non-amenable group; then, the Bernoulli action of Γ is $\mathcal{U}_{\text{fin}}$-cocycle superrigid.

Problem

What is the class of groups whose Bernoulli actions are $\mathcal{U}_{\text{fin}}$-CSR?

Conjecture (I. Chifan, A. Ioana, J. Peterson)

The Bernoulli action of a group Γ is $\mathcal{U}_{\text{fin}}$-CSR if and only if $\beta_1^{(2)}(\Gamma) = 0$.

Results

Theorem A (Peterson-S)

Let Γ be a countable discrete group: if $\beta_1^{(2)}(\Gamma) \neq 0$, then the Bernoulli shift action is not $\mathcal{U}_{\text{fin}}$-cocycle superrigid.

Theorem B (Peterson-S)

Let Γ be a countable discrete group: if $L\Gamma$ is s-L^2-rigid, then the Bernoulli shift action of Γ is $\mathcal{U}_{\text{fin}}$-cocycle superrigid.

Theorem A

Theorem A (Peterson-S)

Let Γ *be a countable discrete group with* $\beta_1^{(2)}(\Gamma) \neq 0$, $\Gamma \curvearrowright^\sigma (X, \mu)$ *the Bernoulli action; then, there exists a cocycle* $c : \Gamma \times X \to \mathbb{T}$ *which does not untwist.*

Corollary

If $\beta_1^{(2)}(\Gamma) \neq 0$, *then* $\Gamma \curvearrowright^\sigma (X, \mu)$ *is not* $\mathcal{U}_{\mathrm{fin}}$*-CSR.*

Sketch of Proof of Theorem A

We will show there is a cocycle not cohomologous to the trivial homomorphism.

- Embed $\ell^2_{\mathbb{R}}\Gamma \subset L^2_{\mathbb{R}}(X, \mu)$ via $\delta_\gamma \mapsto \pi_\gamma$, where $\pi_\gamma : (X, \mu) \to \mathbb{R}$ is the projection on the γ-coordinate.
- Idea of K. Schmidt: given a cocycle $b : \Gamma \to \ell^2_{\mathbb{R}}\Gamma$, exponentiate to obtain cocycle $c_t(\gamma, x) = \exp(\pi itb(\gamma))(x)$.
- Also K. Schmidt: $\int c_t(\gamma, x) d\mu(x) = \exp(-(\pi t||b(\gamma)||)^2/2)$.

Sketch of Proof (cont'd)

- By contradiction, $\forall t > 0$, $\exists u_t : X \to \mathbb{T}$ such that $c_t(\gamma, x) = u_t(\sigma_\gamma(x))u_t(x)^{-1}$.
- $\exists \gamma_n$ such that $\int u_t(\sigma_{\gamma_n}(x))u_t(x)^{-1}d\mu(x) = \exp(-(\pi t||b(\gamma_n)||)^2/2) \to 0 \Rightarrow u_t \perp 1$, $\forall t > 0$.
- But $c_t \to 1$ as $t \to 0$ and σ is ergodic and has spectral gap, a contradiction.

Theorem B

Theorem B (Peterson-S)

Let Γ be a countable discrete group. If $L\Gamma$ is s-L^2-rigid then the Bernoulli shift action of Γ is $\mathcal{U}_{\text{fin}}$-cocycle superrigid.

We need to discuss Peterson's notion of an L^2-rigid von Neumann algebra.

L^2-rigidity

Definition

Let (N, τ) be a finite von Neumann algebra and $\mathcal{H}$ be an N-N correspondence. A **closable derivation** δ is a closable unbounded operator $\delta : L^2(N, \tau) \to \mathcal{H}$ such that $D(\delta)$ contains a $|| \cdot ||_2$-dense *-subalgebra A of N such that $\delta(xy) = x\delta(y) + \delta(x)y, \ \forall x, y \in A$.

To every derivation we associate a semigroup of completely-positive maps $\Phi^t : N \to N$, given by

$$\Phi^t = \exp(-t\delta^*\overline{\delta}).$$

L^2-rigidity(cont'd)

Definition (Peterson)

N is s-L^2-**rigid** if given any inclusion $(N,\tau) \subset (M,\tilde{\tau})$, and any closable derivation $\delta : M \to \mathcal{H}$ such that $\mathcal{H}$ when viewed as an N-N correspondence embeds in $(L^2N\overline{\otimes}L^2N)^{\oplus\infty}$, the associated deformation $\Phi^t = \exp(-t\delta^*\overline{\delta})$ converges uniformly to the identity on $(N)_1$.

L^2-rigidity(cont'd)

Theorem (Peterson)

Let Γ be a countable discrete group. If $\beta_1^{(2)}(\Gamma) \neq 0$, then $L\Gamma$ is not s-L^2-rigid.

Theorem (Peterson)

$L\Gamma$ is s-L^2-rigid if any of the following hold:

- *Γ has Kazhdan's property (T);*
- *$\Gamma = H \times K$, where H is non-amenable and K is infinite;*
- *Γ non-amenable and $L\Gamma$ has property Gamma of Murray and von Neumann;*
- *$\Gamma = A_0 \wr \Gamma_0$, where A_0 is abelian and Γ_0 is not Haagerup.*

Sketch of Proof of Theorem B

- $\Gamma \curvearrowright^{\sigma} (X, \mu)$, the Bernoulli action; $\mathcal{G} \subset \mathcal{U}(N)$;
  $w : \Gamma \times X \to \mathcal{G}$, a cocycle.
- Let $A = L^{\infty}(X, \mu)$,
  $M = (A \otimes 1 \otimes N) \rtimes \Gamma \subset \tilde{M} = (A \otimes A \otimes N) \rtimes \Gamma$.
- $w_\gamma \in \mathcal{U}(A \otimes 1 \otimes N)$ such that $w_\gamma(x) = w(\gamma, x)$.
- $\widetilde{L\Gamma} = \{w_\gamma u_\gamma\}''(\cong L\Gamma)$.

Sketch of Proof (cont'd)

- Popa's "s-malleable deformation" $\alpha : \mathbb{R} \to \mathrm{Aut}(A \otimes A, \tau)$.
- extend α to a deformation $\tilde{\alpha} : \mathbb{R} \to \mathrm{Aut}(\tilde{M}, \tau)$.

Observation

Popa's CSR machinery will untwist w if $\tilde{\alpha}_t$ converges uniformly on $(\widetilde{L\Gamma})_1$.

Sketch of Proof (cont'd)

- $A = \bigotimes_\Gamma A_0$, $\delta_0 : L^2(A_0) \to L^2(A_0 \otimes A_0)$, Voiculescu's difference quotient derivation.
- $\delta = \bigotimes_\Gamma \delta_0 : A \to \mathcal{H} = \bigoplus_{\gamma \in \Gamma} L^2(A_0 \otimes A_0) \otimes (\bigotimes_{\gamma' \neq \gamma} L^2(A_0))$.
- Extend naturally to $\delta : L^2(M) \to \mathcal{H} \otimes N \otimes \ell^2\Gamma$.

Lemma (Peterson-S)

If $\Phi^t = \exp(-t\delta^*\overline{\delta})$ *converges uniformly to the identity on* $(\widetilde{L\Gamma})_1$, *then so does* $\tilde{\alpha}_t$.

Sketch of Proof (cont'd)

- We have that $\mathcal{H} \otimes N \otimes \ell^2\Gamma$ as an $\widetilde{L\Gamma}$-$\widetilde{L\Gamma}$ correspondence embeds in a direct sum of copies of the coarse $L\Gamma$-$L\Gamma$ correspondence.
- Φ^t converges uniformly since $L\Gamma$ is s-L^2-rigid.
- Hence, $\tilde{\alpha}_t$ coverges uniformly on $(\widetilde{L\Gamma})_1$, and w untwists.

Final Remarks

- Theorem A holds more generally:
  If Γ admits an unbounded cocycle into a **nonamenable representation**, then the associated **Gaussian action** will have a $\mathbb{T}$-valued cocycle which does not untwist.

- The use of derivations allows for a unified approach to $\mathcal{U}_{\mathrm{fin}}$-CSR for Gaussian actions. However, Theorem B applies essentially to the Bernoulli action.