The Busemann-Petty Problem in the Complex Hyperbolic Space

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Texas A&M University, July 19, 2012
The Busemann-Petty Problem in $\mathbb{R}^n$

Let $K$ and $L$ be two origin-symmetric convex bodies in $\mathbb{R}^n$. Suppose that for every $\xi \in S^{n-1}$

$$\text{Vol}_{n-1}(K \cap \xi^\perp) \leq \text{Vol}_{n-1}(L \cap \xi^\perp).$$

Does it follow that

$$\text{Vol}_n(K) \leq \text{Vol}_n(L)?$$

The answer is affirmative for $n \leq 4$ and negative for $n \geq 5$.

*posed in 1956
solved in 90’*
The Busemann-Petty Problem in other spaces

Yaskin: in real hyperbolic and spherical spaces

For the real spherical space the answer is the same as for $\mathbb{R}^n$ and for the real hyperbolic space the answer is affirmative for $n \leq 2$ and negative for $n \geq 3$.

Koldobsky, König, Zymonopoulou: $\mathbb{C}^n$

For the complex version of the Busemann-Petty problem the answer is affirmative for the complex dimension $n \leq 3$ and negative for $n \geq 4$. 
The Ball Model of $\mathbb{H}^n_C$

$\mathbb{H}^n_C$ can be identified with the open unit ball in $\mathbb{C}^n$
$B^n := \{ z \in \mathbb{C}^n : (z, z) < 1 \}$.

The volume element on $\mathbb{H}^n_C$ is

$$d\mu_n = 8^n \frac{r^{2n-1} dr d\sigma}{(1 - r^2)^{n+1}},$$

where $d\sigma$ is the volume element on the unit sphere $S^{2n-1}$.

For $K \subset B^n$ denote by $H\text{Vol}_{2n}(K)$ the volume of $K$ with respect to $d\mu_n$. 
Hyperplanes in $\mathbb{C}^n$

For $\xi \in \mathbb{C}^n$, $|\xi| = 1$, denote by $H_\xi := \{ z \in \mathbb{C}^n : (z, \xi) = 0 \}$ the complex hyperplane through the origin perpendicular to $\xi$.

Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the mapping

$$(\xi_1 + i\xi_2, \ldots, \xi_n + i\xi_{n+2}) \mapsto (\xi_1, \xi_2, \ldots, \xi_{n+1}, \xi_{n+2}).$$

Under this mapping the hyperplane $H_\xi$ turns into a $(2n - 2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$\xi = (\xi_1, \xi_2, \ldots, \xi_n, \xi_{n+2}) \text{ and } \xi_\perp := (-\xi_2, \xi_1, \ldots, -\xi_{n+2}, \xi_n).$$
Origin Symmetric Convex Sets in $\mathbb{C}^n$

$K \subset \mathbb{C}^n$ is origin symmetric if $x \in K$ implies $-x \in K$.

Origin symmetric convex bodies in $\mathbb{C}^n$ are unit balls of norms on $\mathbb{C}^n$. We call them complex convex bodies.

Norms on $\mathbb{C}^n$ satisfy: $\|\lambda z\| = |\lambda| \|z\|$ for $\lambda \in \mathbb{C}$.

A star body $K$ in $\mathbb{R}^{2n}$ is called $R_\theta$-invariant, if for every $\theta \in [0, 2\pi]$ and every $\xi = (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$

$$\|\xi\|_K = \|R_\theta(\xi_{11}, \xi_{12}), \ldots, R_\theta(\xi_{n1}, \xi_{n2})\|_K,$$

where $R_\theta$ stands for the counterclockwise rotation by an angle $\theta$ around the origin in $\mathbb{R}^2$.

Thus complex convex bodies in $\mathbb{C}^n$ are $R_\theta$-invariant convex bodies in $\mathbb{R}^{2n}$. 
Formulation of the problem

The Busemann-Petty problem in $\mathbb{H}^n_C$ can be posed as follows.

Given two $R_\theta$-invariant convex bodies $K$ and $L$ in $\mathbb{R}^{2n}$ contained in the unit ball such that

$$\text{HV} \text{Vol}_{2n-2}(K \cap H_\xi) \leq \text{HV} \text{Vol}_{2n-2}(L \cap H_\xi)$$

for all $\xi \in S^{2n-1}$, does it follow that

$$\text{HV} \text{Vol}_{2n}(K) \leq \text{HV} \text{Vol}_{2n}(L)?$$
Basic Definitions

A distribution $f$ on $\mathbb{R}^n$ is even homogeneous of degree $p \in \mathbb{R}$, if

$$\left\langle f(x), \varphi \left( \frac{x}{\alpha} \right) \right\rangle = |\alpha|^{n+p} \left\langle f, \varphi \right\rangle$$

for every test function $\varphi$ and every $\alpha \in \mathbb{R}, \alpha \neq 0$.

The Fourier transform of an even homogeneous distribution of degree $p$ on $\mathbb{R}^n$ is an even homogeneous distribution of degree $-n-p$.

A distribution $f$ is positive definite, if $\hat{f}$ is a positive distribution, i.e. $\left\langle \hat{f}, \varphi \right\rangle \geq 0$ for every non-negative test function $\varphi$.

A star body $K$ is $k$-smooth, $k \in \mathbb{N} \cup \{0\}$, if $\| \cdot \|_K$ belongs to the class $C^k(S^{n-1})$ of $k$ times continuously differentiable functions on the unit sphere.

If $\| \cdot \|_K \in C^k(S^{n-1})$ for any $k \in \mathbb{N}$, then a star body $K$ is said to be infinitely smooth.
Approximation Results

The radial function of $K$ is $\rho_K(x) = \|x\|_K^{-1}$.

One can approximate any convex body $K$ in $\mathbb{R}^n$ in the radial metric

$$\rho(K, L) := \max_{x \in S^{n-1}} |\rho_K(x) - \rho_L(x)|$$

by a sequence of infinitely smooth convex bodies with the same symmetries as $K$. 
Fourier Approach to Sections

For an infinitely smooth origin symmetric star body $K$ in $\mathbb{R}^n$ and $0 < p < n$, the Fourier transform of the distribution $\|x\|^{-p}_K$ is an infinitely smooth function on $\mathbb{R}^n \setminus \{0\}$, homogeneous of degree $-n + p$.

Lemma

Let $K$ and $L$ be infinitely smooth origin symmetric star bodies in $\mathbb{R}^n$, and let $0 < p < n$. Then

$$\int_{S^{n-1}} (\| \cdot \|^{-p}_K)^\wedge(\theta)(\| \cdot \|^{-n+p}_L)^\wedge(\theta) d\theta = (2\pi)^n \int_{S^{n-1}} \|\theta\|^{-p}_K \|\theta\|^{-n+p}_L d\theta.$$
Fourier Approach to Sections

Let $0 < k < n$ and let $H$ be an $(n - k)$-dimensional subspace of $\mathbb{R}^n$. Fix an orthonormal basis $e_1, \ldots, e_k$ in the orthogonal subspace $H^\perp$. For a star body $K$ in $\mathbb{R}^n$, define the $(n - k)$-dimensional parallel section function $A_{K,H}$ as a function on $\mathbb{R}^k$ such that for $u \in \mathbb{R}^k$

$$A_{K,H}(u) = \text{Vol}_{n-k}(K \cap \{H + u_1 e_1 + \cdots + u_k e_k\}).$$

If $K$ is infinitely smooth, the function $A_{K,H}$ is infinitely differentiable at the origin.

Lemma

Let $K$ be an infinitely smooth origin symmetric star body in $\mathbb{R}^n$ and $0 < k < n$. Then for every $(n - k)$-dimensional subspace $H$ of $\mathbb{R}^n$ and for every $m \in \mathbb{N} \cup \{0\}$, $m < (n - k)/2$, 

$$\Delta^mA_{K,H}(0) = \frac{(-1)^m}{(2\pi)^k(n - 2m - k)} \int_{S^{n-1} \cap H^\perp} (\|x\|_K^{-n+2m+k})^\wedge(\xi) \, d\xi,$$

where $\Delta$ denotes the Laplacian on $\mathbb{R}^k$. 
Fourier and Radon Transforms of $R_\theta$-invariant Functions

Lemma

Suppose that $K$ is an infinitely smooth $R_\theta$-invariant star body in $\mathbb{R}^{2n}$. Then for every $0 < p < 2n$ and $\xi \in S^{2n-1}$ the Fourier transform of the distribution $\|x\|_K^{-p}$ is a constant function on $S^{2n-1} \cap H_\xi^\perp$.

Proof: Since $\|x\|_K$ is $R_\theta$-invariant, so is the Fourier transform of $\|x\|_K^{-p}$. The two-dimensional space $H_\xi^\perp$ is spanned by two vectors $\xi$ and $\xi_\perp$.

Every vector in $S^{2n-1} \cap H_\xi^\perp$ is the image of $\xi$ under one of the coordinate-wise rotations $R_\theta$, so the Fourier transform of $\|x\|_K^{-p}$ is a constant function on $S^{2n-1} \cap H_\xi^\perp$. 

□
Fourier and Radon Transforms of $R_\theta$-invariant Functions

Denote by $C_\theta(S^{2n-1})$ the space of $R_\theta$-invariant continuous functions on the unit sphere $S^{2n-1}$, i.e. continuous real-valued functions $f$ satisfying $f(\xi) = f(R_\theta \xi)$ for any $\xi \in S^{2n-1}$ and any $\theta \in [0, 2\pi]$.

The complex spherical Radon transform is an operator $R_c : C_\theta(S^{2n-1}) \rightarrow C_\theta(S^{2n-1})$ defined by

$$R_c f(\xi) = \int_{S^{2n-1} \cap H_\xi} f(x) dx.$$
Fourier and Radon Transforms of $R_{θ}$-invariant Functions

**Lemma**

Let $f \in C_θ(S^{2n-1})$. Extend $f$ to a homogeneous function of degree $-2n + 2$, $f \cdot r^{-2n+2}$, then the Fourier transform of this extension is a homogeneous function of degree $-2$ on $\mathbb{R}^{2n}$, whose restriction to the unit sphere is continuous. Moreover, for every $ξ \in S^{2n-1}$

$$\mathcal{R}_c f(ξ) = \frac{1}{2\pi} (f \cdot r^{-2n+2})^\wedge(ξ).$$
Volume of Sections

Let $K$ be an $R_\theta$-invariant star body contained in the unit ball of $\mathbb{R}^{2n}$ with $n \geq 2$. Let $\xi \in S^{2n-1}$, we compute:

$$\text{HVol}_{2n-2}(K \cap H_\xi)$$

$$= 8^{n-1} \int_{S^{2n-1} \cap H_\xi} \int_0^{\|x\|_K^{-1}} \frac{r^{2n-3}}{(1 - r^2)^n} dr dx$$

$$= 8^{n-1} \int_{S^{2n-1} \cap H_\xi} |x|^{-2n+2} \int_0^{\|x\|_K} \frac{r^{2n-3}}{(1 - r^2)^n} dr dx$$

$$= \frac{8^{n-1}}{2\pi} \left( |x|^{-2n+2} \int_0^{\|x\|_K} \frac{r^{2n-3}}{(1 - r^2)^n} dr \right)^\wedge (\xi).$$
Counterexample for $n \geq 4$

**Theorem**

There exist $R_\theta$-invariant convex bodies $K$ and $L$ contained in the unit ball of $\mathbb{R}^{2n}$ with $n \geq 4$ satisfying

$$\text{HVol}_{2n-2}(K \cap H_\xi) \leq \text{HVol}_{2n-2}(L \cap H_\xi)$$

for every $\xi \in S^{2n-1}$, but

$$\text{HVol}_{2n}(K) > \text{HVol}_{2n}(L).$$

**Proof:** Let $K$, $L$ be $R_\theta$-invariant convex bodies in $\mathbb{R}^{2n}$ with $n \geq 4$ that provide a counterexample to the complex BPP, i.e.

$$\text{Vol}_{2n-2}(K \cap H_\xi) \leq \text{Vol}_{2n-2}(L \cap H_\xi)$$

for every $\xi \in S^{2n-1}$, but

$$\text{Vol}_{2n}(K) > \text{Vol}_{2n}(L).$$

Can assume that bodies $K$ and $L$ are infinitely smooth.
Counterexample for $n \geq 4$

Since (2) is strict, we can dilate the body $L$ by $\alpha > 1$ to make (1) strict as well. By continuity of $\xi \mapsto A_{K,H_\xi}(0)$, there is an $\epsilon > 0$ so that

$$(1 + \epsilon)\text{Vol}_{2n-2}(K \cap H_\xi) \leq \text{Vol}_{2n-2}(L \cap H_\xi)$$

for every $\xi \in S^{2n-1}$, but

$$\text{Vol}_{2n}(K) > (1 + \epsilon)\text{Vol}_{2n}(L).$$

Can dilate $K$ and $L$ by $\alpha > 0$.

Choose $\alpha$ so small that $\alpha K$ and $\alpha L$ lie in the ball of radius $s$ that satisfies the inequality

$$1 \leq \frac{1}{(1 - s^2)^{n+1}} \leq 1 + \epsilon.$$

Then $K$ and $L$ provide a counterexample.
Positive Part I

Lemma

Let $K$ be an $R_\theta$-invariant convex body contained in the unit ball of $\mathbb{R}^{2n}$ such that $\frac{\|x\|^2_K}{1-\left(\frac{|x|}{\|x\|_K}\right)^2}$ is a positive definite distribution on $\mathbb{R}^{2n}$. And let $L$ be an $R_\theta$-invariant star body contained in the unit ball of $\mathbb{R}^{2n}$ so that

$$\text{HVol}_{2n-2}(K \cap H_\xi) \leq \text{HVol}_{2n-2}(L \cap H_\xi)$$

for every $\xi \in S^{2n-1}$. Then

$$\text{HVol}_{2n}(K) \leq \text{HVol}_{2n}(L).$$
Positive Part II

Proof: Since the function $\frac{r^2}{1-r^2}$ is increasing on $(0, 1)$,

$$\frac{a^2}{1-a^2} \int_a^b \frac{r^{2n-3}}{(1-r^2)^n} dr = \int_a^b \frac{r^{2n-1}}{(1-r^2)^{n+1}} \frac{a^2}{1-a^2} \frac{(1-r^2)}{r^2} dr \leq \int_a^b \frac{r^{2n-1}}{(1-r^2)^{n+1}} dr$$

for $a, b \in (0, 1)$. True in case $a \leq b$ as well as in case $b \leq a$.

Integrating both sides in the above inequality over the unit sphere $S^{2n-1}$ with $a = \|x\|_K^{-1}$ and $b = \|x\|_L^{-1}$ we obtain:

$$\int_{S^{2n-1}} \frac{\|x\|_K^{-2}}{1-\|x\|_K^{-2}} \int_{\|x\|_K^{-1}} \frac{\|x\|_L^{-1}}{r^{2n-3}} \frac{r^{2n-3}}{(1-r^2)^n} dr dx \leq \int_{S^{2n-1}} \int_{\|x\|_K^{-1}} \frac{\|x\|_L^{-1}}{r^{2n-1}} \frac{r^{2n-1}}{(1-r^2)^{n+1}} dr dx .$$

Show that the LHS is positive. Let $d\mu_0$ be the measure corresponding to the Fourier transform of the positive definite distribution $\frac{\|x\|_K^{-2}}{1-(\frac{|x|}{\|x\|_K})^2}$, then we obtain:
Positive Part III

\[
(2\pi)^{2n} \int_{S^{2n-1}} \frac{\|x\|^{-2}}{1 - \|x\|^{-2}} \int_{0}^{\|x\|^{-1}} \frac{r^{2n-3}}{(1 - r^2)^n} dr dx
\]

\[
= (2\pi)^{2n} \int_{S^{2n-1}} \frac{\|x\|^{-2}}{1 - \left(\frac{|x|}{\|x\|_K}\right)^2} |x|^{-2n+2} \int_{0}^{\|x\|_K} \frac{|x|}{\|x\|_K} r^{2n-3} \frac{dr}{(1 - r^2)^n} dx
\]

\[
= \int_{S^{2n-1}} \left( |x|^{-2n+2} \int_{0}^{\|x\|_K} \frac{r^{2n-3}}{(1 - r^2)^n} dr \right) (\xi) d\mu_0(\xi)
\]

\[
= \frac{2\pi}{8^{n-1}} \int_{S^{2n-1}} \text{HVol}_{2n-2}(K \cap H_\xi) d\mu_0(\xi)
\]

\[
\leq \frac{2\pi}{8^{n-1}} \int_{S^{2n-1}} \text{HVol}_{2n-2}(L \cap H_\xi) d\mu_0(\xi)
\]

\[
= \int_{S^{2n-1}} \left( |x|^{-2n+2} \int_{0}^{\|x\|_L} \frac{r^{2n-3}}{(1 - r^2)^n} dr \right) (\xi) d\mu_0(\xi)
\]

\[
= (2\pi)^{2n} \int_{S^{2n-1}} \frac{\|x\|^{-2}}{1 - \|x\|^{-2}} \int_{0}^{\|x\|_L^{-1}} \frac{r^{2n-3}}{(1 - r^2)^n} dr dx
\]
Positive Part IV

Thus the RHS is positive as well, which shows

$$\int_{S^{2n-1}} \int_0^{\|x\|^{-1}_K} \frac{r^{2n-1}}{(1 - r^2)^{n+1}} dr dx \leq \int_{S^{2n-1}} \int_0^{\|x\|^{-1}_L} \frac{r^{2n-1}}{(1 - r^2)^{n+1}} dr dx.$$ 

That is,

$$\text{HVol}_{2n}(K) \leq \text{HVol}_{2n}(L).$$
Negative Part I

Lemma

Suppose there is an infinitely smooth complex convex body $K$ in $B^n$ with strictly positive curvature so that \( \frac{||x||_K^{-2}}{1 - \left( \frac{|x|}{||x||_K} \right)^2} \) is not a positive definite distribution on $\mathbb{R}^{2n}$. Then one can perturb the body $K$ to construct another complex convex body $L$ in $B^n$ so that for every $\xi \in S^{2n-1}$

\[
\text{HVol}_{2n-2}(L \cap H_\xi) \leq \text{HVol}_{2n-2}(K \cap H_\xi),
\]

but

\[
\text{HVol}_{2n}(L) > \text{HVol}_{2n}(K).
\]
Negative Part II

**Proof:** There is an open subset $\Omega \subset S^{2n-1}$ on which
\[ \left( \frac{\|x\|_K^{-2}}{1 - \left( \frac{|x|}{\|x\|_K} \right)^2} \right)^\wedge \] is negative. Note: $\Omega$ is $R_\theta$-invariant.

Choose a smooth non-negative $R_\theta$-invariant function $f$ on $S^{2n-1}$ with $\text{supp}(f) \subset \Omega$ and extend $f$ to an $R_\theta$-invariant homogeneous function $f \left( \frac{x}{|x|} \right) |x|^{-2}$ of degree $-2$ on $\mathbb{R}^{2n}$.

The Fourier transform of this extension is an $R_\theta$-invariant infinitely smooth function on $\mathbb{R}^{2n} \setminus \{0\}$, homogeneous of degree $-2n+2$, i.e.
\[ \left( f \left( \frac{x}{|x|} \right) |x|^{-2} \right)^\wedge (y) = g \left( \frac{y}{|y|} \right) |y|^{-2n+2} \]

with $g \in C_\theta^\infty(S^{2n-1})$. 
Negative Part III

Define a body \( L \) in \( B^n \) by

\[
|x|^{-2n+2} \int_0^{\frac{|x|}{\|x\|_L}} \frac{r^{2n-3}}{(1 - r^2)^n} dr = |x|^{-2n+2} \int_0^{\frac{|x|}{\|x\|_K}} \frac{r^{2n-3}}{(1 - r^2)^n} dr - \epsilon g \left( \frac{x}{|x|} \right) |x|^{-2n+2}
\]

for some \( \epsilon > 0 \).

For small enough \( \epsilon \), the body \( L \) is convex.

\( K \) and \( L \) are the bodies we seek.
Parallel section function

For an infinitely smooth $R_\theta$-invariant star body $K \subset \mathbb{R}^{2n}$ and $m \in \mathbb{N} \cup \{0\}, \ m < n - 1,$

$$\Delta^m A_{K,H_\xi}(0) = \frac{(-1)^m}{(2\pi)^2(2n - 2m - 2)} \int_{S^{2n-1} \cap H_{\xi}^\perp} (\|x\|^{-2n+2m+2}_K)^\wedge_2(\nu) d\nu$$

$$= \frac{(-1)^m}{2\pi(2n - 2m - 2)} (\|x\|^{-2n+2m+2}_K)^\wedge_2(\xi).$$

In complex dimension two there is only one choice for $m$, namely $m = 0$, and we get

$$A_{K,H_\xi}(0) = \frac{1}{4\pi} (\|x\|^{-2}_K)^\wedge_2(\xi).$$

In complex dimension three, for $m = 1$, we obtain

$$\Delta A_{K,H_\xi}(0) = \frac{-1}{4\pi} (\|x\|^{-2}_K)^\wedge_2(\xi).$$
Solution of the Problem

\[ n = 2 \]

**Lemma**

*For any \( R_\theta \)-invariant star body \( K \) in \( \mathbb{R}^4 \) contained in the unit ball, the distribution \( \frac{\|x\|_K^{-2}}{1 - \left( \frac{|x|}{\|x\|_K} \right)^2} \) is positive definite.*

**Proof:** Can assume \( K \) is smooth.

Define \( M \) by \( \|x\|_M^{-2} = \frac{\|x\|_K^{-2}}{1 - \left( \frac{|x|}{\|x\|_K} \right)^2} \).

\( M \) is smooth and \( R_\theta \)-invariant, thus

\[ A_{M, H_\xi}(0) = \frac{1}{4\pi} (\|x\|_M^{-2})^\wedge(\xi), \]

and consequently \( \|x\|_M^{-2} \) is positive definite.
Lemma

There is a smooth $R_\theta$-invariant convex body $K$ in $\mathbb{R}^6$ contained in the unit ball for which the distribution $\frac{\|x\|^2_K}{1 - (\frac{|x|}{\|x\|_K})^2}$ is not positive definite.

Proof: For $\xi \in \mathbb{R}^6$, $\xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}, \xi_{31}, \xi_{32})$, denote by $\tilde{\xi} = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22})$ and by $\xi_3 = (\xi_{31}, \xi_{32})$, then $\xi = (\tilde{\xi}, \xi_3)$.

Consider the map $(r, \theta) \mapsto \left( \sqrt{\frac{r^2}{1-r^2}}, \theta \right)$.

This map takes the line $x = \frac{1}{a}$ to the hyperbola $(a^2 - 1)x^2 - y^2 = 1$, provided that $a^2 - 1 > 0$, and it takes the ellipse $x^2 + (1 + b^2)y^2 = 1$ to the line $y = \frac{1}{b}$.

Denote the equation of the elliptic arc by $e$ and the equation of the hyperbolic arc by $h$. Define $K$ by

$$K = \{ \xi \in \mathbb{R}^6 : |\tilde{\xi}| \leq \frac{1}{a} \text{ and } |\xi_3| \leq e(\tilde{\xi}) \}.$$
For $a > 1$, $K$ is contained in the unit ball. Define a star body $M$ by

$$\|x\|_M^{-2} = \frac{\|x\|_K^{-2}}{1 - \left(\frac{|x|}{\|x\|_K}\right)^2}.$$ 

It can be described as

$$M = \{\xi \in \mathbb{R}^6 : \|\tilde{\xi}\| \leq h(|\xi_3|) \text{ with } |\xi_3| \leq \frac{1}{b}\}.$$

By construction both bodies are $R_\theta$-invariant and we may assume that they are smooth.

Let $x = (\tilde{x}, x_3) \in M$ with $x_3 \neq (0,0)$. Choose $\xi \in S^5$ in the direction of $x_3$. Fix an orthonormal basis $\{e_1, e_2\}$ for $H^\perp_\xi$. For $u \in \mathbb{R}^2$ with $|u| \geq \frac{1}{b}$, $A_{M,H_\xi}(u) = 0$, and otherwise
Solution of the Problem

\( n = 3 \)

\[
A_{M, H_\xi}(u) = \operatorname{Vol}_4(M \cap \{ H_\xi + u_1 e_1 + u_2 e_2 \})
\]
\[
= \int \{ x \in \mathbb{R}^{2n} : (x, e_1) = u_1, (x, e_2) = u_2 \} \chi(\| x \|_M) \, dx
\]
\[
= \int_{S^3} \int_0^{h(|u|)} r^3 \, dr \, d\theta
\]
\[
= |S^3| \frac{h(|u|)^4}{4}
\]
\[
= \frac{\pi^2}{2} \, h(|u|)^4 ,
\]

Setting \( a = 2 \), we get \( A_{M, H_\xi}(u) = \frac{\pi^2}{2} \left( \frac{1+|u|^2}{3} \right)^2 \) and consequently

\[
\Delta A_{M, H_\xi}(u) = \frac{4\pi^2}{9} (1 + 2|u|^2) . \]

Since \( M \) is smooth we have

\[
(\| x \|^2_M)^\wedge(\xi) = -4\pi \Delta A_{M, H_\xi}(0) = -\frac{16\pi^3}{9} .
\]