Section 2.1- Systems of Linear Equations

Possible Cases When Solving a System of Linear Equations:

1. Unique Solution
2. No Solution

Example 1: Solve the following system of linear equations:

\[ 2x - y = 1 \]
\[ 6x - 3y = 12 \]

3. Infinitely Many Solutions

Example 2: Solve the following system of linear equations:

\[ 3x - 7y = 4 \]
\[ 6x - 14y = 8 \]
Note: We will first learn how to set-up the system in these word problems. We will then learn the basics about matrices and then return to learn how to solve these problems.

**Example 3:** Jennifer has made it through her ten year class reunion. She is wanting to remember how many former classmates, spouses, and former teachers attended the reunion. She has lost her records but recalls that the ticket sales totaled $4,775. She charged $30 for each former classmate, $25 for each spouse, and $20 for each former teacher. She recalls that the number of former classmates and spouses combined was 130 more than the number of former teachers. She also recalls that there were five times as many former classmates there as spouses. Help Jennifer remember the number of former classmates, spouses, and former teachers that attended the reunion.

Set-up the linear system:

**Example 4:** An investment club has $200,000 earmarked for investment in stocks. To arrive at an acceptable overall level of risk, the stocks that management is considering have been classified into three categories: high-risk, medium-risk, and low-risk. Management estimates that high-risk stocks will have a rate of return of 15%/year; medium-risk stocks, 10%/year; and low-risk stocks, 6%/year. The members have decided that the investment in low-risk stocks should be equal to the sum of the investments in the stocks of the other two categories. Determine how much the club should invest in each type of stock if the investment goal is to have a return of $20,000/year on the total investment. (Assume that all of the money available for investment is invested.)

Set-up the linear system:
Section 2.4 - Matrices

Definitions:

1. A **matrix** is an ordered rectangular array of numbers. A matrix with \( m \) rows and \( n \) columns has size \( m \times n \). The entry in the \( i \)th row and \( j \)th column is denoted by \( a_{ij} \).

2. A **row matrix** is a matrix of size \( 1 \times n \).

3. A **column matrix** is a matrix of size \( m \times 1 \).

4. Two matrices are equal if they have the same size and equal corresponding entries.

**Example 1:** Given the following matrix equation, what are the values of \( a \), \( b \), and \( c \)?
\[
\begin{bmatrix}
a \\
-6
\end{bmatrix}
\begin{bmatrix}
5 \\
3b + 4
\end{bmatrix}
= 
\begin{bmatrix}
8 \\
2c - 15
\end{bmatrix}
\begin{bmatrix}
1 \\
-6
\end{bmatrix}
\begin{bmatrix}
2 \\
19
\end{bmatrix}
\]

Matrix Operations:

1. **Addition** (You can only add matrices that have the same size)

   **Example 2:** Perform the following matrix operation:
   \[
   \begin{bmatrix}
   1 & 2 \\
   3 & 4
   \end{bmatrix}
   + 
   \begin{bmatrix}
   4 & 5 \\
   6 & 7
   \end{bmatrix}
   \]

2. **Subtraction** (You can only subtract matrices that have the same size)

   **Example 3:** Perform the following matrix operation:
   \[
   \begin{bmatrix}
   1 & 2 \\
   3 & 4
   \end{bmatrix}
   - 
   \begin{bmatrix}
   4 & 5 \\
   6 & 7
   \end{bmatrix}
   \]
3. **Transpose:** The transpose of an \( m \times n \) matrix \( A \) with entries \( a_{ij} \) is the \( n \times m \) matrix \( A^T \) with entries \( a_{ji} \).

**Example 4:** Find \( A^T \) if

\[
A = \begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix}
\]

4. **Scalar Product:** For a matrix \( A \) and a real number \( c \), the scalar product \( cA \) is found by multiplying each entry in \( A \) by the real number \( c \).

**Example 5:** Find \( 3A \) if

\[
A = \begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix}
\]

We can allow the calculator to do these basic calculations for us.

(If you have a plain TI-83 (without the Plus), when you see directions to hit 2nd \( x^{-1} \), you need to hit the MATRIX button.)

**Enter the matrix into the calculator:**

- Hit 2nd \( x^{-1} \).
- Cursor right two places to EDIT and hit ENTER on the matrix you wish to edit.
- Enter the size of the matrix.
- Enter the matrix elements.

**Call a matrix for a computation:**

- Make sure you are on the home screen before you begin any calculations. To do this, hit 2nd MODE to quit.
- Press 2nd, \( x^{-1} \) and cursor down under the NAMES list until you get to the matrix you want and hit ENTER. The name of the matrix you need to do computations with will now be on your home-screen.

**Take the Transpose of a Matrix**

- Call the matrix you would like to transpose from the home screen.
- Press 2nd, \( x^{-1} \), cursor right to MATH and select option 2:T.
- You should now see the symbolic representation of transposing your matrix. To actually see the transpose, hit ENTER.
Example 6: Given the following matrices, perform the following operations on your calculator.

\[ A = \begin{bmatrix} 4 & 5 \\ 6 & 9 \\ 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \]

a) Find \( A + C \)

b) Find \( A - B \)

c) Find \( C^T \)

Augmented Matrix: We can combine two matrices into one, visually separating them by a vertical line. This is useful when solving a system of equations.

Example 7:
Section 2.2 - Systems of Linear Equations (The Gauss-Jordan Method)

This method allows us to strategically solve systems of linear equations. We perform operations on an augmented matrix that is formed by combining the coefficient matrix and the constant matrix as shown in the next example.

**Example 1:** Find the initial augmented matrix for the system of equations below:

a) \(2x - 4y = 10\)
\[y = 1 - 3x\]

b) \(x_1 - 2x_2 = 10x_3 + 5\)
\[8x_2 = x_1 - 3x_3\]
\[4x_1 - 3x_3 = x_2\]

The goal of the **Gauss-Jordan Elimination Method** is to get the augmented matrix into **Row Reduced Form**. A matrix is in **Row Reduced Form** when:

1. Each row of the coefficient matrix consisting entirely of zeros lies below any other row having nonzero entries.
2. The first nonzero entry in each row is 1 (called a leading 1)
3. In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.
4. If a column contains a leading 1, then the other entries in that column are zeros.

**Note:** We only consider the coefficient side (left side) of the augmented matrix when determining whether the matrix is in row-reduced form.

**Example 2:** Are the following in Row Reduced form?

a) \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

b) \[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

c) \[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 3 \\
0 & 0 & 2 & 1
\end{bmatrix}
\]
To put a matrix in **Row Reduced Form**, there are three valid **Row Operations**:

1. Interchange any two rows \((R_i \leftrightarrow R_j)\)
2. Replace any row by a nonzero constant multiple of itself \((cR_i)\)
3. Replace any row by the sum of that row and a constant multiple of any other row \((R_i + cR_j)\).

**Steps for Gauss Jordan Elimination:**

1. Begin by transforming the top left corner element, \(a_{11}\), into 1. This is your first pivot element.
2. Next, transform the other elements in its column into zeros using the 3 row operations.
3. Choose the next pivot element (diagonal down from the first pivot element)
4. Turn this 2nd pivot element into a 1, and transform the rest of its column into zeros.
5. Continue until the coefficient matrix resembles the identity matrix (1’s along the main-diagonal and 0’s everywhere else.)

**Example 3:** Solve the following system of equations using Gauss Jordan Elimination:

\[
\begin{align*}
3x + y &= 1 \\
-7x - 2y &= -1
\end{align*}
\]
Example 4: Solve the system that we set-up in Section 2.1 Example 3.
Example 5: Solve the system that we set-up in Section 2.1 Example 4.

Note: Our calculator will put an augmented matrix into row-reduced form for us. This only works when the # of rows is less than or equal to # of columns. Below are the steps you must follow:

1. Enter the augmented matrix into your calculator.
2. Go to your home screen.
3. Press MATRIX, cursor right to MATH, and select B:rref.
4. Call the matrix you want to reduce and hit ENTER.

We will use the calculator for working the problems in the next section. You are responsible for knowing how to do the Gauss-Jordan Method by hand, but always remember that you can check your work with the calculator.
Section 2.3 - The Gauss-Jordan Method - Part II

We are using the same process from Section 2.2 but now our solutions will not always be unique.

**Example 1:** Solve the following system:

\[
\begin{align*}
x + y - 2z &= -3 \\
2x - y + 3z &= 7 \\
x - 2y + 5z &= 0
\end{align*}
\]

**Example 2:** Solve the following system:

\[
\begin{align*}
x + 2y - 3z &= -2 \\
3x - y - 2z &= 1 \\
2x + 3y - 5z &= -3
\end{align*}
\]
Example 3: Solve the following systems:

a) 
\[
\begin{align*}
    x + y &= 7 \\
    2x + 3y &= 8 \\
    4x - y &= 3
\end{align*}
\]

b) 
\[
\begin{align*}
    x + y &= 7 \\
    2x + 3y &= 8 \\
    -5x - 5y &= -35
\end{align*}
\]

Example 4: Solve the following system:

\[
\begin{align*}
    4x_2 - 3x_3 - 5x_4 &= 7 \\
    x_1 - 2x_2 + 3x_4 &= 8
\end{align*}
\]
Section 2.5 - Multiplication of Matrices

How to Multiply Matrices \((C = AB)\):

(a) Check to see if the number of columns of matrix \(A\) is equal to the number of rows in matrix \(B\). If this condition is satisfied, the multiplication is a valid operation. The resulting matrix \(C\) will have the same number of rows as \(A\) and the same number of columns as \(B\).

(b) Compute each entry in the resulting matrix \(C\). The entry \(c_{ij}\) is found by using the \(i\)th row of matrix \(A\) and the \(j\)th column of matrix \(B\) as shown in the next example.

Example 1: Let

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \quad B = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}
\]

Find \(C\) where \(C = AB\).

Note: In general, \(AB \neq BA\)

Example 2: Given the following matrices with given dimensions, determine whether each of the following is a valid matrix operation.

\[
\begin{array}{cccc}
A_{2\times3} & B_{3\times5} & C_{5\times2} & D_{2\times3} \\
\end{array}
\]

a) \(AB - C\)

b) \(3AC - D\)

c) \(CD - 5B^T\)

d) \(DA - 3C\)

e) \(D - BC\)
Definition: The identity matrix of size $n$ has $n$ rows and $n$ columns. It has 1’s along the main diagonal and 0’s everywhere else.

A matrix multiplied by the appropriate identity matrix results in the original matrix.

Example 3: Multiply the matrix below by the appropriate identity matrix.

$$
\begin{bmatrix}
1 & 2 \\
4 & 1 \\
3 & 2
\end{bmatrix}
$$

Example 4: The Yummy Cake Company makes three types of cakes: Angel Food, Italian Cream, and Chocolate. The company produces its cakes in College Station, Santa Cruz, and Austin using two main ingredients, sugar and flour.

a) Each kilogram of Angel Food requires 0.1 kg of sugar and 0.5 kg of flour. Each kilogram of Italian Cream requires 0.6 kg of sugar and 0.2 kg of flour. Each kilogram of Chocolate requires 0.3 kg of sugar and 0.3 kg of flour. Put this information into a $2 \times 3$ matrix.

b) The cost of 1 kg of sugar is $1 in College Station, $4 in Santa Cruz and $2 in Austin. The cost of 1 kg of flour is $0.50 in College Station, $1.50 in Santa Cruz and $1 in Austin. Put this information into a matrix in such a way that when it is multiplied by the matrix in part a) it will tell us the cost of producing each kind of cake in each city. Find the resulting product matrix.
Section 2.6 - The Inverse of a Square Matrix

**Definition:** Let $A$ be a square matrix of size $n$. A square matrix $A^{-1}$ of size $n$ such that

$$A^{-1}A = AA^{-1} = I_n$$

is called the **inverse** of $A$.

**Example 1:** Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ has as its inverse $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

**Definition:** Every square matrix does not have an inverse. We say that a matrix is **singular** if it does not have an inverse. If it does have an inverse, we say that it is **nonsingular**.

You are not responsible for calculating inverses by hand. In the next example, I will show you how to find the inverse of a matrix on the calculator along with matrix multiplication.

**Example 2:** Given

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -2 & 4 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -3 & 0 \\ 5 & 0 & 1 \\ -2 & 8 & 3 \end{bmatrix}$$

Compute:

a) $AB$  

b) $A^{-1}$

**Example 3:** Find $a$, $b$, and $c$ in the following matrix equation.

$$\begin{bmatrix} -32 & 56 \\ 3 & a \end{bmatrix} - 4 \begin{bmatrix} 4/11 & 1/11 \\ -3/11 & 2/11 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -7 & 2b+7 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4c & -1 \\ 3 & 5 \\ 2 & -4 \end{bmatrix}$$
Using Inverses to Solve Systems of Equations

If $AX = B$ is a linear system of $n$ equations with $n$ unknowns where $A$ is the coefficient matrix, $X$ is the matrix of unknowns, and $B$ is the constant matrix and if $A^{-1}$ exists, then

$$X = A^{-1}B$$

is the unique solution of the system.

**Example 4:** Solve the following system:

$$z = 4x - y$$
$$3x = 12 - 4z$$
$$y + 3z = 6$$

**Example 5:** Solve the following system:

$$x_1 + x_2 + x_3 = 5$$
$$x_1 - x_2 + x_3 = -3$$
$$x_1 - 2x_2 - x_3 = -1$$
Section 2.7- Leontief Input-Output Model

Example 1: Consider a simple economy consisting of three sectors: food, clothing, and shelter. The production of $1 worth of food requires the consumption of $0.40 worth of food, $0.20 worth of clothing, and $0.25 worth of shelter. The production of $1 worth of clothing requires the consumption of $0.30 worth of food, $0.25 worth of clothing, and $0.30 worth of shelter. The production of $1 worth of shelter requires the consumption of $0.30 worth of food, $0.10 worth of clothing, and $0.10 worth of shelter. Find the level of production for each sector in order to satisfy the demand for $100 million worth of food, $30 million worth of clothing, and $250 million worth of shelter.

(1) Define the input-output matrix $A$, the total output matrix $X$, and the consumer demand matrix $D$.

(2) Set-up and solve the matrix equation:
(3) Find the solution to this problem:

How much of the total production is consumed internally to satisfy the demand?
Example 2: Consider the economy of a small village consisting of two industries, farming and weaving. The farmers produce food for themselves and the weavers, along with extra food for the remaining people. The weavers produce cloth for themselves and the farmers, along with extra cloth for the remaining people. The villagers have found that to produce $1.00 of food, $0.40 of food and $0.10 worth of cloth is needed locally to feed and clothe the farmers. To produce $1.00 of cloth, $0.30 of food and $0.20 of cloth is needed locally to feed and clothe the weavers. The additional food and cloth produced is then available to export to the city. If the city demands $7,200 worth of food and $2,700 worth of cloth each month, how much food and cloth should be produced by the village to meet its own needs and to supply the city?