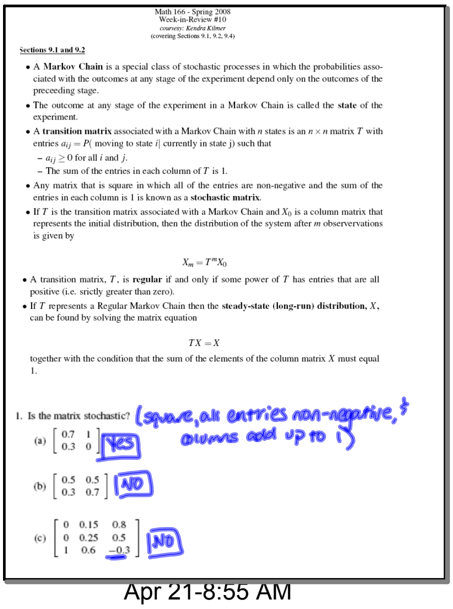


Math 166 - Spring 2008
Week-in-Review #10
courtesy: Kendra Kilmer
(covering Sections 9.1, 9.2, 9.4)

Sections 9.1 and 9.2

- A **Markov Chain** is a special class of stochastic processes in which the probabilities associated with the outcomes at any stage of the experiment depend only on the outcomes of the preceeding stage.
- The outcome at any stage of the experiment in a Markov Chain is called the **state** of the experiment.
- A **transition matrix** associated with a Markov Chain with n states is an $n \times n$ matrix T with entries $a_{ij} = P($ moving to state $i|$ currently in state j) such that
  - $a_{ij} \geq 0$ for all i and j.
  - The sum of the entries in each column of T is 1.
- Any matrix that is square in which all of the entries are non-negative and the sum of the entries in each column is 1 is known as a **stochastic matrix**.
- If T is the transition matrix associated with a Markov Chain and X_0 is a column matrix that represents the initial distribution, then the distribution of the system after m observervations is given by

$$X_m = T^m X_0$$

- A transition matrix, T, is **regular** if and only if some power of T has entries that are all positive (i.e. srictly greater than zero).
- If T represents a Regular Markov Chain then the **steady-state (long-run) distribution,** X, can be found by solving the matrix equation

$$TX = X$$

together with the condition that the sum of the elements of the column matrix X must equal 1.

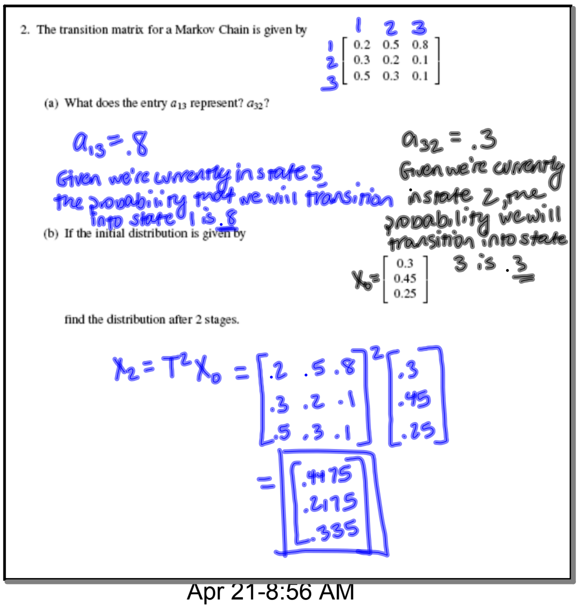
1. Is the matrix stochastic? (square, all entries non-negative, & columns add up to 1)

(a) $\begin{bmatrix} 0.7 & 1 \\ 0.3 & 0 \end{bmatrix}$ $\boxed{\text{Yes}}$

(b) $\begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}$ $\boxed{\text{NO}}$

(c) $\begin{bmatrix} 0 & 0.15 & 0.8 \\ 0 & 0.25 & 0.5 \\ 1 & 0.6 & \underline{-0.3} \end{bmatrix}$ $\boxed{\text{NO}}$

Apr 21-8:55 AM



2. The transition matrix for a Markov Chain is given by
$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.2 & 0.5 & 0.8 \\ 2 & 0.3 & 0.2 & 0.1 \\ 3 & 0.5 & 0.3 & 0.1 \end{array}$$

(a) What does the entry a_{13} represent? a_{32}?

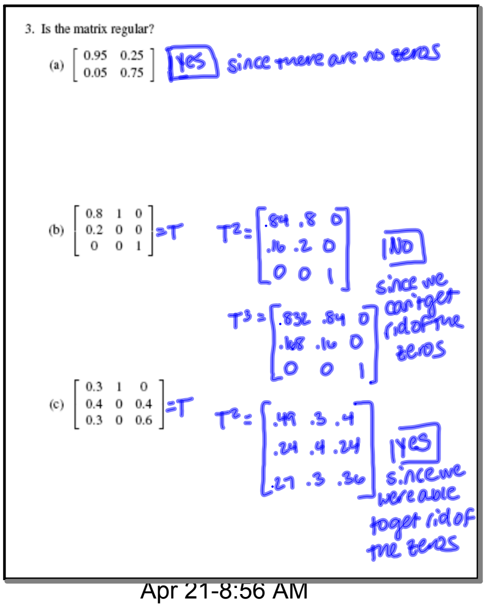
$a_{13} = .8$
Given we're currently in state 3 the probability that we will transition into state 1 is $\underline{.8}$

$a_{32} = .3$
Given we're currently in state 2, the probability we will transition into state 3 is $\underline{\underline{.3}}$

(b) If the initial distribution is given by
$$X_0 = \begin{bmatrix} 0.3 \\ 0.45 \\ 0.25 \end{bmatrix}$$
find the distribution after 2 stages.

$$X_2 = T^2 X_0 = \begin{bmatrix} .2 & .5 & .8 \\ .3 & .2 & .1 \\ .5 & .3 & .1 \end{bmatrix}^2 \begin{bmatrix} .3 \\ .45 \\ .25 \end{bmatrix}$$
$$= \boxed{\begin{bmatrix} .4475 \\ .2175 \\ .335 \end{bmatrix}}$$

Apr 21-8:56 AM



3. Is the matrix regular?

(a) $\begin{bmatrix} 0.95 & 0.25 \\ 0.05 & 0.75 \end{bmatrix}$ $\boxed{\text{Yes}}$ since there are no zeros

(b) $\begin{bmatrix} 0.8 & 1 & 0 \\ 0.2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T$

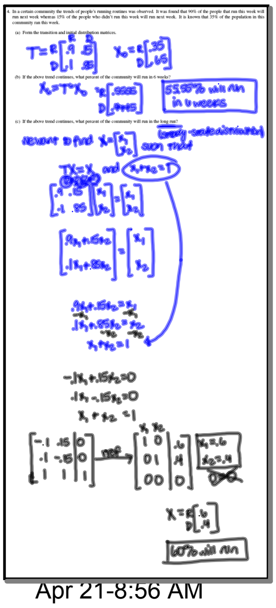
$T^2 = \begin{bmatrix} .84 & .8 & 0 \\ .16 & .2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\boxed{\text{NO}}$ since we can't get rid of the zeros

$T^3 = \begin{bmatrix} .832 & .84 & 0 \\ .168 & .16 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0.3 & 1 & 0 \\ 0.4 & 0 & 0.4 \\ 0.3 & 0 & 0.6 \end{bmatrix} = T$

$T^2 = \begin{bmatrix} .49 & .3 & .4 \\ .24 & .4 & .24 \\ .27 & .3 & .36 \end{bmatrix}$ $\boxed{\text{Yes}}$ since we were able to get rid of the zeros

Apr 21-8:56 AM



4. In a certain community the trends of people's running routines was observed. It was found that 90% of the people that run this week will run next week whereas 15% of the people who didn't run this week will run next week. It is known that 35% of the population in this community ran this week.

(a) Form the transition and initial distribution matrices.

$$T = \begin{array}{c|cc} & R & D \\ \hline R & .9 & .15 \\ D & .1 & .85 \end{array} \qquad X_0 = \begin{array}{c|c} R & .35 \\ D & .65 \end{array}$$

(b) If the above trend continues, what percent of the community will run in 6 weeks?

$X_6 = T^6 X_0 = \begin{array}{c|c} R & .5555 \\ D & .4445 \end{array}$ $\boxed{\text{55.55\% will run in 6 weeks}}$

(c) If the above trend continues, what percent of the community will run in the long run?

We want to find $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that (steady-state distribution)

$TX = X$ and $x_1 + x_2 = 1$

$$\begin{bmatrix} .9 & .15 \\ .1 & .85 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} .9x_1 + .15x_2 \\ .1x_1 + .85x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$.9x_1 + .15x_2 = x_1$
$-x_1 \qquad -x_1$
$.1x_1 + .85x_2 = x_2$
$-x_2 \qquad -x_2$
$x_1 + x_2 = 1$

$-.1x_1 + .15x_2 = 0$
$.1x_1 - .15x_2 = 0$
$x_1 + x_2 = 1$

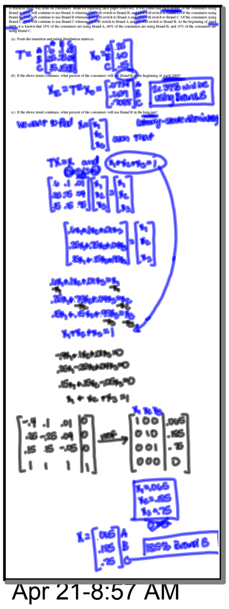
$$\left[\begin{array}{cc|c} -.1 & .15 & 0 \\ .1 & -.15 & 0 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & .6 \\ 0 & 1 & .4 \\ 0 & 0 & 0 \end{array}\right]$$

$\boxed{x_1 = .6 \quad x_2 = .4}$ ~~0=0~~

$X = \begin{array}{c|c} R & .6 \\ D & .4 \end{array}$

$\boxed{\text{60\% will run}}$

Apr 21-8:56 AM



A research study was done on consumers' behavior regarding their paper towel use. It was found that each month, of the consumers using Brand A, [illegible] will continue to use Brand A whereas [illegible] will switch to Brand B, and [illegible] will switch to Brand C. Of the consumers using Brand B, [illegible] will continue to use Brand B whereas [illegible] will switch to Brand A and [illegible] will switch to Brand C. Of the consumers using Brand C, [illegible] will continue to use Brand C whereas [illegible] will switch to Brand A and [illegible] will switch to Brand B. At the beginning of April 2008, it is known that 25% of the consumers are using Brand A, 60% of the consumers are using Brand B, and 15% of the consumers are using Brand C.

(a) Form the transition and initial distribution matrices.

$$T = \begin{array}{c|ccc} & A & B & C \\ \hline A & .6 & .1 & .01 \\ B & .25 & .75 & .04 \\ C & .15 & .15 & .95 \end{array} \qquad X_0 = \begin{array}{c|c} A & .25 \\ B & .60 \\ C & .15 \end{array}$$

(b) If the above trend continues, what percent of the consumers will use Brand B at the beginning of April 2009?

$X_{12} = T^{12} X_0 = \begin{array}{c|c} .0774 & A \\ .2139 & B \\ .7088 & C \end{array}$ $\boxed{\text{21.39\% will be using Brand B}}$

(c) If the above trend continues, what percent of the consumers will use Brand B in the long run?

We want to find $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that (steady-state distribution)

$TX = X$ and $x_1 + x_2 + x_3 = 1$

$$\begin{bmatrix} .6 & .1 & .01 \\ .25 & .75 & .04 \\ .15 & .15 & .95 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} .6x_1 + .1x_2 + .01x_3 \\ .25x_1 + .75x_2 + .04x_3 \\ .15x_1 + .15x_2 + .95x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$.6x_1 + .1x_2 + .01x_3 = x_1$
$-x_1 \qquad -x_1$
$.25x_1 + .75x_2 + .04x_3 = x_2$
$-x_2 \qquad -x_2$
$.15x_1 + .15x_2 + .95x_3 = x_3$
$-x_3 \qquad -x_3$
$x_1 + x_2 + x_3 = 1$

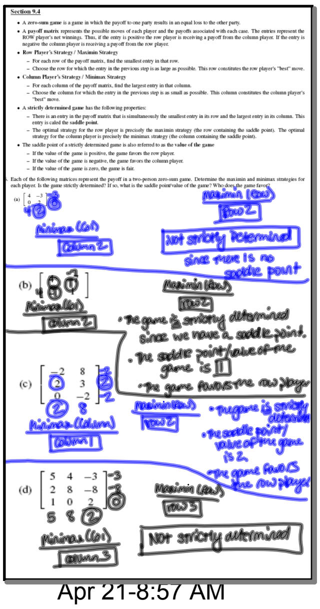
$-.4x_1 + .1x_2 + .01x_3 = 0$
$.25x_1 - .25x_2 + .04x_3 = 0$
$.15x_1 + .15x_2 - .05x_3 = 0$
$x_1 + x_2 + x_3 = 1$

$$\left[\begin{array}{ccc|c} -.4 & .1 & .01 & 0 \\ .25 & -.25 & .04 & 0 \\ .15 & .15 & -.05 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & .065 \\ 0 & 1 & 0 & .185 \\ 0 & 0 & 1 & .75 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{x_1 = .065 \quad x_2 = .185 \quad x_3 = .75}$ ~~0=0~~

$X = \begin{array}{c|c} .065 & A \\ .185 & B \\ .75 & C \end{array}$ $\boxed{\text{18.5\% Brand B}}$

Apr 21-8:57 AM



Section 9.4

- A **zero-sum game** is a game in which the payoff to one party results in an equal loss to the other party.
- A **payoff matrix** represents the possible moves of each player and the payoffs associated with each case. The entries represent the ROW player's net winnings. Thus, if the entry is positive the row player is receiving a payoff from the column player. If the entry is negative the column player is receiving a payoff from the row player.
- **Row Player's Strategy / Maximin Strategy**
  - For each row of the payoff matrix, find the smallest entry in that row.
  - Choose the row for which the entry in the previous step is as large as possible. This row constitutes the row player's "best" move.
- **Column Player's Strategy / Minimax Strategy**
  - For each column of the payoff matrix, find the largest entry in that column.
  - Choose the column for which the entry in the previous step is as small as possible. This column constitutes the column player's "best" move.
- A **strictly determined game** has the following properties:
  - There is an entry in the payoff matrix that is simultaneously the smallest entry in its row and the largest entry in its column. This entry is caled the **saddle point**.
  - The optimal strategy for the row player is precisely the maximin strategy (the row containing the saddle point). The optimal strategy for the column player is precisely the minimax strategy (the column containing the saddle point).
- The saddle point of a strictly determined game is also referred to as the **value of the game**
  - If the value of the game is positive, the game favors the row player.
  - If the value of the game is negative, the game favors the column player.
  - If the value of the game is zero, the game is fair.

5. Each of the following matrices represent the payoff in a two-person zero-sum game. Determine the maximin and minimax strategies for each player. Is the game strictly determined? If so, what is the saddle point/value of the game? Who does the game favor?

(a) $\begin{bmatrix} 4 & -3 \\ 0 & 2 \end{bmatrix} \begin{matrix} -3 \\ ⓪ \end{matrix}$
$4 \quad ②$

Minimax (Col) $\boxed{\text{Column 2}}$

Maximin (Row) $\boxed{\text{Row 2}}$

$\boxed{\text{Not strictly Determined}}$ since there is no saddle point

(b) [illegible]

Minimax (Col) $\boxed{\text{Column 2}}$

Maximin (Row) $\boxed{\text{row 2}}$

- The game $\underline{\underline{\text{is}}}$ strictly determined since we have a saddle point.
- The saddle point/value of the game is $\boxed{1}$
- The game favors the row player

(c) $\begin{bmatrix} -2 & 8 \\ ② & 3 \\ 0 & -2 \end{bmatrix} \begin{matrix} -2 \\ ② \\ -2 \end{matrix}$
$② \quad 8$

Minimax (Column) $\boxed{\text{Column 1}}$

Maximin (Row) $\boxed{\text{row 2}}$

- The game $\underline{\underline{\text{is}}}$ strictly determined
- The saddle point/value of the game is 2.
- The game favors the row player

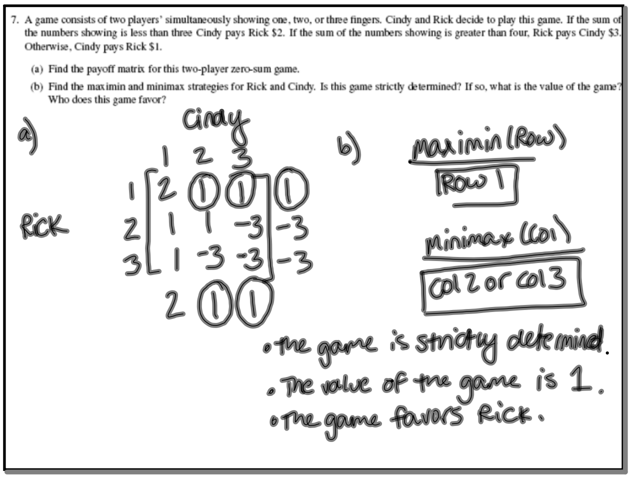
(d) $\begin{bmatrix} 5 & 4 & -3 \\ 2 & 8 & -8 \\ 1 & 0 & 2 \end{bmatrix} \begin{matrix} -3 \\ -8 \\ ⓪ \end{matrix}$
$5 \quad 8 \quad ②$

Minimax (Col) $\boxed{\text{column 3}}$

Maximin (Row) $\boxed{\text{row 3}}$

$\boxed{\text{Not strictly determined}}$

Apr 21-8:57 AM



7. A game consists of two players' simultaneously showing one, two, or three fingers. Cindy and Rick decide to play this game. If the sum of the numbers showing is less than three Cindy pays Rick $2. If the sum of the numbers showing is greater than four, Rick pays Cindy $3. Otherwise, Cindy pays Rick $1.

(a) Find the payoff matrix for this two-player zero-sum game.

(b) Find the maximin and minimax strategies for Rick and Cindy. Is this game strictly determined? If so, what is the value of the game? Who does this game favor?

a)

$$\text{Rick}\quad \begin{array}{c|ccc|c} & \multicolumn{3}{c}{\text{Cindy}} & \\ & 1 & 2 & 3 & \\ \hline 1 & 2 & \text{①} & \text{①} & \text{①} \\ 2 & 1 & 1 & -3 & -3 \\ 3 & 1 & -3 & -3 & -3 \\ \hline & 2 & \text{①} & \text{①} & \end{array}$$

b) maximin (Row)

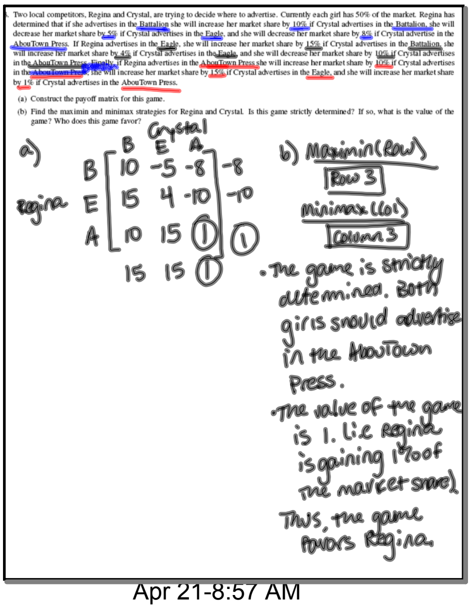
$\boxed{\text{Row 1}}$

minimax (Col)

$\boxed{\text{Col 2 or Col 3}}$

- The game is strictly determined.
- The value of the game is 1.
- The game favors Rick.

Apr 21-8:57 AM



Two local competitors, Regina and Crystal, are trying to decide where to advertise. Currently each girl has 50% of the market. Regina has determined that if she advertises in the Battalion she will increase her market share by 10% if Crystal advertises in the Battalion, she will decrease her market share by 5% if Crystal advertises in the Eagle, and she will decrease her market share by 8% if Crystal advertise in the AbouTown Press. If Regina advertises in the Eagle, she will increase her market share by 15% if Crystal advertises in the Battalion, she will increase her market share by 4% if Crystal advertises in the Eagle, and she will decrease her market share by 10% if Crystal advertises in the AbouTown Press. Finally, if Regina advertises in the AbouTown Press she will increase her market share by 10% if Crystal advertises in the ~~AbouTown Press~~ Battalion, she will increase her market share by 15% if Crystal advertises in the Eagle, and she will increase her market share by 1% if Crystal advertises in the AbouTown Press.

(a) Construct the payoff matrix for this game.

(b) Find the maximin and minimax strategies for Regina and Crystal. Is this game strictly determined? If so, what is the value of the game? Who does this game favor?

a)

$$\text{Regina}\quad \begin{array}{c|ccc|c} & \multicolumn{3}{c}{\text{Crystal}} & \\ & B & E & A & \\ \hline B & 10 & -5 & -8 & -8 \\ E & 15 & 4 & -10 & -10 \\ A & 10 & 15 & \text{①} & \text{①} \\ \hline & 15 & 15 & \text{①} & \end{array}$$

b) Maximin (Row)

$\boxed{\text{Row 3}}$

minimax (Col)

$\boxed{\text{Column 3}}$

- The game is strictly determined. Both girls should advertise in the AbouTown Press.
- The value of the game is 1. (i.e Regina is gaining 1% of the market share).

Thus, the game favors Regina.

Apr 21-8:57 AM