Week-in-Review #2
Math 166 - Spring 2008
courtesy: Kendra Kilmer
(covering Sections 2.4-2.7)

Section 2.4

- A matrix is an ordered rectangular array of numbers. A matrix with m rows and n columns has dimensions $m \times n$.
- Matrix Operations
  - **Addition and Subtraction:** Matrices must have the same dimensions. To find resulting matrix, add or subtract corresponding entries.
  - **Transpose** (A^T)**:** Each row in A becomes a column in A^T.
  - **Scalar Multiplication:** Multiply each entry by the constant.

1. Given A= $\begin{bmatrix} -2 & 8 & 2 & 11 \\ 3 & -7 & 5 & 6 \\ 8 & 2 & -5 & 3 \end{bmatrix}$ What are the dimensions of A? Find a_{34}, a_{12}, and a_{23}.

(3×4)

$a_{34} = 3$

$a_{12} = 8$

$a_{23} = 5$

2. Given $A=\begin{bmatrix} 2 & 1 \\ -5 & 7 \\ 8 & f \end{bmatrix}$ $B=\begin{bmatrix} 1 & 9 & -7 \\ 4 & k & 8 \end{bmatrix}$ Find C where $C = 3A + B^T$.

$$C = 3\begin{bmatrix} 2 & 1 \\ -5 & 7 \\ 8 & f \end{bmatrix} + \begin{bmatrix} 1 & 9 & -7 \\ 4 & k & 8 \end{bmatrix}^T$$

$$= \begin{bmatrix} 6 & 3 \\ -15 & 21 \\ 24 & 3f \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 9 & k \\ -7 & 8 \end{bmatrix} = \boxed{\begin{bmatrix} 7 & 7 \\ -6 & 21+k \\ 17 & 3f+8 \end{bmatrix}}$$

Section 2.5

- To multiply two matrices, the number of columns in the first matrix must equal the number of rows in the second matrix. The resulting matrix will have the same number of rows as the first matrix and the same number of columns as the second matrix.

$(4\times 2)(2\times 5) = 4\times 5$

- Be able to multiply matrices by hand.
- Matrix Multiplication is NOT Commutative!
- The identity matrix is a square matrix with 1's along the main diagonal and 0's everywhere else. Any matrix multiplied by the appropriate identity matrix results in the original matrix.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Perform the following operation by hand:

$$\begin{bmatrix} 1 & -4 \\ 5 & 2 \\ 8 & -1 \\ 12 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 5 \\ 1 & 3 & 4 \end{bmatrix}$$

$(4\times 2)(2\times 3)$

$$= \begin{bmatrix} 1(2)+(-4)(1) & 1(-1)+(-4)(3) & 1(5)+(-4)(4) \\ 5(2)+2(1) & 5(-1)+2(3) & 5(5)+2(4) \\ 8(2)+(-1)(1) & 8(-1)+(-1)(3) & 8(5)+(-1)(4) \\ 12(2)+3(1) & 12(-1)+3(3) & 12(5)+3(4) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -2 & -13 & -11 \\ 12 & 1 & 33 \\ 15 & -11 & 36 \\ 27 & -3 & 72 \end{bmatrix}}$$

4. Given the follwing matrices with the indicated dimensions, which of the following are valid matrix operations?

$$A_{3\times 2}, B_{4\times 2}, C_{2\times 3}, D_{3\times 4}$$

(a) $AB^T - 6D$ VALID $(3\times 2)(2\times 4) - (3\times 4) = (3\times 4) - (3\times 4)$ ✓

(b) $C - DB$ NOT VALID $(2\times 3) - (3\times 4)(4\times 2) = (2\times 3) - (3\times 2)$ X

(c) AC^T NOT VALID $(3\times 2)(3\times 2)$

(d) $5D + 3B$ NOT VALID

$(3\times 4) + (4\times 2)$ X

5. Let $A = \begin{bmatrix} 2 & -7 \\ s & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 9 & -4 \\ -5 & 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 43 & 11 & t \\ -25 & 5 & 15 \end{bmatrix}$ If $AB = C$, find the values of s and t.

$(2\times 2)(2\times 3)$

$$\begin{bmatrix} 2 & -7 \\ s & 5 \end{bmatrix}\begin{bmatrix} 4 & 9 & -4 \\ -5 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 43 & 11 & t \\ -25 & 5 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 2(4)+(-7)(-5) & 2(9)+(-7)(1) & 2(-4)+(-7)(3) \\ s(4)+5(-5) & s(9)+5(1) & s(-4)+5(3) \end{bmatrix} = \begin{bmatrix} 43 & 11 & t \\ -25 & 5 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 43 & 11 & -29 \\ 4s-25 & 9s+5 & -4s+15 \end{bmatrix} = \begin{bmatrix} 43 & 11 & t \\ -25 & 5 & 15 \end{bmatrix}$$

$$\boxed{t=-29}$$
$$\boxed{s=0}$$

$4s-25=-25 \qquad 9s+5=5 \qquad -4s+15=15$

$+25 \quad +25$

$\frac{4s}{4}=\frac{0}{4}$

$s=0$

6. The Crazy Cat Food Company makes three types of cat food: Meow Mix, Kitty Kibbles, and Fishy Flavor. The company produces the cat food in Texas, California, and Wyoming using three main ingredients: fish, chicken, and rice.

(a) Each bag of Meow Mix requires 2 ounces of fish, 3 ounces of chicken, and 1 ounce of rice. Each bag of Kitty Kibbles requires 2 ounces of fish, 2 ounces of chicken and 3 ounces of rice. Each bag of Fishy Flavor requires 3 ounces of fish, 2 ounces of chicken, and 2 ounces of rice. Put this information into a 3 × 3 matrix.

$$A = \begin{array}{c} \\ MM \\ KK \\ FF \end{array}\begin{array}{c} \begin{array}{ccc} F & C & R \end{array} \\ \begin{bmatrix} 2 & 3 & 1 \\ 2 & 2 & 3 \\ 3 & 2 & 2 \end{bmatrix} \end{array}$$

$$\underset{(\text{types of cat food} \times \text{ingred})}{(3 \times 3)} \; \underset{(\text{ingred} \times \text{states})}{(3 \times 3)}$$

(b) The cost of 1 ounce of fish is \$0.75 in Texas, \$0.50 in California, and \$1 in Wyoming. The cost of 1 ounce of chicken is \$0.25 in Texas, \$1.00 in California, and \$0.75 in Wyoming. The cost of 1 ounce of rice is \$0.25 in Texas, \$0.35 in California, and \$0.40 in Wyoming. Put this information into a matrix in such a way that when it is multiplied by the matrix in part **a)** it will tell us the cost of producing a bag of each variety of cat food in each city. Find the resulting product matrix.

$$B = \begin{array}{c} \\ F \\ C \\ R \end{array}\begin{array}{c} \begin{array}{ccc} T & C & W \end{array} \\ \begin{bmatrix} .75 & .5 & 1 \\ .25 & 1 & .75 \\ .25 & .35 & .4 \end{bmatrix} \end{array}$$

$$\boxed{AB = \begin{array}{c} \\ MM \\ KK \\ FF \end{array}\begin{array}{c} \begin{array}{ccc} T & C & W \end{array} \\ \begin{bmatrix} 2.4 & 4.35 & 4.65 \\ 2.75 & 4.05 & 4.7 \\ 3.25 & 4.2 & 5.3 \end{bmatrix} \end{array}}$$

Section 2.6

- If a square matrix A has an inverse then $AA^{-1} = A^{-1}A = I_n$ where I_n is the identity matrix of size n.
- A square matrix that has an inverse is nonsingular. If a square matrix does not have an inverse, then it is singular.
- Be able to solve systems of linear equations using inverses.

7. If $A = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$, find B so that $AB = I_2$. Hint: Let $B = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$

$(2\times 2)(2\times 2)$

$$\begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2c+3e & -2d+3f \\ c+4e & d+4f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$-2c+3e=1$

$c+4e=0$

$c=-4e$

$-2(-4e)+3e=1$

$8e+3e=1$

$11e=1$

$e=1/11$

$c=-4(1/11)=-4/11$

$-2d+3f=0$

$d+4f=1 \Rightarrow d=1-4f$

$-2(1-4f)+3f=0$

$-2+8f+3f=0$

$-2+11f=0$

$\frac{11f}{11}=\frac{2}{11}$

$f=2/11$

$d=1-4\left(\frac{2}{11}\right)$

$=1-\frac{8}{11}=\frac{3}{11}$

Thus, $\boxed{B=\begin{bmatrix} -4/11 & 3/11 \\ 1/11 & 2/11 \end{bmatrix}}$

8. Determine whether each of the following statments is True or False.

(a) It is possible for a 3×4 matrix to have an inverse. FALSE

(b) If a matrix has an inverse, then we say the matrix is nonsingular. TRUE

(c) In solving the matrix equation $AX = B$ which represents a system of linear equations, if the matrix A is singular, we can conclude that the system of linear equations has no solution. FALSE

(d) A matrix mutliplied by the appropriate identity matrix results in the original matrix. TRUE

$AX = B$

If A^{-1} exists

$X = A^{-1}B$

9. Solve the following system of linear equations using inverses.

$$3x - 8y + 9z = 7$$
$$2x + 3y - z = -2$$
$$x - 2y + 4z = 3$$

$AX = B$

$$\begin{array}{ccc} A & X & B \\ \begin{bmatrix} 3 & -8 & 9 \\ 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} & \begin{bmatrix} x \\ y \\ z \end{bmatrix} = & \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} \end{array}$$

If A^{-1} exists

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5/13 \\ -2/13 \\ 10/13 \end{bmatrix}$$

$\boxed{\left(\frac{-5}{13}, \frac{-2}{13}, \frac{10}{13}\right)}$ OR $\boxed{\begin{matrix} x = -5/13 \\ y = -2/13 \\ z = 10/13 \end{matrix}}$

10. ~~At the~~ Austin City Limits Music Festival, a three day event held on Friday, Saturday, ~~and Sunday, YoSoy Candle Company managed to sell $2400 worth of candles~~. ~~Amazingly, the sales made Sunday ended up being the same as the combined sales for Friday and Saturday~~. ~~Even more astonishing, the sales made Sunday were three times the amount of sales made Friday~~. What were YoSoy's total sales for each day of the festival? (Use inverses to solve the system)

x = total sales on Friday (in $)
y = total sales on Saturday (in $)
z = total sales on Sunday (in $)

$$x+y+z=2400$$
$$z=x+y$$
$$z=3x$$

$$\begin{aligned} x+y+z&=2400\\ -x-y+z&=0\\ -3x\quad\;\; +z&=0 \end{aligned}$$

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1\\ -1 & -1 & 1\\ -3 & 0 & 1 \end{bmatrix}}\underset{X}{\begin{bmatrix} x\\ y\\ z \end{bmatrix}}=\underset{B}{\begin{bmatrix} 2400\\ 0\\ 0 \end{bmatrix}}$$

If A^{-1} exists

$$X=A^{-1}B$$

$$\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} 400\\ 800\\ 1200 \end{bmatrix}$$

$400 of sales on Fri
$800 of sales on Sat
$1200 of sales on Sun

11. Consider a simple economy consisting of three sectors: food, clothing, and shelter. The production of 1 unit of food requires the consumption of 0.2 unit of food, 0.3 unit of clothing, and 0.1 unit of shelter. The production of 1 unit of clothing requires the consumption of 0.2 unit of food, 0.3 unit of clothing, and 0.2 unit of shelter. The production of 1 unit of shelter requires the consumption of 0.2 unit of food, 0.2 unit of clothing, and 0.1 unit of shelter.

(a) Find the level of production for each sector in order to satisfy the demand for \$150 million worth of food, \$40 million worth of clothing, and \$350 million worth of shelter.

$X = (I-A)^{-1}D \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 413.1443 \\ 382.7320 \\ 519.8454 \end{bmatrix}$

We should produce
\$413.1443 million of food
\$382.7320 " " clothing
\$519.8454 " " shelter

(b) How much of each sector is consumed internally in satisfying the above demand?

$X-D$ or $AX = \begin{bmatrix} 263.1443 \\ 342.7320 \\ 169.8454 \end{bmatrix}$

① Form 3 matrices: A, X, D

A ← input-output matrix
X ← total production matrix
D ← demand matrix

output

$$A = \begin{array}{c|ccc} & F & C & S \\ \hline F & .2 & .2 & .2 \\ C & .3 & .3 & .2 \\ S & .1 & .2 & .1 \end{array}$$

input

x = amount of food we produce (in millions of \$)
y = " " clothing " " "
z = " " shelter " " "

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad D = \begin{bmatrix} 150 \\ 40 \\ 350 \end{bmatrix}$$

② Set-up and solve the equation

Amount we produce − Amount we consume = Demand

$$X - AX = D$$

$$\underbrace{(I-A)^{-1}(I-A)}X = (I-A)^{-1}D$$

$$IX = (I-A)^{-1}D$$

$$\boxed{X = (I-A)^{-1}D}$$

$x - ax = d$
$x(1-a) = d$
$x = \frac{d}{1-a}$

$$\overset{A}{\begin{array}{c|ccc} & F & C & S \\ \hline F & .2 & .2 & .2 \\ C & .3 & .3 & .2 \\ S & .1 & .2 & .1 \end{array}} \overset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} .2x+.2y+.2z \\ .3x+.3y+.2z \\ .1x+.2y+.1z \end{bmatrix}$$

Ans to (b)

Internal Consumption
\$263.1443 million of food
\$342.7320 " " clothing
\$169.8454 " " shelter