

ON THE ORTHOGONALITY OF FRAMES AND THE DENSITY AND CONNECTIVITY OF WAVELET FRAMES

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ABSTRACT. We examine some recent results of Bownik on density and connectivity of the wavelet frames. We use orthogonality (strong disjointness) properties of frame and Bessel sequences, and also properties of Bessel multipliers (operators that map wavelet Bessel functions to wavelet Bessel functions). In addition we obtain an asymptotically tight approximation result for wavelet frames.

1. INTRODUCTION

This article is motivated by Bownik's recent solutions to two problems posed by the second author on the density and connectivity of the wavelet frames. We examine these results from the point of view of orthogonality (also called strong disjointness) of frame and Bessel sequences, and the corresponding properties for generators of wavelet frame and wavelet Bessel sequences. Along the way we obtain a new approximation result. A sequence of wavelet frames $\{\psi_n\}$ is called *asymptotically tight* if $\lim_{n \rightarrow \infty} \frac{B_n}{A_n} = 1$, where B_n and A_n are the upper and lower frame bounds of ψ_n . We show that every function in $L^2(\mathbb{R})$ is the limit of an asymptotically tight sequence of wavelet frames.

About twelve years ago [36] the second author raised the question of whether the set of all Riesz wavelets for the dyadic wavelet system on $L^2(R)$ is a norm-dense path-wise connected subset of $L^2(R)$. This is related to the problem that was posed earlier by Guido Weiss and his group [31, 32] and independently by Dai and Larson ([11], problem 1) of whether the set of all orthonormal dyadic wavelets is connected. Neither conjecture has been settled to date, although it has been shown [44] that the set of *MRA* orthonormal dyadic wavelets is connected. Shortly after this, the two authors developed an operator-theoretic approach to frame theory in [27], and the same problems (density and connectivity) were posed for wavelet frames (also called frame wavelets or framelets). This density problem was not posed formally in [27] but was alluded to in that memoir, and also in the semi-expository papers [37] and [39], and it was finally posed formally (along with some other related problems) in [38]. In [27] it was pointed out that for the Gabor-frame case, and for the group-frame case (i.e. frame sequences generated by the action of a unitary group on a single generator vector), both the density and the connectivity problems have positive solutions. This follows immediately from the parametrization theorem for frame vectors. It was later

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proven that the parametrization result presented in [27] is also valid for the more general *projective* unitary group representations (cf. [16, 18, 19, 25]). These systems were also called *group-like* systems, and they include the Gabor systems as special cases. The frame density and connectivity problems also have positive solutions for these systems.

There has been steady progress on the connectivity and density problems for special classes of wavelets and wavelet frames, including the MSF (s-elementary) and MRA (multiresolution analysis) wavelets and their analogues for wavelet frames and especially Parseval wavelets. The interested reader should refer to (c.f. [7, 9, 10, 20, 21, 22, 29, 34, 40, 42, 44]) for details and exposition of these results. However, the general connectivity and density problems for the set of all wavelet frames remained open until recently Marcin Bownik [5] settled both problems affirmatively. His argument was clever, elegant, and short. In fact Bownik proved the density of wavelet frames not only in the Hilbert space norm but also in another natural logmodular norm associated with wavelet theory (see [41] and [23] for the definition of this norm). While the general connectivity and density problems are settled for the case of wavelet **frames**, they are still open for the case of wavelets.

In this paper, we examine the density and connectedness results obtained in [5]. We observed that the essential ingredient that *makes the proofs work* seems to be a very clever use of *frame-orthogonality (or strong-disjointness)*, a concept that arose simultaneously and independently to Balan [2] and the authors some time ago [27]. This is a natural geometric concept in frame theory which was formally introduced and studied in [2, 27], and it has proven useful for developing the theory of frames and its applications (cf. [1, 2, 3, 4, 12, 13, 14, 15, 26, 27, 30, 35, 43]). We note that Bownik does not explicitly use this term, or this property, in his argument; it is just that we have observed that the essential reason the argument works seems to involve the orthogonality concept. The point of this paper is to try to make this observation *crystal-clear*, to borrow a phrase of Kadison. The key observation is that if a wavelet Bessel function f is strongly disjoint with a wavelet frame ψ , then $f + t\psi$ is a wavelet frame for all $t \neq 0$ and hence f is the limit of a sequence of wavelet frames. This allows us to obtain a new type of approximation result for wavelet Bessel sequence generators. And we also pose some new questions.

A *frame* for a Hilbert space H is a sequence of vectors $\{f_n\}$ in H such that there exist positive constants A and B such that

$$(1.1) \quad A\|f\|^2 \leq \sum_{n \in \mathbb{N}} |\langle f, f_n \rangle|^2 \leq B\|f\|^2$$

holds for every $f \in H$, and we call the optimal constants A and B the lower frame bound and the upper frame bound, respectively. A *tight frame* refers to the case when $A = B$, and a *Parseval frame* refers to the case when $A = B = 1$. If we only require the right-hand side of the inequality (1.1), then $\{x_n\}$ is called a *Bessel sequence*. The *analysis operator* Θ for a Bessel sequence is defined by

$$\Theta(f) = \sum_{n \in \mathbb{N}} \langle f, f_n \rangle e_n, \quad f \in H,$$

where $\{e_n\}$ is the standard orthonormal basis for the $\ell^2(\mathbb{N})$ -sequence space. It is easy to verify that $\Theta^*\Theta = \sum_n f_n \otimes f_n$, where the convergence is in the strong operator topology (*SOT* for short), and $f \otimes g$ is the elementary tensor rank-one operator defined by $(f \otimes g)(h) =$

$\langle h, g \rangle f$. Moreover, it is true that a sequence $\{f_n\}$ is a frame (resp. a Bessel sequence) if and only if $\sum_n f_n \otimes f_n$ is *SOT*-convergent to a bounded invertible operator (resp. bounded operator) on H .

Let Θ_1 and Θ_2 be the analysis operators for Bessel sequences $\{f_n\}$ and $\{g_n\}$, respectively. We say that the two Bessel sequences are *strongly disjoint (or orthogonal)* if the two range spaces $\text{ran}(\Theta_1)$ and $\text{ran}(\Theta_2)$ are orthogonal. Clearly, $\{f_n\}$ and $\{g_n\}$ are strongly disjoint if and only if

$$\sum_{n \in \mathbb{N}} \langle f, f_n \rangle \langle g_n, g \rangle = 0$$

for all $f, g \in H$. Two Parseval frames $\{x_i\}, \{y_i\}$ are strongly disjoint if and only if the inner direct sum $\{x_i \oplus y_i\}$ is also a Parseval frame, and this was the basis for defining strong disjointness in [27], where it was shown that the strongly disjointness of two frames is equivalent to the orthogonality of the ranges of the analysis operators for the two frames.

A special class of frames that has been extensively studied is the class of wavelet frames. Let $H = L^2(\mathbb{R})$, and let T, D be the translation and dilation unitary operators on $L^2(\mathbb{R})$ defined $(Tf)(t) = f(T^{-1}t)$, $(Df)(t) = \sqrt{2}f(2t)$. A *wavelet frame* is a function $\psi \in L^2(\mathbb{R})$ such that the *affine system* $\{D^n T^\ell \psi\}_{n, \ell \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Tight wavelet frames and Parseval wavelet frames are defined similarly as generators of tight or Parseval (resp.) frame sequences under the action of the affine system. In the case that $\{D^n T^\ell \psi\}_{n, \ell \in \mathbb{Z}}$ is a Bessel sequence, we say that ψ is a *wavelet Bessel function*. If $\{D^n T^\ell \psi\}_{n, \ell \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, then ψ is called an (*orthonormal*) *wavelet*.

We will show that every function in $L^2(\mathbb{R})$ is a limit of an asymptotically tight sequence of wavelet frames. This extends to arbitrary expansive wavelet systems on $L^2(\mathbb{R}^n)$. (See Section 3). This property can be false for group-frames (see Example 1 in Section 3 for a Fourier type system).

2. BESSEL MULTIPLIERS

Let \mathcal{F} denote the Fourier transform defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi t} f(t) dt, \quad f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

This operator is isometric and so can be extended to a unitary operator on $L^2(\mathbb{R})$. Moreover, we have $\hat{T} := \mathcal{F}T\mathcal{F}^{-1} = M_{e^{-i\xi}}$ and $\hat{D} := \mathcal{F}D\mathcal{F}^{-1} = D^{-1}$.

By a *Fourier Bessel multiplier* for the wavelet system we mean an L^∞ -function h such that the inverse Fourier transform of the multiplication operator M_h maps every wavelet Bessel function to a wavelet Bessel function. More generally, a *Bessel multiplier operator* for the wavelet system is a bounded linear operator B on $L^2(\mathbb{R})$ such that B maps wavelet Bessel functions to wavelet Bessel functions.

The family of Fourier Bessel Multipliers is quite rich, as the following proposition and the remark following it show. The set of all the Fourier Bessel Multipliers the wavelet system is clearly a linear subspace and is closed under pointwise multiplication. So it is a function algebra. We do not have a complete characterization of it, however, and we leave that as an open problem. See also the remark after the proof of the next proposition, where we

indicate some other Fourier Bessel multipliers. The following proposition will be sufficient for our purpose in this article.

Proposition 2.1. If $h \in L^\infty(\mathbb{R})$ is either 2π - translation-periodic or 2- dilation-periodic, then h is a Fourier Bessel multiplier.

Proof. First assume that h is 2π -translation periodic. Let f and g be arbitrary compactly supported bounded functions. Then we have

$$\sum_{\ell \in \mathbb{Z}} \langle f, \hat{T}^\ell \hat{\psi} \rangle \langle \hat{T}^\ell \hat{\psi}, g \rangle = \int_{[0, 2\pi]} \left(\sum_{k \in \mathbb{Z}} \overline{f(\xi - 2k\pi)} \hat{\psi}(\xi - 2k\pi) \cdot \sum_{k \in \mathbb{Z}} g(\xi - 2k\pi) \overline{\hat{\psi}(\xi - 2k\pi)} \right) d\xi.$$

Replacing both f and g by $\hat{D}^{-n}f$, we obtain

$$\sum_{\ell \in \mathbb{Z}} |\langle \hat{D}^{-n}f, \hat{T}^\ell \hat{\psi} \rangle|^2 = 2^n \int_{[0, 2\pi]} \left| \sum_{k \in \mathbb{Z}} \overline{f(2^n(\xi - 2k\pi))} \hat{\psi}(\xi - 2k\pi) \right|^2 d\xi,$$

and so

$$\sum_{n \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} |\langle f, \hat{D}^{-n} \hat{T}^\ell \hat{\psi} \rangle|^2 = \sum_{n \in \mathbb{Z}} 2^n \int_{[0, 2\pi]} \left| \sum_{k \in \mathbb{Z}} \overline{f(2^n(\xi - 2k\pi))} \hat{\psi}(\xi - 2k\pi) \right|^2 d\xi.$$

Assume that $|h(\xi)| \leq K$ and let $\hat{\varphi}(\xi) = h(\xi)\hat{\psi}(\xi)$. Replace ψ by φ in the above equality and use the π -periodic property of h , we obtain

$$\begin{aligned} \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}} |\langle f, \hat{D}^{-n} \hat{T}^\ell \hat{\varphi} \rangle|^2 &= \sum_{n \in \mathbb{Z}} 2^n \int_{[0, 2\pi]} \left| \sum_{k \in \mathbb{Z}} \overline{f(2^n(\xi - 2k\pi))} h(\xi - 2k\pi) \hat{\psi}(\xi - 2k\pi) \right|^2 d\xi \\ &= \sum_{n \in \mathbb{Z}} 2^n \int_{[0, 2\pi]} |h(\xi)|^2 \left| \sum_{k \in \mathbb{Z}} \overline{f(2^n(\xi - 2k\pi))} \hat{\psi}(\xi - 2k\pi) \right|^2 d\xi \\ &\leq K^2 \sum_{n \in \mathbb{Z}} 2^n \int_{[0, 2\pi]} \left| \sum_{k \in \mathbb{Z}} \overline{f(2^n(\xi - 2k\pi))} \hat{\psi}(\xi - 2k\pi) \right|^2 d\xi \\ &= K^2 \sum_{n \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} |\langle f, \hat{D}^{-n} \hat{T}^\ell \hat{\psi} \rangle|^2 \end{aligned}$$

Thus φ is Bessel, as claimed.

For the case when h is a 2 -periodic L^∞ function, the statement follows immediately from the fact that the multiplication operator M_h commutes with both \hat{D} and \hat{T} [11]. \square

Remark 1. There are many other Fourier Bessel multipliers. For example, any function h satisfying one of the following two conditions is a Fourier Bessel multiplier.

(i) $h \in L^\infty(\mathbb{R})$ with the property that $h(\xi) = \mathcal{O}(|\xi|^\delta)$ as $\xi \rightarrow 0$ and $h(\xi) = \mathcal{O}(|\xi|^{-1/2-\delta})$ as $|\xi| \rightarrow \infty$ (In this case the statement that h is a Fourier Bessel multiplier follows from Theorem 10.0.1 in [33] and the fact that all the Bessel functions are $L^\infty(\mathbb{R})$ -functions).

(ii) $h(\xi)$ and $1/h(\xi)$ are bounded, and $h(\xi)/h(2\xi)$ is 2π -periodic (We will not include the proof for this case since it will not be used in this paper. This will be included in a subsequent paper, where we develop the theory of these operators for wavelets and more general unitary systems).

3. ASYMPTOTIC TIGHT FRAME SEQUENCE APPROXIMATION

We require some results that involve strongly disjoint wavelet Bessel functions. For wavelet systems, there is a general equation characterization for strongly disjoint Bessel functions [43]. In this paper we will only need a simple sufficient condition (Lemma 3.2) which is given in terms of the support of the Fourier transform of Bessel functions. Although this sufficient condition can be easily derived from the general characterization of Eric Weber, we provide an elementary proof for self-completeness of the paper.

Two measurable subsets E, F of \mathbb{R} are called *essentially disjoint* if $E \cap F$ is a Lebesgue measure zero set. For any set $E \subseteq \mathbb{R}$, we write $\tau(E) = \cup_{n \in \mathbb{Z}} (E + 2n\pi)$. We say that E and F are *translation disjoint* if $\tau(E)$ and $\tau(F)$ are essentially disjoint. By a *Bessel shift sequence* we mean a Bessel sequence of the form $\{\psi(\cdot - \ell)\}_{\ell \in \mathbb{Z}}$. The following is well-known and we insert a proof for the reader's convenience.

Lemma 3.1. Two Bessel shift sequences $\{\psi(x - \ell)\}_{\ell \in \mathbb{Z}}$ and $\{\varphi(x - \ell)\}_{\ell \in \mathbb{Z}}$ are strongly disjoint if and only if $\text{supp}(\hat{\psi})$ and $\text{supp}(\hat{\varphi})$ are translation disjoint.

Proof. The two sequences are strongly disjoint if and only if

$$\sum_{\ell \in \mathbb{Z}} \langle f, e^{-i\ell\xi} \hat{\varphi} \rangle \langle e^{-i\ell\xi} \hat{\psi}, g \rangle = 0, \quad \forall f, g \in L^2(\mathbb{R}).$$

A standard argument shows that this is equivalent to

$$0 = \sum_{\ell \in \mathbb{Z}} \left(\int_{[0, 2\pi]} e^{i\ell\xi} \overline{F(\xi)} d\xi \right) \cdot \left(\int_{[0, 2\pi]} e^{-i\ell\xi} G(\xi) d\xi \right),$$

where $F(\xi) = \sum_{k \in \mathbb{Z}} \overline{f(\xi - 2k\pi)} \hat{\varphi}(\xi - 2k\pi)$ and $G(\xi) = \sum_{k \in \mathbb{Z}} \overline{g(\xi - 2k\pi)} \hat{\psi}(\xi - 2k\pi)$. This says that the Fourier coefficient sequences for F and G are orthogonal, so F is orthogonal to G in $L^2[0, 2\pi]$. Equivalently,

$$0 = \int_{[0, 2\pi]} \left(\sum_{k \in \mathbb{Z}} \overline{f(\xi - 2k\pi)} \hat{\varphi}(\xi - 2k\pi) \right) \cdot \sum_{k \in \mathbb{Z}} g(\xi - 2k\pi) \overline{\hat{\psi}(\xi - 2k\pi)} d\xi.$$

This is trivially satisfied if $\tau(\text{supp}(\hat{\psi}))$ and $\tau(\text{supp}(\hat{\varphi}))$ are essentially disjoint. Conversely, if this integral is 0 for all f, g , we can fix k and j , and set

$$f = \chi_{[2k\pi, 2(k+1)\pi]} \cdot \text{sgn}(\hat{\varphi}), \quad g = \chi_{[2j\pi, 2(j+1)\pi]} \cdot \text{sgn}(\hat{\psi}).$$

So we have

$$\int_{[0, 2\pi]} |\hat{\varphi}(\xi - 2k\pi)| \cdot |\hat{\psi}(\xi - 2k\pi)| d\xi$$

for all k, j . It follows that $\tau(\text{supp}(\hat{\psi}))$ and $\tau(\text{supp}(\hat{\varphi}))$ are essentially disjoint. \square

The above strongly disjointness condition for two Bessel shift sequences is not necessary for the corresponding *affine* sequences (indexed by $\mathbb{Z} \times \mathbb{Z}$) to be strongly disjoint. However, it is *sufficient* for strongly disjointness of affine sequences, which is enough for our purpose.

Lemma 3.2. Let $\psi, \phi \in L^2(\mathbb{R})$ be two wavelet Bessel functions, and let $E = \text{supp}\hat{\psi}$ and $F = \text{supp}\hat{\phi}$. If $\tau(E)$ and $\tau(F)$ are disjoint, then ψ and ϕ are strongly disjoint, i.e., $\{D^n T^\ell \psi\}_{n, \ell \in \mathbb{Z}}$ and $\{D^n T^\ell \phi\}_{n, \ell \in \mathbb{Z}}$ are strongly disjoint.

Proof. By Lemma 3.1 we have that the “partial” Bessel sequences $\{\psi(x-\ell)\}_{\ell \in \mathbb{Z}}$ and $\{\varphi(x-\ell)\}_{\ell \in \mathbb{Z}}$ are strongly disjoint. So

$$\sum_{\ell \in \mathbb{Z}} \langle f, T^\ell \varphi \rangle T^\ell \psi = 0, \quad \forall f \in L^2(\mathbb{R}).$$

For each fixed integer n , replace f by $D^{-n}f$ and apply D^n to both sides of the above equality we get

$$\sum_{\ell \in \mathbb{Z}} \langle f, D^n T^\ell \varphi \rangle D^n T^\ell \psi = 0, \quad \forall f \in L^2(\mathbb{R}),$$

and so

$$\sum_{n \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \langle f, D^n T^\ell \varphi \rangle D^n T^\ell \psi = 0, \quad \forall f \in L^2(\mathbb{R}).$$

Therefore ψ and φ are strongly disjoint. \square

Remark 2. As mentioned above, the translation disjointness condition on $\text{supp}\hat{\psi}$ and $\text{supp}\hat{\phi}$ is not necessary in general for the strong disjointness (orthogonality) of the two Bessel sequences $\{\psi_{k,\ell}\}$ and $\{\phi_{k,\ell}\}$. However, the condition will become necessary if we assume that one of these two Bessel sequences is *semi-orthogonal*. For example, assume that $\{\psi_{k,\ell}\}$ is semi-orthogonal, i.e., $\psi_{k,\ell} \perp \psi_{k',\ell'}$ whenever $k \neq k'$. Then the orthogonality definition

$$\sum_{k \in \mathbb{Z}, l \in \mathbb{Z}} \langle f, D^k T^l \phi \rangle D^k T^l \psi = 0 \quad (\forall f \in L^2(\mathbb{R}))$$

and the semi-orthogonality condition of the Bessel sequence $\{\psi_{k,\ell}\}$ imply that

$$\sum_{l \in \mathbb{Z}} \langle f, D^k T^l \phi \rangle D^k T^l \psi = 0 \quad (\forall f \in L^2(\mathbb{R}), k \in \mathbb{Z}).$$

In particular, we have $\sum_{l \in \mathbb{Z}} \langle f, T^l \phi \rangle T^l \psi = 0 \quad (\forall f \in L^2(\mathbb{R}))$, which implies that $\text{supp}\hat{\psi}$ and $\text{supp}\hat{\phi}$ are translation disjoint.

We also need the following proposition. This was proven in [27] for the case of two disjoint (a condition that is weaker than the strongly disjointness) general frames for the same Hilbert space, and so in particular, it holds for strongly disjoint frames. The proof goes through in fact for the case when one is a frame and one is a Bessel sequence. The proof for wavelet system case is simple, and so we include it for completeness.

Proposition 3.3. Assume that ψ is wavelet frame and g is Bessel. If ψ and g are strongly disjoint, then $\varphi = \psi + g$ is also a wavelet frame. Moreover, if ψ has upper and lower frame bounds A and B , respectively, and g has a Bessel bound C , then φ has a lower frame bound A , and upper frame bound $B + C$.

Proof. By the strong disjointness of ψ and g , we have for any $f \in L^2(\mathbb{R})$ that

$$\begin{aligned} \sum_{j,k \in \mathbb{Z}} |\langle f, \varphi_{jk} \rangle|^2 &= \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{jk} \rangle + \langle f, g_{jk} \rangle|^2 \\ &= \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{jk} \rangle|^2 + \sum_{j,k \in \mathbb{Z}} |\langle f, g_{jk} \rangle|^2. \end{aligned}$$

Thus we get

$$A\|f\|^2 \leq \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{jk} \rangle|^2 \leq \sum_{j,k \in \mathbb{Z}} |\langle f, \varphi_{jk} \rangle|^2 \leq B\|f\|^2 + C\|f\|^2.$$

So $\varphi = \psi + g$ is a wavelet frame with the claimed frame bounds. \square

Now we prove that every L^2 -function can be approximated by almost tight wavelet frames (in the L^2 -norm).

Theorem 3.4. Every function $f \in L^2(\mathbb{R})$ is a limit of an asymptotic tight sequence of wavelet frames.

Proof. Since the set of all wavelet Bessel functions is dense in $L^2(\mathbb{R})$, it suffices to show that for any wavelet Bessel function ψ with Bessel bound $D > 0$ and any given $\epsilon > 0$ and $A > 0$, there exists a wavelet frame η with frame bounds in the interval $[A, A + D]$ such that

$$\|\psi - \eta\| < \epsilon.$$

Choose a small neighborhood G of 0 such that $\hat{\psi}|_{\tau(G)}$ has norm less than $\frac{\epsilon}{2}$. Let $E = \tau(G)^c$. Then, by the Proposition 2.1, $\hat{\psi}\chi_E$ and $\hat{\psi}\chi_{\tau(G)}$ are Bessel functions. Since their support are 2π -translation disjoint, we get by Proposition 3.2 that $\hat{\psi}\chi_E$ and $\hat{\psi}\chi_{\tau(G)}$ are strongly disjoint Bessel functions.

Assume that D_1 is the Bessel bound for $\hat{\psi}\chi_E$ and D_2 is the Bessel bound for $\hat{\psi}\chi_{\tau(G)}$. Then, from Proposition 3.3, since these two functions are strongly disjoint and add to $\hat{\psi}$ we have $D \leq D_1 + D_2$ and $D_1, D_2 \leq D$.

Let $N \geq 1$ be any positive integer such that

$$2^{-N}([-\pi, -\pi/2] \cup [\pi/2, \pi]) \subset G.$$

Then for any $n \geq N$ the function φ_n with

$$\hat{\varphi}_n(\xi) = \frac{1}{\sqrt{2\pi}} \chi_{2^{-n}([-\pi, -\pi/2] \cup [\pi/2, \pi])}$$

is a Parseval wavelet frame, and $\|\varphi_n\| = 2^{-n}$.

Let $A > 0$ be any positive number, and choose $n > N$ large enough so that $A2^{-n} < \epsilon/2$. Let $\sigma = A\varphi_n$. Then σ is a tight wavelet frame with frame bound A , and $\|\sigma\| \leq \epsilon/2$. Moreover, $\hat{\psi} \cdot \chi_E$ and $\hat{\sigma}$ are strongly disjoint. So

$$\hat{\eta} := \hat{\psi} \cdot \chi_E + \hat{\sigma}$$

is a wavelet frame. By Proposition 3.3, η has the lower frame bound no smaller than A , and the upper frame bound no bigger than $A + D$. We also have

$$\begin{aligned} \|\psi - \eta\| &= \|\hat{\psi} - (\hat{\psi} \cdot \chi_E + \hat{\sigma})\| \\ &= \|\hat{\psi}|_{\tau(G)} - \hat{\sigma}\| \\ &\leq \|\hat{\psi}|_{\tau(G)}\| + \|\hat{\sigma}\| < \epsilon. \end{aligned}$$

\square

In many cases, frames associated with group structures have better topological properties than wavelet frames. However this is not true in general. The following example shows that the above theorem fails for Fourier frames:

Example 1. Let $H = L^2(E)$ where E is a subset of $[0, 2\pi)$ with positive measure. Let $G = \mathbb{Z}$, and U be the multiplication unitary operator by e^{it} . Then $\{U^n f\}_{n \in \mathbb{Z}}$ is a frame if and only if both f and f^{-1} are $L^\infty(E)$ -functions (Here f^{-1} denotes the reciprocal of f on E). Moreover, the frame bounds are given by $A = \|f\|_{L^\infty(E)}$ and $B = \|f^{-1}\|_{L^\infty(E)}$. We usually refer f as a Fourier frame generator. Now assume that $\{\psi_n\}$ are Fourier frame generators with frame bounds A_n and B_n such that $\|\psi_n - g\| \rightarrow 0$ and $\lim_{n \rightarrow \infty} \frac{B_n}{A_n} = 1$. Since $A_n \mu(E) \leq \|\psi_n\|^2 \leq B_n \mu(E)$, we get

$$\frac{A_n}{B_n} \mu(E) \leq \frac{\|\psi_n\|^2}{B_n} \leq \mu(E).$$

So, by the conditions $\lim_{n \rightarrow \infty} \frac{B_n}{A_n} = 1$ and $\lim_{n \rightarrow \infty} \|\psi_n\|^2 = \|g\|^2$, we have $\{B_n\}$ is a bounded sequence. Similarly we can show that $\{A_n\}$ is a sequence bounded from below. Therefore $\{U^n g\}$ is a frame, and hence both g and g^{-1} are $L^\infty(E)$ -functions. Thus the above theorem fails for Fourier frames.

We remark that similar argument can be applied to show that this is also true for frames of the form $\{U\xi : U \in \mathcal{U}\}$, where \mathcal{U} is an abelian group of unitary operators on a Hilbert space H .

Theorem 3.4 suggests the question of whether the set of all tight wavelet frames is dense in $L^2(\mathbb{R})$. When we restrict ourselves to the set of MRA wavelet frames, then the this question has an negative answer. In fact, a much stronger result is proved in [29]: *The set of all MRA (not necessarily tight) wavelet frames is nowhere dense in $L^2(\mathbb{R})$.* The next example shows that the answer to the question on density of tight frames is negative if we replace wavelet frames by frames induced by infinite dimensional projective unitary representations of groups (these include Gabor frames, and frames obtained from group representations). We refer to [26] for the definitions of projective unitary representations.

In an early version of the present article we formally posed the question ‘‘Is the set of all tight wavelet frames dense in $L^2(\mathbb{R})$?’’ In response to our question, recently Bownik proved that the answer is negative by showing that every function in the norm closure of the set of tight wavelet frames satisfies the second equation in the equation-characterization of the tight wavelet frames. His paper will appear in the same special volume as the present article.

Example 2. Let π be a projective unitary representation of a countable group G on an infinite dimensional Hilbert space H . A (tight) frame generator is a vector ξ such that $\{\pi(g)\xi\}_{g \in G}$ is (tight) frame for H . Then the set of all tight frame generators is not dense in H . In fact we will prove that the limit of a sequence of tight frame vectors must be a Bessel vector, which will lead to a contradiction since not every vector is Bessel for an infinite dimensional projective unitary representation.

Fix a Parseval frame vector ψ for a projective unitary representation of a countable group G . Let $\{\psi_n\}$ be a sequence of tight frame vectors for π with the tight frame bound B_n such that $\|\psi_n - \eta\| \rightarrow 0$. To show that η is Bessel, it suffice to prove that $\{B_n\}$ is a bounded

sequence. In fact, by the parametrization result of tight frame vectors (c.f. [25]), for each n there exists a unitary operator U_n in the von Neumann algebra generated by $\pi(G)$ such that $\psi_n = \sqrt{B_n}U_n\psi$. So $B_n\|\psi\| = \|\psi_n\| \rightarrow \|\eta\|$. This implies that $\{B_n\}$ is convergent and hence bounded. Thus η is Bessel.

We remark that the statement in the above example is false when π is a finite dimensional projective unitary representation. For instance, assume that π is an irreducible unitary representation of a finite group G on a finite dimensional Hilbert space H . Then every vector $x \in H$ is a Bessel vector for π . Let $x \in H$ be any non-zero vector and Θ_x be the analysis operator for $\{\pi(g)x : g \in G\}$. Then the frame operator $S = \Theta_x^*\Theta_x$ commutes with $\pi(G)$ and hence $S = \alpha I$ for some $\alpha \neq 0$ since π is irreducible. Hence every nonzero vector x is a tight frame vector for π , and therefore set of all tight frame vectors for π is dense in H . However, it can be shown that this (the irreducibility of π) is the only case when the set of all tight frame vectors for a finite dimensional projective unitary representation π is dense in H .

Using Proposition 3.3 we also provide a slightly more constructive proof for the connectivity result.

Proposition 3.5. Assume that ψ is a wavelet frame and $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then there exists a continuous path ψ_t such that $\psi_0 = \psi$, $\psi_1 = f$ and ψ_t is a wavelet frame for every $0 \leq t < 1$. In particular, this also implies that the set of all wavelet frames is path-connected and is dense in $L^2(\mathbb{R})$.

Proof. We will work in the frequency domain, all the functions involved are considered to be functions in frequency domain. Let $f \in L^2(\mathbb{R})$ and $\psi_0 = \chi_E$, where $E = [-\pi, -\pi/2) \cup [\pi/2, \pi)$. Then ψ_0 is wavelet frame. It suffices to show that there is a continuous path connection ψ_0 and f and satisfying the requirements of the theorem.

For each n , let $\{\delta_n\}_{n=1}^\infty$ be a decreasing sequence such that $\int_G |f(t)|^2 dt \leq \frac{1}{n}$ whenever $\mu(G) < \delta_n$, where μ is the Lebesgue measure on \mathbb{R} . Pick a positive integer increasing sequence $\{m_n\}_{n=1}^\infty$ such that

$$\mu([-2(n+1)\pi, 2(n+1)\pi] \cap \tau(2^{-m_n}E)) \leq \frac{1}{2}\delta_n.$$

Let $\psi_n = \chi_{2^{-m_n}E}$ for $n \geq 1$. Then ψ_n is a wavelet frame for $L^2(\mathbb{R})$. Moreover ψ_n is strongly disjoint with ψ_k when $n \neq k$. Thus $t\psi_n + (1-t)\psi_k$ will be always a wavelet frame for each t .

For each $n \geq 0$, pick a small interval $I_n = (-\epsilon_n, \epsilon_n)$ such that $\epsilon_n < \frac{1}{2}\delta_n$, and then define a compactly supported function $g_n \in L^2(\mathbb{R})$ by

$$g_n(t) = \begin{cases} 0 & |t| > 2(n+1)\pi \text{ or } t \in \tau(2^{-m_n}E) \cup \tau(2^{-m_{n+1}}E) \text{ or } t \in I_n \\ f(t) & \text{otherwise} \end{cases}$$

where we set $m_0 = 0$. Then clearly we have $\tau(\text{supp}(g_n)) \cap \tau(\text{supp}(\psi_n)) = \emptyset$ and $\tau(\text{supp}(g_n)) \cap \tau(\text{supp}(\psi_{n+1})) = \emptyset$ for all $n \geq 0$, and hence g_n is strongly disjoint with both ψ_n and ψ_{n+1} . This implies that every function in the following list is a wavelet frame:

$$(3.2) \quad \psi_0, \psi_1 + g_0, \psi_1 + g_1, \dots, \psi_n + g_n, \psi_{n+1} + g_n, \psi_{n+1} + g_{n+1}, \dots$$

Moreover, we have

$$\begin{aligned} \|g_n - f\|^2 &= \int_{|t|>2n\pi} |f(t)|^2 dt + \int_{[-2n\pi, 2n\pi] \cap [\tau(2^{-m_n}E) \cup \tau(2^{-m_{n+1}}E)]} |f(t)|^2 dt + \int_{I_n} |f(t)|^2 dt \\ &\leq \int_{|t|>2n\pi} |f(t)|^2 dt + \frac{1}{n} + \frac{1}{n}, \end{aligned}$$

where in the last inequality we use the fact that $\mu(I_n) \leq \delta_n$ and

$$\mu([-2n\pi, 2n\pi] \cap [\tau(2^{-m_n}E) \cup \tau(2^{-m_{n+1}}E)]) \leq \frac{1}{2}\delta_n + \frac{1}{2}\delta_{n+1} \leq \delta_n.$$

Thus $\|g_n - f\| \rightarrow 0$ as $n \rightarrow \infty$. Our next step is to construct a continuous wavelet frame path between any two neighboring wavelet frames from the list (3.2): Let

$$\varphi_t^{(0)} = t\psi_0 + (1-t)(\psi_1 + g_0),$$

and

$$\begin{cases} h_t^{(n)} = \psi_n + tg_{n-1} + (1-t)g_n \\ \varphi_t^{(n)} = t\psi_n + (1-t)\psi_{n+1} + g_n \end{cases}$$

Since ψ_0 and $\psi_0 + g_0$ are strongly disjoint wavelet frames, we get that $\varphi_t^{(0)}$ is a wavelet frame. Similarly, because of the strong disjointness of ψ_n, ψ_{n+1} and g_n , we have that $h_t^{(n)}$ and $\varphi_t^{(n)}$ are wavelet frames for $n \geq 1$.

To complete the proof, it suffices to show that $\|h_t^{(n)} - f\| \rightarrow 0$ and $\|\varphi_t^{(n)} - f\| \rightarrow 0$ uniformly for $1 \leq t \leq 1$ as $n \rightarrow \infty$. Note that

$$\|h_t^{(n)} - f\| \leq \|\psi_n\| + \|g_{n-1} - f\| + \|g_n - f\|$$

and

$$\|\varphi_t^{(n)} - f\| \leq \|\psi_n\| + \|\psi_{n+1}\| + \|g_n - f\|.$$

Thus the statement follows immediately since we already have $\|\psi_n\| = \mu(2^{-m_n}E) \rightarrow 0$ and $\|g_n - f\| \rightarrow 0$. \square

Remark 3. Both Theorem 3.4 and Proposition 3.5 in this section also hold for higher dimensions and any dilations. Let A be an expansive $d \times d$ real matrix and D_A be defined on $L^2(\mathbb{R}^d)$ by $D_A f(t) = \sqrt{|\det A|} f(At)$. Then $\psi \in L^2(\mathbb{R}^d)$ is called wavelet frame for $L^2(\mathbb{R}^d)$ if there exist two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}^d} |\langle f, D_A^n T_\ell \psi \rangle|^2 \leq B\|f\|^2$$

holds for all $f \in L^2(\mathbb{R}^d)$. It can be shown (c.f. [17]) that there exists $F \in [-\pi, \pi]^d$ such that $A^{-n}F \in [-\pi, \pi]^d$ for $n \geq 0$, and $\{A^{-n}F\}_{n=0}^\infty$ is disjoint. Replacing E by F , and $[-2n\pi, 2n\pi]$ by $[-2n\pi, 2n\pi]^d$ in the proofs of Theorem 3.4 and Proposition 3.5 and keeping the rest of the argument, then we get the same results for the high dimension and arbitrary dilation wavelet frames. Similar techniques can be used to show that both theorem are valid for subspace wavelet frames.

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