

Unitary Systems, Wavelet Sets, and Operator-Theoretic Interpolation of Wavelets and Frames

David R. Larson

*Department of Mathematics
Texas A&M University
College Station, TX 77845
E-mail: larson@math.tamu.edu*

A wavelet is a special case of a vector in a separable Hilbert space that generates a basis under the action of a collection, or system, of unitary operators. We will describe the operator-interpolation approach to wavelet theory using the local commutant of a system. This is really an abstract application of the theory of operator algebras to wavelet theory. The concrete applications of this method include results obtained using specially constructed families of wavelet sets. A frame is a sequence of vectors in a Hilbert space which is a compression of a basis for a larger space. This is not the usual definition in the frame literature, but it is easily equivalent to the usual definition. Because of this compression relationship between frames and bases, the unitary system approach to wavelets (and more generally: wandering vectors) is perfectly adaptable to frame theory. The use of the local commutant is along the same lines as in the wavelet theory. Finally, we discuss constructions of frames with special properties using targeted decompositions of positive operators, and related problems.

1. Introduction

This is a write-up of of a tutorial series of three talks which I gave as part of the "Workshop on Functional and Harmonic Analyses of Wavelets and Frames" held August 4-7, 2004 at the National University of Singapore. I will first give the titles and abstracts essentially as they appeared in the workshop schedule. I will say that the actual style of write-up of these notes will be structured a bit differently, but only in that more than three sections will be given, and subsections indicated, to (hopefully) improve expositional quality.

1.1. *Talks and Abstracts*

(a) "Unitary Systems and Wavelet Sets": A wavelet is a special case of a vector in a separable Hilbert space that generates a basis under the action of a collection, or "system", of unitary operators defined in terms of translation and dilation operations. This approach to wavelet theory goes back, in particular, to earlier work of Goodman, Lee and Tang [25] in the context of multiresolution analysis. We will begin by describing the operator-interpolation approach to wavelet theory using the local commutant of a system that was worked out by the speaker and his collaborators a few years ago. This is really an abstract application of the theory of operator algebras, mainly von Neumann algebras, to wavelet theory. The concrete applications of operator-interpolation to wavelet theory include results obtained using specially constructed families of wavelet sets. In fact X. Dai and the speaker had originally developed our theory of wavelet sets [11] specifically to take advantage of their natural and elegant relationships with these wavelet unitary systems. We will also discuss some new results and open questions.

(b) "Unitary Systems and Frames": A frame is a sequence of vectors in a Hilbert space which is a compression of a basis for a larger space. (This is not the usual definition in the frame literature, but it is equivalent to the usual definition. In this spirit, the usual "inequality" definition can be thought of as an abstract characterization of a compression of a basis.) Because of this compression relationship between frames and bases, the unitary system approach to wavelets (and more generally: wandering vectors) is perfectly adaptable to frame theory. This idea was developed into a theory a few years ago by D. Han and the speaker [33]. The use of the local commutant is along the same lines.

(c) "Decompositions of Operators and Operator-Valued Frames": We will discuss some joint work with K. Kornelson and others on construction of frames with targeted properties [16, 42]. These are related to targeted decompositions of positive operators.

1.2. *Some Background*

It might be appropriate to give some comments of a personal-historical nature, before continuing with the technical aspects. My particular point of view on "wavelet theory", which was developed jointly with my good friend and colleague Xingde Dai, began in the summer of 1992. Before then, I was strictly an operator theorist. I had heard my approximation theory

colleagues and friends at Texas A&M University talk about wavelets and frames, and Xingde had frequently mentioned these topics to me when he was finishing his Ph.D. at A&M (he was my student, graduating in 1990, with a thesis [9] on the subject of *nest algebras*). But it was this meeting of minds we had in June of 1992 that was the turning point for me. We formulated an approach to wavelet theory (and as it turned out ultimately, to frame-wavelet theory as well) that I felt we could "really understand" as operator algebraists. This was the abstract *unitary system* approach. Dai knew the unitary operator approach to multiresolution analysis that had been recently (at that time) published by Goodman, Lee and Tang [25], and he suggested to me that we should try to go further with these ideas in an attempt to get a some type of tractable classification of *all* wavelets. We went, in fact, in some completely different directions. The first paper that came out of this was the AMS Memoir [11] with Dai. The second paper was our paper [12] with Dai and Speegle, which proved the existence of single wavelets in higher dimensions, for arbitrary expansive dilations. After that, several papers followed including the AMS Memoir [33] with Deguang Han, and the Wutam Consortium paper [52], as well as the papers [10, 13, 26, 27, 28, 30, 34, 43, 44] by my students Dai, Gu, Han and Lu, and their collaborators, and the papers [2, 39, 40] with my colleagues Azoff, Ionascu, and Percy.

The paper [11] with Dai mentioned above, which was published in 1998, culminated about two years of work on this topic by the authors. It contained our entire operator-theoretic approach to wavelet theory, and was completed in December 1994. This work, while theoretical, had much hands-on experimentation in its development, and resulted in certain theorems we were able to prove concerning constructions of new families of wavelets. In order to conduct successful *experiments* with our operator techniques, we needed a supply of easily computable *test* wavelets: that is, wavelets which were very amenable to *paper and pencil computations*. We discovered that certain sets, we called *wavelet sets*, existed in abundance, and we computed many concrete examples of them along the way toward proving our results of [11]. Several of these were given as examples in [11, Example 4.5, items (i) \rightarrow (xi)]. Some of these are given in Section 2.6.1 of the present article.

Most of our work in [11] on the local commutant and the theory of wavelet sets was accomplished in the two-month period July-September 1992. The first time Dai and I used the terms *wavelet set* and *local commutant*, as well as the first time we discussed what we referred to as the *connectedness problem for wavelets*, was in a talk in a Special Session on

Operator Algebras in the October 1992 AMS Sectional Meeting in Dayton.

Along the way a graduate student at A&M, Darrin Speegle, who was enrolled in a seminar course of Larson on the manuscript of [11], answered an open question Larson gave out in class by proving that the set of all wavelet sets for a given wavelet system is *connected* in the symmetric difference metric on the class of measurable sets of finite measure. That resulted in a paper [50] which became part of his thesis (which was directed by William Johnson of Texas A&M), and Speegle subsequently joined forces with Dai and Larson [12] to prove that wavelet sets (and indeed, wavelets) exist in much greater generality than the prevailing *folklore* dictated. We received some attention for our work, and especially we thank Guido Weiss and John Benedetto for recognizing our work. This led to a flurry of papers by a number of authors, notably [4, 6, 13], and also led to the paper [52] by the Wutam Consortium, which was a group led by Guido Weiss and Larson, consisting of 14 researchers—students and postdocs of Weiss and Larson—based at Washington University and Texas A&M University, for the purpose of doing basic research on wavelet theory.

1.2.1. *Interpolation*

The main point of the *operator-theoretic interpolation* of wavelets (and frames) that Dai and I developed is that new wavelets can be obtained as linear combinations of known ones using *coefficients* which are not necessarily scalars but can be taken to be *operators* (in fact, *Fourier multipliers*) in a certain class. The ideas involved in this, and the essential computations, all extend naturally to more general unitary systems and *wandering vectors*, and I think that much of the theory is best-put in this abstract setting because clarity is enhanced, and because many of the methods work for more involved systems that are important to applied harmonic analysis, such as Gabor and generalized Gabor systems, and various types of *frame* unitary systems.

1.2.2. *Some Basic Terminology*

This article will concern bounded linear operators on separable Hilbert spaces. The set of all bounded linear operators on a Hilbert space H will be denoted by $B(H)$. By a *bilateral shift* U on H we mean a unitary operator U for which there exists a closed linear subspace $E \subset H$ with the property that the family of subspaces $\{U^n E : n \in \mathbb{Z}\}$ are orthogonal and give a direct-sum decomposition of H . The subspace E is called a *complete wandering*

subspace for U . The *multiplicity* of U is defined to be the dimension of E .

The *strong operator topology* on $B(H)$ is the topology of pointwise convergence, and the *weak operator topology* is the weakest topology such that the vector functionals $\omega_{x,y}$ on $B(H)$ defined by $A \mapsto \langle Ax, y \rangle$, $A \in B(H)$, $x, y \in H$, are all continuous. An *algebra of operators* is a linear subspace of $B(H)$ which is closed under multiplication. An *operator algebra* is an algebra of operators which is *norm-closed*. A subset $\mathcal{S} \subset B(H)$ is called *selfadjoint* if whenever $A \in \mathcal{S}$ then also $A^* \in \mathcal{S}$. A *C^* -algebra* is a selfadjoint operator algebra. A *von Neumann algebra* is a C^* -algebra which is closed in the weak operator topology. For a unital operator algebra, it is well known that being closed in the weak operator topology is equivalent to being closed in the closed in the strong operator topology.

The *commutant* of a set \mathcal{S} of operators in $B(H)$ is the family of all operators in $B(H)$ that *commute* with every operator in \mathcal{S} . It is closed under addition and multiplication, so is an algebra. And it is clearly closed in both the weak operator topology and the strong operator topology. We use the standard *prime* notation for the commutant. So the commutant of a subset $\mathcal{S} \subset B(H)$ is denoted: $\mathcal{S}' := \{A \in B(H) : AS = SA, S \in \mathcal{S}\}$.

The commutant of a selfadjoint set of operators is clearly a von Neumann algebra. Moreover, by a famous theorem of Fuglede every operator which commutes with a normal operator N also commutes with its adjoint N^* , and hence the commutant of any set of *normal* operators is also a von Neumann algebra. So, of particular relevance to this work, the commutant of any set of *unitary* operators is a von Neumann algebra.

One of the main tools in this work is the *local commutant* of a system of unitary operators. (See section 2.4.) This is a natural generalization of the commutant of the system, and like the commutant it is a linear space of operators which is closed in the weak and the strong operator topologies, but unlike the commutant it is usually not selfadjoint, and is usually not closed under multiplication. It contains the commutant of the system, but can be much larger than the commutant. The local commutant of a wavelet unitary system captures all the information about the wavelet system in an essential way, and this gives the *flavor* of our approach to the subject.

If U is a unitary operator and \mathcal{A} is an operator algebra, then U is said to *normalize* \mathcal{A} if $U^* \cdot \mathcal{A} \cdot U = \mathcal{A}$. In the most interesting cases of operator-theoretic interpolation: that is, those cases that yield the strongest structural results, the relevant unitaries in the local commutant of the system normalize the commutant of the system.

1.2.3. Acknowledgements

I want to take the opportunity to thank the organizers of this wonderful workshop at the National University of Singapore for their splendid hospitality and great organization, and for inviting me to give the series of tutorial-style talks that resulted in this write-up. I also want to state that the work discussed in this article was supported by grants from the United States National Science Foundation.

2. Unitary Systems and Wavelet Sets

We define a *unitary system* to be simply a collection of unitary operators \mathcal{U} acting on a Hilbert space H which contains the identity operator. The *interesting* unitary systems all have additional structural properties of various types. We will say that a vector $\psi \in H$ is *wandering* for \mathcal{U} if the set

$$\mathcal{U}\psi := \{U\psi : U \in \mathcal{U}\} \quad (1)$$

is an orthonormal set, and we will call ψ a *complete wandering vector* for \mathcal{U} if $\mathcal{U}\psi$ spans H . This (abstract) point of view can be useful. Write

$$\mathcal{W}(\mathcal{U})$$

for the set of complete wandering vectors for \mathcal{U} .

2.1. The One-Dimensional Wavelet System

For simplicity of presentation, much of the work in this article will deal with one-dimensional wavelets, and in particular, the dyadic case. The other cases: non-dyadic and in higher dimensions, are well-described in the literature and are at least notationally more complicated.

2.1.1. Dyadic Wavelets

A *dyadic orthonormal* wavelet in one dimension is a unit vector $\psi \in L^2(\mathbb{R}, \mu)$, with μ Lebesgue measure, with the property that the set

$$\{2^{\frac{n}{2}}\psi(2^n t - l) : n, l \in \mathbb{Z}\} \quad (2)$$

of all integral translates of ψ followed by dilations by arbitrary integral powers of 2, is an orthonormal basis for $L^2(\mathbb{R}, \mu)$. The term *dyadic* refers

to the dilation factor "2". The term *mother wavelet* is also used in the literature for ψ . Then the functions

$$\psi_{n,l} := 2^{\frac{n}{2}}\psi(2^n t - l)$$

are called elements of the wavelet basis generated by the "mother". The functions $\psi_{n,l}$ will not themselves be mother wavelets unless $n = 0$.

Let T and D be the translation (by 1) and dilation (by 2) unitary operators in $B(L^2(\mathbb{R}))$ given by $(Tf)(t) = f(t - 1)$ and $(Df)(t) = \sqrt{2}f(2t)$. Then

$$2^{\frac{n}{2}}\psi(2^n t - l) = (D^n T^l \psi)(t)$$

for all $n, l \in \mathbb{Z}$. Operator-theoretically, the operators T, D are *bilateral shifts* of *infinite multiplicity*. It is obvious that $L^2([0, 1])$, considered as a subspace of $L^2(\mathbb{R})$, is a complete wandering subspace for T , and that $L^2([-2, -1] \cup [1, 2])$ is a complete wandering subspace for D .

2.1.2. The Dyadic Unitary System

Let $\mathcal{U}_{D,T}$ be the unitary system defined by

$$\mathcal{U}_{D,T} = \{D^n T^l : n, l \in \mathbb{Z}\} \quad (3)$$

where D and T are the operators defined above. Then ψ is a dyadic orthonormal wavelet if and only if ψ is a complete wandering vector for the unitary system $\mathcal{U}_{D,T}$. This was our original motivation for developing the abstract unitary system theory. Write

$$\mathcal{W}(D, T) := \mathcal{W}(\mathcal{U}_{D,T}) \quad (4)$$

to denote the set of all dyadic orthonormal wavelets in one dimension.

An abstract interpretation is that, since D is a bilateral shift it has (many) complete wandering subspaces, and a wavelet for the system is a vector ψ whose translation space (that is, the closed linear span of $\{T^k : k \in \mathbb{Z}\}$) is a complete wandering subspace for D . Hence ψ must generate an orthonormal basis for the entire Hilbert space under the action of the unitary system.

2.1.3. Non-Dyadic Wavelets in One Dimension

In one dimension, there are non-dyadic orthonormal wavelets: i.e. wavelets for all possible dilation factors besides 2 (the dyadic case). We said "possible", because the scales $\{0, 1, -1\}$ are excluded as scales because the dilation

operators they would introduce are not bilateral shifts. All other real numbers for scales yield wavelet theories. In [11, Example 4.5 (x)] a family of examples is given of three-interval wavelet sets (and hence wavelets) for all scales $d \geq 2$, and it was noted there that such a family also exists for dilation factors $1 < d \leq 2$. There is some recent (yet unpublished) work that has been done, by REU students and mentors, building on this, classifying finite-interval wavelet sets for all possible real (positive and negative scale factors). I mentioned this work, in passing, in my talk.

2.2. *N dimensions*

2.2.1. *The Expansive-Dilation Case*

Let $1 \leq m < \infty$, and let A be an $n \times n$ real matrix which is *expansive* (equivalently, all (complex) eigenvalues have modulus > 1). By a *dilation - A regular-translation orthonormal wavelet* we mean a function $\psi \in L^2(\mathbb{R}^n)$ such that

$$\{ |\det(A)|^{\frac{n}{2}} \psi(A^n t - (l_1, l_2, \dots, l_n)^t) : n, l \in \mathbb{Z} \} \tag{5}$$

where $t = (t_1, \dots, t_n)^t$, is an orthonormal basis for $L^2(\mathbb{R}^n; m)$. (Here m is product Lebesgue measure, and the superscript "t" means transpose.)

If $A \in M_n(\mathbb{R})$ is invertible (so in particular if A is expansive), then it is very easy to verify that the operator defined by

$$(D_A f)(t) = |\det A|^{\frac{1}{2}} f(At) \tag{6}$$

for $f \in L^2(\mathbb{R}^n)$, $t \in \mathbb{R}^n$, is *unitary*. For $1 \leq i \leq n$, let T_i be the unitary operator determined by translation by 1 in the i^{th} coordinate direction.

The set (5) above is then

$$\{ D_A^k T_1^{l_1} \dots T_n^{l_n} \psi : k, l_i \in \mathbb{Z} \} \tag{7}$$

If the dilation matrix A is expansive, but the translations are along some oblique lattice, then there is an invertible real $n \times n$ matrix T such that conjugation with D_T takes the entire wavelet system to a regular-translation expansive-dilation matrix. This is easily worked out, and was shown in detail in [39] in the context of working out a complete theory of unitary equivalence of wavelet systems. Hence the wavelet theories are equivalent.

2.2.2. *The Non-Expansive Dilation Case*

Much work has been accomplished concerning the existence of wavelets for dilation matrices A which are not expansive. Some of the original work was

accomplished in the Ph.D. theses of Q. Gu and D. Speegle, when they were together finishing up at Texas A&M. Some significant additional work was accomplished by Speegle in [49], and also by others. In [39], with Ionascu and Pearcy we proved that if an $n \times n$ real invertible matrix A is not similar (in the $n \times n$ complex matrices) to a unitary matrix, then the corresponding dilation operator D_A is in fact a bilateral shift of infinite multiplicity. If a dilation matrix were to admit any type of wavelet (or frame-wavelet) theory, then it is well-known that a necessary condition would be that the corresponding dilation operator would have to be a bilateral shift of infinite multiplicity. I am happy to report that in very recent work [45], with E. Schulz, D. Speegle, and K. Taylor, we have succeeded in showing that this minimal condition is in fact sufficient: such a matrix, with regular translation lattice, admits a (perhaps infinite) tuple of functions, which collectively generates a frame-wavelet under the action of this unitary system.

2.3. Abstract Systems

2.3.1. Restrictions on Wandering Vectors

We note that *most* unitary systems \mathcal{U} do not have complete wandering vectors. For $\mathcal{W}(\mathcal{U})$ to be nonempty, the set \mathcal{U} must be very special. It must be *countable* if it acts separably (i.e. on a separable Hilbert space), and it must be *discrete* in the strong operator topology because if $U, V \in \mathcal{U}$ and if x is a wandering vector for \mathcal{U} then

$$\|U - V\| \geq \|Ux - Vx\| = \sqrt{2}$$

Certain other properties are forced on \mathcal{U} by the presence of a wandering vector. One purpose of [11] was to study such properties. Indeed, it was a matter of some surprise to us to discover that such a theory is viable even in some considerable generality. For perspective, it is useful to note that while $\mathcal{U}_{D,T}$ has complete wandering vectors, the reversed system

$$\mathcal{U}_{T,D} = \{T^l D^n : n, l \in \mathbb{Z}\}$$

fails to have a complete wandering vector. (A proof of this was given in the introduction to [11].)

2.3.2. Group Systems

An example which is important to the theory is the following: let G be an arbitrary countable group, and let $H = \ell^2(G)$. Let π be the (left) regular representation of G on H . Then every element of G gives a complete

wandering vector for the unitary system

$$\mathcal{U} := \pi(G).$$

(If $h \in G$ it is clear that the vector $\lambda_h \in l^2(G)$, which is defined to have 1 in the h position and 0 elsewhere, is in $\mathcal{W}(\mathcal{U})$.) If a unitary system is a *group*, and if it has a complete wandering vector, it is not hard to show that it is unitarily equivalent to this example.

2.4. The Local Commutant

2.4.1. The Local Commutant of the System $\mathcal{U}_{D,T}$

Computational aspects of operator theory can be introduced into the wavelet framework in an elementary way. Here is the way we originally did it: Fix a wavelet ψ and consider the set of all operators $S \in B(L^2(\mathbb{R}))$ which *commute* with the *action* of dilation and translation on ψ . That is, require

$$(S\psi)(2^n t - l) = S(\psi(2^n t - l)) \tag{8}$$

or equivalently

$$D^n T^l S \psi = S D^n T^l \psi \tag{9}$$

for all $n, l \in \mathbb{Z}$. Call this the *local commutant of the wavelet system $\mathcal{U}_{D,T}$ at the vector ψ* . (In our first preliminary writings and talks we called it the *point commutant* of the system. Formally, the local commutant of the dyadic wavelet system on $L^2(\mathbb{R})$ is:

$$\mathcal{C}_\psi(\mathcal{U}_{D,T}) := \{S \in B(L^2(\mathbb{R})) : (S D^n T^l - D^n T^l S) \psi = 0, \forall n, l \in \mathbb{Z}\} \tag{10}$$

This is a linear subspace of $B(H)$ which is closed in the strong operator topology, and in the weak operator topology, and it clearly contains the *commutant* of $\{D, T\}$.

A motivating example is that if η is any other wavelet, let $V := V_\psi^\eta$ be the unitary (we call it the *interpolation unitary*) that takes the basis $\psi_{n,l}$ to the basis $\eta_{n,l}$. That is, $V \psi_{n,l} = \eta_{n,l}$ for all $n, l \in \mathbb{Z}$. Then $\eta = V \psi$, so $V D^n T^l \psi = D^n T^l S \psi$ hence $V \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$.

In the case of a pair of complete wandering vectors ψ, η for a general unitary system \mathcal{U} , we will use the same notation V_ψ^η for the unitary that takes the vector $U \psi$ to $U \eta$ for all $U \in \mathcal{U}$.

This simple-minded idea is reversible, so for every unitary V in $\mathcal{C}_\psi(\mathcal{U}_{D,T})$ the vector $V \psi$ is a wavelet. This correspondence between unitaries in

$\mathcal{C}_\psi(D, T)$ and dyadic orthonormal wavelets is one-to-one and onto (see Proposition 1.) This turns out to be useful, because it leads to some new formulas relating to decomposition and factorization results for wavelets, making use of the *linear* and *multiplicative* properties of $\mathcal{C}_\psi(D, T)$.

It turns out (a proof is required) that the entire local commutant of the system $\mathcal{U}_{D, T}$ at a wavelet ψ is *not* closed under multiplication, but it also turns out (also via a proof) that for *most* (and perhaps *all*) wavelets ψ the local commutant at ψ contains many noncommutative operator algebras (in fact von Neumann algebras) as subsets, and their unitary groups *parameterize* norm-arcwise-connected families of wavelets. Moreover, $\mathcal{C}_\psi(D, T)$ is closed under *left multiplication* by the commutant $\{D, T\}'$, which turns out to be an abelian nonatomic von Neumann algebra. The fact that $\mathcal{C}_\psi(D, T)$ is a *left module* under $\{D, T\}'$ leads to a method of obtaining new wavelets from old, and of obtaining connectedness results for wavelets, which we called *operator-theoretic interpolation* of wavelets in [DL], (or simply *operator-interpolation*).

2.4.2. The Local Commutant of an Abstract Unitary System

More generally, let $\mathcal{S} \subset B(H)$ be a set of operators, where H is a separable Hilbert space, and let $x \in H$ be a nonzero vector, and *formally* define the *local commutant* of \mathcal{S} at x by

$$\mathcal{C}_x(\mathcal{S}) := \{A \in B(H) : (AS - SA)x = 0, S \in \mathcal{S}\}$$

As in the wavelet case, this is a weakly and strongly closed linear subspace of $B(H)$ which contains the commutant \mathcal{S}' of \mathcal{S} . If x is *cyclic* for \mathcal{S} in the sense that $\text{span}(\mathcal{S}x)$ is dense in H , then x *separates* $\mathcal{C}_x(\mathcal{S})$ in the sense that for $S \in \mathcal{C}_x(\mathcal{S})$, we have $Sx = 0$ iff $x = 0$. Indeed, if $A \in \mathcal{C}_x(\mathcal{S})$ and if $Ax = 0$, then for any $S \in \mathcal{S}$ we have $ASx = SAx = 0$, so $ASx = 0$, and hence $A = 0$.

If $A \in \mathcal{C}_x(\mathcal{S})$ and $B \in \mathcal{S}'$, let $C = BA$. Then for all $S \in \mathcal{S}$,

$$(CS - SC)x = B(AS)x - (SB)Ax = B(SA)x - (BS)Ax = 0$$

because $ASx = SAx$ since $A \in \mathcal{C}_x(\mathcal{S})$, and $SB = BS$ since $B \in \mathcal{S}'$. Hence $\mathcal{C}_x(\mathcal{S})$ is closed under left multiplication by operators in \mathcal{S}' . That is, $\mathcal{C}_x(\mathcal{S})$ is a *left module* over \mathcal{S}' .

It is interesting that, if in addition \mathcal{S} is a multiplicative semigroup, then in fact $\mathcal{C}_x(\mathcal{S})$ is identical with the commutant \mathcal{S}' so in this case the commutant is not a new structure. To see this, suppose $A \in \mathcal{C}_x(\mathcal{S})$. Then

for each $S, T \in \mathcal{S}$ we have $ST \in \mathcal{S}$, and so

$$AS(Tx) = (ST)Ax = S(ATx) = (S)Tx$$

So since $T \in \mathcal{S}$ was arbitrary and $\text{span}(\mathcal{S}x) = H$, it follows that $AS = SA$.

Proposition 1: *If \mathcal{U} is any unitary system for which $\mathcal{W}(\mathcal{U}) \neq \emptyset$, then for any $\psi \in \mathcal{W}(\mathcal{U})$*

$$\mathcal{W}(\mathcal{U}) = \{U\psi : U \text{ is a unitary operator in } \mathcal{C}_\psi(\mathcal{U})\}$$

and the correspondence $U \rightarrow U\psi$ is one-to-one.

A *Riesz basis* for a Hilbert space H is the image under a bounded invertible operator of an orthonormal basis. Proposition 1 generalizes to generators of Riesz bases. A *Riesz vector* for a unitary system \mathcal{U} is defined to be a vector ψ for which $\mathcal{U}\psi := \{U\psi : U \in \mathcal{U}\}$ is a Riesz basis for the closed linear span of $\mathcal{U}\psi$, and it is called *complete* if $\overline{\text{span}} \mathcal{U}\psi = H$. Let $\mathcal{RW}(\mathcal{U})$ denote the set of all complete Riesz vectors for \mathcal{U} .

Proposition 2: *Let \mathcal{U} be a unitary system on a Hilbert space H . If ψ is a complete Riesz vector for \mathcal{U} , then*

$$\mathcal{RW}(\mathcal{U}) = \{A\psi : A \text{ is an operator in } \mathcal{C}_\psi(\mathcal{U}) \text{ that is invertible in } B(H)\}.$$

2.4.3. Operator-Theoretic Interpolation

Now suppose \mathcal{U} is a unitary system, such as $\mathcal{U}_{D,T}$, and suppose $\{\psi_1, \psi_2, \dots, \psi_m\} \subset \mathcal{W}(\mathcal{U})$. (In the case of $\mathcal{U}_{D,T}$, this means that $(\psi_1, \psi_2, \dots, \psi_n)$ is an n -tuple of wavelets.

Let (A_1, A_2, \dots, A_n) be an n -tuple of operators in the commutant \mathcal{U}' of \mathcal{U} , and let η be the vector

$$\eta := A_1\psi_1 + A_2\psi_2 + \dots + A_n\psi_n .$$

Then

$$\begin{aligned} \eta &= A_1\psi_1 + A_2V_{\psi_1}^{\psi_2}\psi_1 + \dots + A_nV_{\psi_1}^{\psi_n}\psi_1 \\ &= (A_1 + A_2V_{\psi_1}^{\psi_2} + \dots + A_nV_{\psi_1}^{\psi_n})\psi_1 . \end{aligned} \tag{11}$$

We say that η is obtained by *operator interpolation* from $\{\psi_1, \psi_2, \dots, \psi_m\}$. Since $\mathcal{C}_{\psi_1}(\mathcal{U})$ is a left \mathcal{U}' -module, it follows that the operator

$$A := A_1 + A_2V_{\psi_1}^{\psi_2} + \dots + A_nV_{\psi_1}^{\psi_n} \tag{12}$$

is an element of $\mathcal{C}_{\psi_1}(\mathcal{U})$. Moreover, if B is another element of $\mathcal{C}_{\psi_1}(\mathcal{U})$ such that $\eta = B\psi_1$, then $A - B \in \mathcal{C}_{\psi_1}(\mathcal{U})$ and $(A - B)\psi_1 = 0$. So since ψ_1 separates $\mathcal{C}_{\psi_1}(\mathcal{U})$ it follows that $A = B$. Thus A is the unique element of $\mathcal{C}_{\psi_1}(\mathcal{U})$ that takes ψ_1 to η . Let $\mathcal{S}_{\psi_1, \dots, \psi_n}$ be the family of all finite sums of the form

$$\sum_{i=0}^n A_i V_{\psi_1}^{\psi_i} .$$

This is the left module of \mathcal{U}' generated by $\{I, V_{\psi_1}^{\psi_2}, \dots, V_{\psi_1}^{\psi_n}\}$. It is the \mathcal{U}' -linear span of $\{I, V_{\psi_1}^{\psi_2}, \dots, V_{\psi_1}^{\psi_n}\}$. Let

$$\mathcal{M}_{\psi_1, \dots, \psi_n} := (\mathcal{S}_{\psi_1, \dots, \psi_n})\psi_1 \tag{13}$$

So

$$\mathcal{M}_{\psi_1, \dots, \psi_n} = \left\{ \sum_{i=0}^n A_i \psi_i : A_i \in \mathcal{U}' \right\} .$$

We call this the *interpolation space* for \mathcal{U} generated by (ψ_1, \dots, ψ_n) . From the above discussion, it follows that for every vector $\eta \in \mathcal{M}_{\psi_1, \psi_2, \dots, \psi_n}$ there exists a unique operator $A \in \mathcal{C}_{\psi_1}(\mathcal{U})$ such that $\eta = A\psi_1$, and moreover this A is an element of $\mathcal{S}_{\psi_1, \dots, \psi_n}$.

2.4.4. Normalizing the Commutant

In certain essential cases (and we are not sure how general this type of case is) one can prove that an interpolation unitary V_{ψ}^{η} normalizes the commutant \mathcal{U}' of the system in the sense that $V_{\psi}^{\eta} \mathcal{U}' V_{\psi}^{\eta} = \mathcal{U}'$. (Here, it is easily seen that $(V_{\psi}^{\eta})^* = V_{\eta}^{\psi}$.) Write $V := V_{\psi}^{\eta}$. If V normalizes \mathcal{U}' , then the algebra, before norm closure, generated by \mathcal{U}' and V is the set of all finite sums (trig polynomials) of the form $\sum A_n V^n$, with coefficients $A_n \in \mathcal{U}'$, $n \in \mathbb{Z}$. The closure in the strong operator topology is a von Neumann algebra. Now suppose further that every power of V is contained in $\mathcal{C}_{\psi}(\mathcal{U})$. This occurs only in special cases, yet it occurs frequently enough to yield some general methods. Then since $\mathcal{C}_{\psi}(\mathcal{U})$ is a SOT-closed linear subspace which is closed under left multiplication by \mathcal{U}' , this von Neumann algebra is contained in $\mathcal{C}_{\psi}(\mathcal{U})$, so its unitary group parameterizes a norm-path-connected subset of $\mathcal{W}(\mathcal{U})$ that contains ψ and η via the correspondence $U \rightarrow U\psi$.

In the special case of *wavelets*, this is the basis for the work that Dai and I did in [11, Chapter 5] on operator-theoretic interpolation of wavelets.

In fact, we specialized there and *reserved* the term *operator-theoretic interpolation* to refer explicitly to the case when the interpolation unitaries normalize the commutant. In some subsequent work, we *loosened* this restriction yielding our more general definition given in this article, because there are cases of interest in which we weren't able to prove normalization. However, it turns out that if ψ and η are s -elementary wavelets (see section 2.5.4), then indeed V_ψ^η normalizes $\{D, T\}'$. (See Proposition 14.) Moreover, V_ψ^η has a very special form: after conjugating with the Fourier transform, it is a composition operator with symbol a natural and very computable measure-preserving transformation of \mathbb{R} . In fact, it is precisely this special form for V_ψ^η that allows us to make the computation that it normalizes $\{D, T\}'$. On the other hand, we know of no pair (ψ, η) of wavelets for which V_ψ^η fails to normalize $\{D, T\}'$. The difficulty is simply that in general it is very hard to do the computations.

Problem: If $\{\psi, \eta\}$ is a pair of dyadic orthonormal wavelets, does the interpolation unitary V_ψ^η normalize $\{D, T\}'$? As mentioned above, the answer is yes if ψ and η are s -elementary wavelets.

2.4.5. An Elementary Interpolation Result

The following result is the most elementary case of operator-theoretic interpolation.

Proposition 3: *Let \mathcal{U} be a unitary system on a Hilbert space H . If ψ_1 and ψ_2 are in $\mathcal{W}(\mathcal{U})$, then*

$$\psi_1 + \lambda\psi_2 \in \mathcal{RW}(\mathcal{U})$$

for all complex scalars λ with $|\lambda| \neq 1$. More generally, if ψ_1 and ψ_2 are in $\mathcal{RW}(\mathcal{U})$ then there are positive constants $b > a > 0$ such that $\psi_1 + \lambda\psi_2 \in \mathcal{RW}(\mathcal{U})$ for all $\lambda \in \mathbb{C}$ with either $|\lambda| < a$ or with $|\lambda| > b$.

Proof: If $\psi_1, \psi_2 \in \mathcal{W}(\mathcal{U})$, let V be the unique unitary in $\mathcal{C}_{\psi_2}(\mathcal{U})$ given by Proposition 1 such that $V\psi_2 = \psi_1$. Then

$$\psi_1 + \lambda\psi_2 = (V + \lambda I)\psi_2.$$

Since V is unitary, $(V + \lambda I)$ is an invertible element of $\mathcal{C}_{\psi_2}(\mathcal{U})$ if $|\lambda| \neq 1$, so the first conclusion follows from Proposition 2. Now assume $\psi_1, \psi_2 \in \mathcal{RW}(\mathcal{U})$. Let A be the unique invertible element of $\mathcal{C}_{\psi_2}(\mathcal{U})$ such that $A\psi_2 =$

ψ_1 , and write $\psi_1 + \lambda\psi_2 = (A + \lambda I)\psi_2$. Since A is bounded and invertible there are $b > a > 0$ such that

$$\sigma(A) \subseteq \{z \in \mathbb{C} : a < |z| < b\}$$

where $\sigma(A)$ denotes the spectrum of A , and the same argument applies. \square

2.4.6. Interpolation Pairs of Wandering Vectors

In some cases where a pair ψ, η of vectors in $\mathcal{W}(\mathcal{U})$ are given it turns out that the unitary V in $\mathcal{C}_\psi(\mathcal{U})$ with $V\psi = \eta$ happens to be a *symmetry* (i.e. $V^2 = I$). Such pairs are called *interpolation pairs* of wandering vectors, and in the case where \mathcal{U} is a wavelet system, they are called interpolation pairs of wavelets. Interpolation pairs are more prevalent in the theory, and in particular the wavelet theory, than one might expect. In this case (and in more complex generalizations of this) certain linear combinations of complete wandering vectors are themselves complete wandering vectors – not simply complete Riesz vectors.

Proposition 4: *Let \mathcal{U} be a unitary system, let $\psi, \eta \in \mathcal{W}(\mathcal{U})$, and let V be the unique operator in $\mathcal{C}_\psi(\mathcal{U})$ with $V\psi = \eta$. Suppose*

$$V^2 = I.$$

Then

$$\cos \alpha \cdot \psi + i \sin \alpha \cdot \eta \in \mathcal{W}(\mathcal{U})$$

for all $0 \leq \alpha \leq 2\pi$.

The above result can be thought of as the *prototype* of our operator-theoretic interpolation results. It is the second most elementary case. More generally, the scalar α in Proposition 4 can be replaced with an appropriate *self-adjoint operator* in the commutant of \mathcal{U} . In the wavelet case, after conjugating with the Fourier transform, which is a unitary operator, this means that α can be replaced with a wide class of nonnegative dilation-periodic (see definition below) bounded measurable functions on \mathbb{R} .

2.4.7. A Test For Interpolation Pairs

The following converse to Proposition 4 is typical of the type of computations encountered in some wandering vector proofs.

Proposition 5: Let \mathcal{U} be a unitary system, let $\psi, \eta \in \mathcal{W}(\mathcal{U})$, and let V be the unique unitary in $\mathcal{C}_\psi(\mathcal{U})$ with $V\psi = \eta$. Suppose for some $0 < \alpha < \frac{\pi}{2}$ the vector

$$\rho := \cos \alpha \cdot \psi + i \sin \alpha \cdot \eta$$

is contained in $\mathcal{W}(\mathcal{U})$. Then

$$V^2 = I.$$

Proof: Since $\mathcal{U}\psi$ is a basis it will be enough to show that $VU_1\psi = V^*U_1\psi$ for all $U_1 \in \mathcal{U}$. So it will suffice to prove that for all $U_1, U_2 \in \mathcal{U}$ we have

$$\langle VU_1\psi, U_2\psi \rangle = \langle V^*U_1\psi, U_2\psi \rangle.$$

Using the fact that V locally commutes with \mathcal{U} at ψ we have

$$\langle VU_1\psi, U_2\psi \rangle = \langle U_1V\psi, U_2\psi \rangle = \langle U_1\eta, U_2\psi \rangle \quad \text{and}$$

$$\langle V^*U_1\psi, U_2\psi \rangle = \langle U_1\psi, VU_2\psi \rangle = \langle U_1\psi, U_2V\psi \rangle = \langle U_1\psi, U_2\eta \rangle.$$

So we must show that $\langle U_1\eta, U_2\psi \rangle = \langle U_1\psi, U_2\eta \rangle$ for all $U_1, U_2 \in \mathcal{U}$.

Write $\rho := \rho_\alpha$. By hypothesis ψ, η and ρ are unit vectors. So compute

$$\begin{aligned} 1 = \langle \rho, \rho \rangle &= \cos^2 \alpha \cdot \langle \psi, \psi \rangle + i \sin \alpha \cos \alpha \cdot \langle \eta, \psi \rangle \\ &\quad - i \sin \alpha \cos \alpha \cdot \langle \psi, \eta \rangle + \sin^2 \alpha \cdot \langle \eta, \eta \rangle \\ &= 1 + i \sin \alpha \cos \alpha \cdot (\langle \eta, \psi \rangle - \langle \psi, \eta \rangle). \end{aligned}$$

Thus, since $\sin \alpha \cos \alpha \neq 0$, we must have $\langle \eta, \psi \rangle = \langle \psi, \eta \rangle$. Also, for $U_1, U_2 \in \mathcal{U}$ with $U_1 \neq U_2$ we have

$$\begin{aligned} 0 = \langle U_1\rho, U_2\rho \rangle &= \cos^2 \alpha \cdot \langle U_1\psi, U_2\psi \rangle + i \sin \alpha \cos \alpha \cdot \langle U_1\eta, U_2\psi \rangle \\ &\quad - i \sin \alpha \cos \alpha \cdot \langle U_1\psi, U_2\eta \rangle + \sin^2 \alpha \cdot \langle U_1\eta, U_2\eta \rangle \\ &= i \sin \alpha \cos \alpha \cdot (\langle U_1\eta, U_2\psi \rangle - \langle U_1\psi, U_2\eta \rangle), \end{aligned}$$

which implies $\langle U_1\eta, U_2\psi \rangle = \langle U_1\psi, U_2\eta \rangle$ as required. \square

The above result gives an *experimental method* of checking whether $V^2 = I$ for a given pair $\psi, \eta \in \mathcal{W}(\mathcal{U})$. One just checks whether

$$\rho := \frac{1}{\sqrt{2}}\psi + \frac{i}{\sqrt{2}}\eta$$

is an element of $\mathcal{W}(\mathcal{U})$, which is much simpler than attempting to work with the infinite matrix of V with respect to the basis $\mathcal{U}\psi$ (or some other basis for H).

2.4.8. Connectedness

If we consider again the example of the left regular representation π of a group G on $H := l^2(G)$, then the local commutant of $\mathcal{U} := \pi(G)$ at a vector $\psi \in \mathcal{W}(\pi(G))$ is just the commutant of $\pi(G)$. So since the unitary group of the von Neumann algebra $(\pi(G))'$ is norm-arcwise-connected, it follows that $\mathcal{W}(\pi(G))$ is norm-arcwise-connected.

Problem A in [11] asked whether $\mathcal{W}(D, L)$ is norm-arcwise-connected. It turned out that this conjecture was also formulated independently by Guido Weiss ([38], [37]) from a harmonic analysis point of view (our point of view was purely functional analysis), and this problem (and related problems) was the primary stimulation for the creation of the WUTAM CONSORTIUM – a team of 14 researchers based at Washington University and Texas A&M University. (See [52].)

This *connectedness conjecture* was answered yes in [52] for the special case of the family of dyadic orthonormal MRA wavelets in $L^2(\mathbb{R})$, but still remains open for the family of *arbitrary* dyadic orthonormal wavelets in $L^2(\mathbb{R})$.

In the wavelet case $\mathcal{U}_{D,T}$, if $\psi \in \mathcal{W}(D, T)$ then it turns out that $\mathcal{C}_\psi(\mathcal{U}_{D,T})$ is in fact *much larger* than $(\mathcal{U}_{D,T})' = \{D, T\}'$, underscoring the fact that $\mathcal{U}_{D,T}$ is NOT a group. In particular, $\{D, T\}'$ is abelian while $\mathcal{C}_\psi(D, T)$ is nonabelian for every wavelet ψ . (The proof of these facts are contained in [11].)

2.5. Wavelet Sets

2.5.1. The Fourier Transform

We will use the following form of the Fourier–Plancherel transform \mathcal{F} on $\mathcal{H} = L^2(\mathbb{R})$, because it is a form *normalized* so it is a unitary transformation. Although there is another such *normalized* form that is frequently used, and actually simpler, the present form is the one we used in our original first paper [11] involving operator theory and wavelets, and so we will stick with it in these notes to avoid any confusion to a reader of both.

If $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then

$$(\mathcal{F}f)(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t) dt := \hat{f}(s), \quad (14)$$

and

$$(\mathcal{F}^{-1}g)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ist} g(s) ds. \quad (15)$$

We have

$$(\mathcal{F}T_\alpha f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t - \alpha) dt = e^{-is\alpha} (\mathcal{F}f)(s).$$

So $\mathcal{F}T_\alpha \mathcal{F}^{-1}g = e^{-is\alpha}g$. For $A \in \mathcal{B}(\mathcal{H})$ let \hat{A} denote $\mathcal{F}A\mathcal{F}^{-1}$. Thus

$$\hat{T}_\alpha = M_{e^{-i\alpha s}}, \tag{16}$$

where for $h \in L^\infty$ we use M_h to denote the multiplication operator $f \rightarrow hf$. Since $\{M_{e^{-i\alpha s}} : \alpha \in \mathbb{R}\}$ generates the m.a.s.a. $\mathcal{D}(\mathbb{R}) := \{M_h : h \in L^\infty(\mathbb{R})\}$ as a von Neumann algebra, we have

$$\mathcal{F}A_T \mathcal{F}^{-1} = \mathcal{D}(\mathbb{R}).$$

Similarly,

$$\begin{aligned} (\mathcal{F}D^n f)(s) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} (\sqrt{2})^n f(2^n t) dt \\ &= (\sqrt{2})^{-n} \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i2^{-n}st} f(t) dt \\ &= (\sqrt{2})^{-2} (\mathcal{F}f)(2^{-n}s) = (D^{-n} \mathcal{F}f)(s). \end{aligned}$$

So $\hat{D}^n = D^{-n} = D^{*n}$. Therefore,

$$\hat{D} = D^{-1} = D^*. \tag{17}$$

2.5.2. The Commutant of $\{D, T\}$

We have $\mathcal{F}\{D, T\}'\mathcal{F}^{-1} = \{\hat{D}, \hat{T}\}'$. It turns out that $\{\hat{D}, \hat{T}\}'$ has an easy characterization.

Theorem 6:

$$\{\hat{D}, \hat{T}\}' = \{M_h : h \in L^\infty(\mathbb{R}) \text{ and } h(s) = h(2s) \text{ a.e.}\}.$$

Proof: Since $\hat{D} = D^*$ and D is unitary, it is clear that $M_h \in \{\hat{D}, \hat{T}\}'$ if and only if M_h commutes with D . So let $g \in L^2(\mathbb{R})$ be arbitrary. Then (a.e.) we have

$$\begin{aligned} (M_h Dg)(s) &= h(s)(\sqrt{2}g(2s)), \quad \text{and} \\ (DM_h g)(s) &= D(h(s)g(s)) = \sqrt{h}(2s)g(2s). \end{aligned}$$

Since these must be equal a.e. for arbitrary g , we must have $h(s) = h(2s)$ a.e. \square

Now let $E = [-2, -1) \cup [1, 2)$, and for $n \in \mathbb{Z}$ let $E_n = \{2^n x : x \in E\}$. Observe that the sets E_n are disjoint and have union $\mathbb{R} \setminus \{0\}$. So if g is any uniformly bounded function on E , then g extends uniquely (a.e.) to a function $\tilde{g} \in L^\infty(\mathbb{R})$ satisfying

$$\tilde{g}(s) = \tilde{g}(2s), \quad s \in \mathbb{R},$$

by setting

$$\tilde{g}(2^n s) = g(s), \quad s \in E, n \in \mathbb{Z},$$

and $\tilde{g}(0) = 0$. We have $\|\tilde{g}\|_\infty = \|g\|_\infty$. Conversely, if h is any function satisfying $h(s) = h(2s)$ a.e., then h is uniquely (a.e.) determined by its restriction to E . This 1-1 mapping $g \rightarrow M_{\tilde{g}}$ from $L^\infty(E)$ onto $\{\widehat{D}, \widehat{T}\}'$ is a *-isomorphism.

We will refer to a function h satisfying $h(s) = h(2s)$ a.e. as a *2-dilation periodic function*. This gives a simple algorithm for computing a large class of wavelets from a given one, by simply modifying the *phase*:

Given ψ , let $\widehat{\psi} = \mathcal{F}(\psi)$, choose a real-valued function $h \in L^\infty(E)$ arbitrarily, let $g = \exp(ih)$, extend to a 2-dilation periodic function \tilde{g} as above, and compute $\psi_{\tilde{g}} = \mathcal{F}^{-1}(\tilde{g}\widehat{\psi})$. (18)

In the description above, the set E could clearly be replaced with $[-2\pi, -\pi) \cup [\pi, 2\pi)$, or with any other “dyadic” set $[-2a, a) \cup [a, 2a)$ for some $a > 0$.

2.5.3. Wavelets of Computationally Elementary Form

We now give an account of *s*-elementary and *MSF*-wavelets. The two most elementary dyadic orthonormal wavelets are the *Haar wavelet* and *Shannon’s wavelet* (also called the Littlewood–Paley wavelet).

The Haar wavelet is the function

$$\psi_H(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} \leq t \leq 1 \\ 0, & \text{otherwise.} \end{cases} \quad (19)$$

In this case it is very easy to see that the dilates/translates

$$\{2^{\frac{n}{2}} \psi_H(2^n - \ell) : n, \ell \in \mathbb{Z}\}$$

are orthonormal, and an elementary argument shows that their span is dense in $L^2(\mathbb{R})$.

Shannon's wavelet is the $L^2(\mathbb{R})$ -function with Fourier transform $\widehat{\psi}_S = \frac{1}{\sqrt{2\pi}}\chi_{E_0}$ where

$$E_0 = [-2\pi, -\pi) \cup [\pi, 2\pi). \quad (20)$$

The argument that $\widehat{\psi}_S$ is a wavelet is in a way even more transparent than for the Haar wavelet. And it has the advantage of generalizing nicely. For a simple argument, start from the fact that the exponents

$$\{e^{i\ell s} : \ell \in \mathbb{Z}\}$$

restricted to $[0, 2\pi]$ and normalized by $\frac{1}{\sqrt{2\pi}}$ is an orthonormal basis for $L^2[0, 2\pi]$. Write $E_0 = E_- \cup E_+$ where $E_- = [-2\pi, -\pi)$, $E_+ = [\pi, 2\pi)$. Since $\{E_- + 2\pi, E_+\}$ is a partition of $[0, 2\pi)$ and since the exponentials $e^{i\ell s}$ are invariant under translation by 2π , it follows that

$$\left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}} \Big|_{E_0} : \ell \in \mathbb{Z} \right\} \quad (21)$$

is an orthonormal basis for $L^2(E_0)$. Since $\widehat{T} = M_{e^{-is}}$, this set can be written

$$\{\widehat{T}^\ell \widehat{\psi}_s : \ell \in \mathbb{Z}\}. \quad (22)$$

Next, note that any "dyadic interval" of the form $J = [b, 2b)$, for some $b > 0$ has the property that $\{2^n J : n \in \mathbb{Z}\}$, is a partition of $(0, \infty)$. Similarly, any set of the form

$$\mathcal{K} = [-2a, -a) \cup [b, 2b) \quad (23)$$

for $a, b > 0$, has the property that

$$\{2^n \mathcal{K} : n \in \mathbb{Z}\}$$

is a partition of $\mathbb{R} \setminus \{0\}$. It follows that the space $L^2(\mathcal{K})$, considered as a subspace of $L^2(\mathbb{R})$, is a complete wandering subspace for the dilation unitary $(Df)(s) = \sqrt{2} f(2s)$. For each $n \in \mathbb{Z}$,

$$D^n(L^2(\mathcal{K})) = L^2(2^{-n}\mathcal{K}). \quad (24)$$

So $\bigoplus_n D^n(L^2(\mathcal{K}))$ is a direct sum decomposition of $L^2(\mathbb{R})$. In particular E_0 has this property. So

$$D^n \left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}} \Big|_{E_0} : \ell \in \mathbb{Z} \right\} = \left\{ \frac{e^{2^n i\ell s}}{\sqrt{2\pi}} \Big|_{2^{-n}E_0} : \ell \in \mathbb{Z} \right\} \quad (25)$$

is an orthonormal basis for $L^2(2^{-n}E_0)$ for each n . It follows that

$$\{D^n \widehat{T}^\ell \widehat{\psi}_s : n, \ell \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. Hence $\{D^n T^\ell \psi_s : n, \ell \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, as required.

The Haar wavelet can be generalized, and in fact Daubechie's well-known continuous compactly-supported wavelet is a generalization of the Haar wavelet. However, known generalization of the Haar wavelet are all more complicated and difficult to work with in hand-computations.

For our work, in order to proceed with developing an operator algebraic theory that had a chance of directly impacting concrete function-theoretic wavelet theory we needed a large supply of examples of wavelets which were elementary enough to work with. First, we found another "Shannon-type" wavelet in the literature. This was the Journe wavelet, which we found described on p. 136 in Daubechie's book [14]. Its Fourier transform is $\widehat{\psi}_J = \frac{1}{\sqrt{2\pi}} \chi_{E_J}$, where

$$E_J = \left[-\frac{32\pi}{7}, -4\pi\right) \cup \left[-\pi, -\frac{4\pi}{7}\right) \cup \left[\frac{4\pi}{7}, \pi\right) \cup \left[4\pi, \frac{32\pi}{7}\right).$$

Then, thinking the old adage "where there's smoke there's fire!", we painstakingly worked out many more examples. So far, these are the basic building blocks in the *concrete* part of our theory. By this we mean the part of our theory that has had some type of direct impact on function-theoretic wavelet theory.

2.5.4. Definition of Wavelet Set

We define a *wavelet set* to be a measurable subset E of \mathbb{R} for which $\frac{1}{\sqrt{2\pi}} \chi_E$ is the Fourier transform of a wavelet. The wavelet $\widehat{\psi}_E := \frac{1}{\sqrt{2\pi}} \chi_E$ is called *s-elementary* in [11].

It turns out that this class of wavelets was also discovered and systematically explored completely independently, and in about the same time period, by Guido Weiss (Washington University), his colleague and former student E. Hernandez (U. Madrid), and his students X. Fang and X. Wang. In [17,37, 38] they are called MSF (minimally supported frequency) wavelets. In signal processing, the parameter s , which is the independent variable for $\widehat{\psi}$, is the *frequency* variable, and the variable t , which is the independent variable for ψ , is the *time* variable. No function with support a subset of a wavelet set E of strictly smaller measure can be the Fourier transform of a wavelet.

Problem. Must the support of the Fourier transform of a wavelet contain a wavelet set? This question is open for dimension 1. It makes sense for any

finite dimension.

2.5.5. The Spectral Set Condition

From the argument above describing why Shannon's wavelet is, indeed, a wavelet, it is clear that *sufficient* conditions for E to be a wavelet set are

- (i) the normalized exponential $\frac{1}{\sqrt{2\pi}}e^{i\ell s}$, $\ell \in \mathbb{Z}$, when restricted to E should constitute an orthonormal basis for $L^2(E)$ (in other words E is a *spectral set* for the integer lattice \mathbb{Z}),

and

- (ii) The family $\{2^n E: n \in \mathbb{Z}\}$ of dilates of E by integral powers of 2 should constitute a measurable partition (i.e. a partition modulo null sets) of \mathbb{R} .

These conditions are also necessary. In fact if a set E satisfies (i), then for it to be a wavelet set it is obvious that (ii) must be satisfied. To show that (i) must be satisfied by a wavelet set E , consider the vectors

$$\widehat{D}^n \widehat{\psi}_E = \frac{1}{\sqrt{2\pi}} \chi_{2^{-n}E}, \quad n \in \mathbb{Z}.$$

Since $\widehat{\psi}_E$ is a wavelet these must be orthogonal, and so the sets $\{2^n E: n \in \mathbb{Z}\}$ must be disjoint modulo null sets. It follows that $\{\frac{1}{\sqrt{2\pi}}e^{i\ell s}|_E: \ell \in \mathbb{Z}\}$ is not only an orthonormal set of vectors in $L^2(E)$, it must also *span* $L^2(E)$.

It is known from the theory of *spectral sets* (as an elementary special case) that a measurable set E satisfies (i) if and only if it is a generator of a measurable partition of \mathbb{R} under translation by 2π (i.e. iff $\{E+2\pi n: n \in \mathbb{Z}\}$ is a measurable partition of \mathbb{R}). This result generalizes to spectral sets for the integral lattice in \mathbb{R}^n . For this elementary special case a direct proof is not hard.

2.5.6. Translation and Dilation Congruence

We say that measurable sets E, F are *translation congruent modulo 2π* if there is a measurable bijection $\phi: E \rightarrow F$ such that $\phi(s) - s$ is an integral multiple of 2π for each $s \in E$; or equivalently, if there is a measurable partition $\{E_n: n \in \mathbb{Z}\}$ of E such that

$$\{E_n + 2n\pi: n \in \mathbb{Z}\} \tag{26}$$

is a measurable partition of F . Analogously, define measurable sets G and H to be *dilation congruent modulo 2* if there is a measurable bijection $\tau: G \rightarrow H$ such that for each $s \in G$ there is an integer n , depending on s , such that $\tau(s) = 2^n s$; or equivalently, if there is a measurable partition $\{G_n\}_{-\infty}^{\infty}$ of G such that

$$\{2^n G\}_{-\infty}^{\infty} \tag{27}$$

is a measurable partition of H . (Translation and dilation congruency modulo other positive numbers of course make sense as well.)

The following lemma is useful.

Lemma 7: *Let $f \in L^2(\mathbb{R})$, and let $E = \text{supp}(f)$. Then f has the property that*

$$\{e^{ins} f: n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(E)$ if and only if

- (i) *E is congruent to $[0, 2\pi)$ modulo 2π , and*
- (ii) *$|f(s)| = \frac{1}{\sqrt{2\pi}}$ a.e. on E .*

If E is a measurable set which is 2π -translation congruent to $[0, 2\pi)$, then since

$$\left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}} \Big|_{[0, 2\pi)} : \ell \in \mathbb{Z} \right\}$$

is an orthonormal basis for $L^2[0, 2\pi]$ and the exponentials $e^{i\ell s}$ are 2π -invariant, as in the case of Shannon's wavelet it follows that

$$\left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}} \Big|_E : \ell \in \mathbb{Z} \right\}$$

is an orthonormal basis for $L^2(E)$. Also, if E is 2π -translation congruent to $[0, 2\pi)$, then since

$$\{[0, 2\pi) + 2\pi n: n \in \mathbb{Z}\}$$

is a measurable partition of \mathbb{R} , so is

$$\{E + 2\pi n: n \in \mathbb{Z}\}.$$

These arguments can be reversed.

We say that a measurable subset $G \subset \mathbb{R}$ is a *2-dilation generator* of a *partition* of \mathbb{R} if the sets

$$2^n G := \{2^n s: s \in G\}, \quad n \in \mathbb{Z} \tag{28}$$

are disjoint and $\mathbb{R} \setminus \cup_n 2^n G$ is a null set. Also, we say that $E \subset \mathbb{R}$ is a 2π -translation generator of a partition of \mathbb{R} if the sets

$$E + 2n\pi := \{s + 2n\pi : s \in E\}, \quad n \in \mathbb{Z}, \quad (29)$$

are disjoint and $\mathbb{R} \setminus \cup_n (E + 2n\pi)$ is a null set.

Lemma 8: *A measurable set $E \subseteq \mathbb{R}$ is a 2π -translation generator of a partition of \mathbb{R} if and only if, modulo a null set, E is translation congruent to $[0, 2\pi)$ modulo 2π . Also, a measurable set $G \subseteq \mathbb{R}$ is a 2-dilation generator of a partition of \mathbb{R} if and only if, modulo a null set, G is a dilation congruent modulo 2 to the set $[-2\pi, -\pi) \cup [\pi, 2\pi)$.*

2.5.7. A Criterion

The following is a useful criterion for wavelet sets. It was published independently by Dai–Larson in [11] and by Fang–Wang in [17] at about the same time in December, 1994. In fact, it is amusing that the two papers had been submitted within two days of each other; only much later did we even learn of each others work and of this incredible timing.

Proposition 9: *Let $E \subseteq \mathbb{R}$ be a measurable set. Then E is a wavelet set if and only if E is both a 2-dilation generator of a partition (modulo null sets) of \mathbb{R} and a 2π -translation generator of a partition (modulo null sets) of \mathbb{R} . Equivalently, E is a wavelet set if and only if E is both translation congruent to $[0, 2\pi)$ modulo 2π and dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$ modulo 2.*

Note that a set is 2π -translation congruent to $[0, 2\pi)$ iff it is 2π -translation congruent to $[-2\pi, \pi) \cup [\pi, 2\pi)$. So the last sentence of Proposition 9 can be stated: A measurable set E is a wavelet set if and only if it is both 2π -translation and 2-dilation congruent to the Littlewood–Paley set $[-2\pi, -\pi) \cup [\pi, 2\pi)$.

2.6. Phases

If E is a wavelet set, and if $f(s)$ is any function with support E which has constant modulus $\frac{1}{\sqrt{2\pi}}$ on E , then $\mathcal{F}^{-1}(f)$ is a wavelet. Indeed, by Lemma 7 $\{\widehat{T}^\ell f : \ell \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(E)$, and since the sets $2^n E$ partition \mathbb{R} , so $L^2(E)$ is a complete wandering subspace for \widehat{D} , it follows that $\{\widehat{D}^n \widehat{T}^\ell f : n, \ell \in \mathbb{Z}\}$ must be an orthonormal basis for $L^2(\mathbb{R})$, as required. In [17, 37, 38] the term MSF-wavelet includes this type of wavelet.

So MSF-wavelets can have arbitrary phase and s -elementary wavelets have phase 0. *Every* phase is attainable in the sense of chapter 3 for an MSF or s -elementary wavelet.

2.6.1. Some Examples of One-Dimensional Wavelet Sets

It is usually easy to determine, using the dilation-translation criteria, in Proposition 9, whether a given finite union of intervals is a wavelet set. In fact, to verify that a given “candidate” set E is a wavelet set, it is clear from the above discussion and criteria that it suffices to do two things.

- (1) Show, by appropriate partitioning, that E is 2-dilation-congruent to a set of the form $[-2a, -a) \cup [b, 2b)$ for some $a, b > 0$.

and

- (2) Show, by appropriate partitioning, that E is 2π -translation-congruent to a set of the form $[c, c + 2\pi)$ for some real number c .

On the other hand, wavelet sets suitable for testing hypotheses, can be quite difficult to construct. There are very few “recipes” for wavelet sets, as it were. Many families of such sets have been constructed for reasons including perspective, experimentation, testing hypotheses, etc., including perhaps the pure enjoyment of doing the computations – which are somewhat “puzzle-like” in nature. In working with the theory it is nice (and in fact necessary) to have a large supply of wavelets on hand that permit relatively simple analysis.

For this reason we take the opportunity here to present for the reader a collection of such sets, mainly taken from [11], leaving most of the “fun” in verifying that they are indeed wavelet sets to the reader.

We refer the reader to [12] for a proof of the existence of wavelet sets in $\mathbb{R}^{(n)}$, and a proof that there are sufficiently many to generate the Borel structure of $\mathbb{R}^{(n)}$. These results are true for arbitrary expansive dilation factors. Some concrete examples in the plane were subsequently obtained by Soardi and Weiland, and others were obtained by Gu and Speegle. Two had also been obtained by Dai for inclusion in the revised concluding remarks section of our Memoir [11].

In these examples we will usually write intervals as half-open intervals $[\cdot, \cdot)$ because it is easier to verify the translation and dilation congruency relations (1) and (2) above when wavelet sets are written thus, even though

in actuality the relations need only hold modulo null sets.

(i) As mentioned above, an example due to Journé of a wavelet which admits no multiresolution analysis is the s -elementary wavelet with wavelet set

$$\left[-\frac{32\pi}{7}, -4\pi\right) \cup \left[-\pi, \frac{4\pi}{7}\right) \cup \left[\frac{4\pi}{7}, \pi\right) \cup \left[4\pi, \frac{32\pi}{7}\right).$$

To see that this satisfies the criteria, label these intervals, in order, as J_1, J_2, J_3, J_4 and write $J = \cup J_i$. Then

$$J_1 \cup 4J_2 \cup 4J_3 \cup J_4 = \left[-\frac{32\pi}{7}, -\frac{16\pi}{7}\right) \cup \left[\frac{16\pi}{7}, \frac{32\pi}{7}\right).$$

This has the form $[-2a, a) \cup [b, 2b)$ so is a 2-dilation generator of a partition of $\mathbb{R} \setminus \{0\}$. Then also observe that

$$\{J_1 + 6\pi, J_2 + 2\pi, J_3, J_4 - 4\pi\}$$

is a partition of $[0, 2\pi)$.

(ii) The Shannon (or Littlewood–Paley) set can be generalized. For any $-\pi < \alpha < \pi$, the set

$$E_\alpha = [-2\pi + 2\alpha, -\pi + \alpha) \cup [\pi + \alpha, 2\pi + 2\alpha)$$

is a wavelet set. Indeed, it is clearly a 2-dilation generator of a partition of $\mathbb{R} \setminus \{0\}$, and to see that it satisfies the translation congruency criterion for $-\pi < \alpha \leq 0$ (the case $0 < \alpha < \pi$ is analogous) just observe that

$$\{[-2\pi + 2\alpha, 2\pi) + 4\pi, [-2\pi, -\pi + \alpha) + 2\pi, [\pi + \alpha, 2\pi + 2\alpha)\}$$

is a partition of $[0, 2\pi)$. It is clear that ψ_{E_α} is then a continuous (in $L^2(\mathbb{R})$ -norm) path of s -elementary wavelets. Note that

$$\lim_{\alpha \rightarrow \pi} \widehat{\psi}_{E_\alpha} = \frac{1}{\sqrt{2\pi}} \chi_{[2\pi, 4\pi)}.$$

This is *not* the Fourier transform of a wavelet because the set $[2\pi, 4\pi)$ is not a 2-dilation generator of a partition of $\mathbb{R} \setminus \{0\}$. So

$$\lim_{\alpha \rightarrow \pi} \psi_{E_\alpha}$$

is not an orthogonal wavelet. (It is what is known as a Hardy wavelet because it generates an orthonormal basis for $H^2(\mathbb{R})$ under dilation and translation.) This example demonstrates that $\mathcal{W}(D, T)$ is *not* closed in $L^2(\mathbb{R})$.

(iii) Journé's example above can be extended to a path. For $-\frac{\pi}{7} \leq \beta \leq \frac{\pi}{7}$ the set

$$J_\beta = \left[-\frac{32\pi}{7}, -4\pi + 4\beta\right) \cup \left[-\pi + \beta, -\frac{4\pi}{7}\right) \cup \left[\frac{4\pi}{7}, \pi + \beta\right) \cup \left[4\pi + 4\beta, 4\pi + \frac{4\pi}{7}\right)$$

is a wavelet set. The same argument in (i) establishes dilation congruency. For translation, the argument in (i) shows congruency to $[4\beta, 2\pi + 4\beta)$ which is in turn congruent to $[0, 2\pi)$ as required. Observe that here, as opposed to in (ii) above, the limit of ψ_{J_β} as β approaches the boundary point $\frac{\pi}{7}$ is a wavelet. Its wavelet set is a union of 3 disjoint intervals.

(iv) Let $A \subseteq [\pi, \frac{3\pi}{2})$ be an arbitrary measurable subset. Then there is a wavelet set W , such that $W \cap [\pi, \frac{3\pi}{2}) = A$. For the construction, let

$$B = [2\pi, 3\pi) \setminus 2A,$$

$$C = \left[-\pi, -\frac{\pi}{2}\right) \setminus (A - 2\pi)$$

$$\text{and } D = 2A - 4\pi.$$

Let

$$W = \left[\frac{3\pi}{2}, 2\pi\right) \cup A \cup B \cup C \cup D.$$

We have $W \cap [\pi, \frac{3\pi}{2}) = A$. Observe that the sets $[\frac{3\pi}{2}, 2\pi)$, A, B, C, D , are disjoint. Also observe that the sets

$$\left[\frac{3\pi}{2}, 2\pi\right), A, \frac{1}{2}B, 2C, D,$$

are disjoint and have union $[-2\pi, -\pi) \cup [\pi, 2\pi)$. In addition, observe that the sets

$$\left[\frac{3\pi}{2}, 2\pi\right), A, B - 2\pi, C + 2\pi, D + 2\pi,$$

are disjoint and have union $[0, 2\pi)$. Hence W is a wavelet set.

(v) Wavelet sets for arbitrary (not necessarily integral) dilation factors other than 2 exist. For instance, if $d \geq 2$ is arbitrary, let

$$A = \left[-\frac{2d\pi}{d+1}, -\frac{2\pi}{d+1}\right),$$

$$B = \left[\frac{2\pi}{d^2-1}, \frac{2\pi}{d+1}\right),$$

$$C = \left[\frac{2d\pi}{d+1}, \frac{2d^2\pi}{d^2-1}\right)$$

and let $G = A \cup B \cup C$. Then G is d -wavelet set. To see this, note that $\{A + 2\pi, B, C\}$ is a partition of an interval of length 2π . So G is 2π -translation-congruent to $[0, 2\pi)$. Also, $\{A, B, d^{-1}C\}$ is a partition of the set $[-d\alpha, -\alpha) \cup [\beta, d\beta)$ for $\alpha = \frac{2\pi}{d^2-1}$, and $\beta = \frac{2\pi}{d^2-1}$, so from this form it follows that $\{d^n G : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R} \setminus \{0\}$. Hence if $\psi := \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}\chi_G)$, it follows that $\{d^{\frac{n}{2}}\psi(d^n t - \ell) : n, \ell \in \mathbb{Z}\}$ is orthonormal basis for $L^2(\mathbb{R})$, as required.

2.7. Operator-Theoretic Interpolation of Wavelets: The Special Case of Wavelet Sets

Let E, F be a pair of wavelet sets. Then for (a.e.) $x \in E$ there is a unique $y \in F$ such that $x - y \in 2\pi\mathbb{Z}$. This is the *translation congruence* property of wavelet sets. Also, for (a.e.) $x \in E$ there is a unique $z \in F$ such that $\frac{x}{z}$ is an integral power of 2. This is the *dilation congruence* property of wavelet sets. (See section 2.5.6.)

There is a natural *closed-form algorithm* for the *interpolation unitary* $V_{\psi_E}^{\psi_F}$ which maps the wavelet basis for $\widehat{\psi}_E$ to the wavelet basis for $\widehat{\psi}_F$. Indeed, using both the translation and dilation congruence properties of $\{E, F\}$, one can explicitly compute a (unique) measure-preserving transformation $\sigma := \sigma_E^F$ mapping \mathbb{R} onto \mathbb{R} which has the property that $V_{\psi_E}^{\psi_F}$ is identical with the *composition operator* defined by:

$$f \mapsto f \circ \sigma^{-1}$$

for all $f \in L^2(\mathbb{R})$. With this formulation, compositions of the maps σ between different pairs of wavelet sets are not difficult to compute, and thus products of the corresponding interpolation unitaries can be computed in terms of them.

2.7.1. *The Interpolation Map σ*

Let E and F be arbitrary wavelet sets. Let $\sigma : E \rightarrow F$ be the 1-1, onto map implementing the 2π -translation congruence. Since E and F both generated partitions of $\mathbb{R} \setminus \{0\}$ under dilation by powers of 2, we may extend σ to a 1-1 map of \mathbb{R} onto \mathbb{R} by defining $\sigma(0) = 0$, and

$$\sigma(s) = 2^n \sigma(2^{-n}s) \quad \text{for } s \in 2^n E, \quad n \in \mathbb{Z}. \tag{30}$$

We adopt the notation σ_E^F for this, and call it the *interpolation map* for the ordered pair (E, F) .

Lemma 10: *In the above notation, σ_E^F is a measure-preserving transformation from \mathbb{R} onto \mathbb{R} .*

Proof: Let $\sigma := \sigma_E^F$. Let $\Omega \subseteq \mathbb{R}$ be a measurable set. Let $\Omega_n = \Omega \cap 2^n E$, $n \in \mathbb{Z}$, and let $E_n = 2^{-n}\Omega_n \subseteq E$. Then $\{\Omega_n\}$ is a partition of Ω , and we have $m(\sigma(E_n)) = m(E_n)$ because the restriction of σ to E is measure-preserving. So

$$\begin{aligned} m(\sigma(\Omega)) &= \sum_n m(\sigma(\Omega_n)) = \sum_n m(2^n \sigma(E_n)) \\ &= \sum_n 2^n m(\sigma(E_n)) = \sum_n 2^n m(E_n) \\ &= \sum_n m(2^n E_n) = \sum_n m(\Omega_n) = m(\Omega). \end{aligned} \quad \square$$

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called *2-homogeneous* if $f(2s) = 2f(s)$ for all $s \in \mathbb{R}$. Equivalently, f is 2-homogeneous iff $f(2^n s) = 2^n f(s)$, $s \in \mathbb{R}$, $n \in \mathbb{Z}$. Such a function is completely determined by its values on any subset of \mathbb{R} which generates a partition of $\mathbb{R} \setminus \{0\}$ by 2-dilation. So σ_E^F is the (unique) 2-homogeneous extension of the 2π -transition congruence $E \rightarrow F$. The set of all 2-homogeneous measure-preserving transformations of \mathbb{R} clearly forms a group under composition. Also, the composition of a 2-dilation-periodic function f with a 2-homogeneous function g is (in either order) 2-dilation periodic. We have $f(g(2s)) = f(2g(s)) = f(g(s))$ and $g(f(2s)) = g(f(s))$. These facts will be useful.

2.7.2. An Algorithm For The Interpolation Unitary

Now let

$$U_E^F := U_{\sigma_E^F}, \tag{31}$$

where if σ is any measure-preserving transformation of \mathbb{R} then U_σ denotes the composition operator defined by $U_\sigma f = f \circ \sigma^{-1}$, $f \in L^2(\mathbb{R})$. Clearly $(\sigma_E^F)^{-1} = \sigma_F^E$ and $(U_E^F)^* = U_F^E$. We have $U_E^F \widehat{\psi}_E = \widehat{\psi}_F$ since $\sigma_E^F(E) = F$. That is,

$$U_E^F \widehat{\psi}_E = \widehat{\psi}_E \circ \sigma_F^E = \frac{1}{\sqrt{2\pi}} \chi_E \circ \sigma_F^E = \frac{1}{\sqrt{2\pi}} \chi_F = \widehat{\psi}_F.$$

Proposition 11: *Let E and F be arbitrary wavelet sets. Then $U_E^F \in \mathcal{C}_{\widehat{\psi}_E}(\widehat{D}, \widehat{T})$. Hence $\mathcal{F}^{-1} U_E^F \mathcal{F}$ is the interpolation unitary for the ordered pair (ψ_E, ψ_F) .*

Proof: Write $\sigma = \sigma_E^F$ and $U_\sigma = U_E^F$. We have $U_\sigma \widehat{\psi}_E = \widehat{\psi}_F$ since $\sigma(E) = F$. We must show

$$U_\sigma \widehat{D}^n \widehat{T}^l \widehat{\psi}_E = \widehat{D}^n \widehat{T}^l U_\sigma \widehat{\psi}_E, \quad n, l \in \mathbb{Z}.$$

We have

$$\begin{aligned} (U_\sigma \widehat{D}^n \widehat{T}^l \widehat{\psi}_E)(s) &= (U_\sigma \widehat{D}^n e^{-ils} \widehat{\psi}_E)(s) \\ &= U_\sigma 2^{-\frac{n}{2}} e^{-il2^{-n}s} \widehat{\psi}_E(2^{-n}s) \\ &= 2^{-\frac{n}{2}} e^{-il2^{-n}\sigma^{-1}(s)} \widehat{\psi}_E(2^{-n}\sigma^{-1}(s)) \\ &= 2^{-\frac{n}{2}} e^{-il\sigma^{-1}(2^{-n}s)} \widehat{\psi}_E(\sigma^{-1}(2^{-n}s)) \\ &= 2^{-\frac{n}{2}} e^{-il\sigma^{-1}(2^{-n}s)} \widehat{\psi}(2^{-n}s). \end{aligned}$$

This last term is nonzero iff $2^{-n}s \in F$, in which case $\sigma^{-1}(2^{-n}s) = \sigma_F^E(2^{-n}s) = 2^{-n}s + 2\pi k$ for some $k \in \mathbb{Z}$ since σ_F^E is a 2π -translation-congruence on F . It follows that $e^{-il\sigma^{-1}(2^{-n}s)} = e^{-il2^{-n}s}$. Hence we have

$$\begin{aligned} (U_\sigma \widehat{D}^n \widehat{T}^l \widehat{\psi}_E)(s) &= 2^{-\frac{n}{2}} e^{-ils-2^n s} \widehat{\psi}_F(2^{-n}s) \\ &= (\widehat{D}^n \widehat{T}^l \widehat{\psi}_F)(s) \\ &= (\widehat{D}^n \widehat{T}^l U_\sigma \widehat{\psi}_E)(s). \end{aligned}$$

We have shown $U_E^F \in \mathcal{C}_{\widehat{\psi}_E}(\widehat{D}, \widehat{T})$. Since $U_E^F \widehat{\psi}_E = \widehat{\psi}_F$, the uniqueness part of Proposition 1 shows that $\mathcal{F}^{-1} U_E^F \mathcal{F}$ must be the interpolation unitary for (ψ_E, ψ_F) . \square

2.8. The Interpolation Unitary Normalizes The Commutant

Proposition 12: *Let E and F be arbitrary wavelet sets. Then the interpolation unitary for the ordered pair (ψ_E, ψ_F) normalizes $\{D, T\}'$.*

Proof: By Proposition 11 we may work with U_E^F in the Fourier transform domain. By Theorem 6, the generic element of $\{\widehat{D}, \widehat{T}\}'$ has the form M_h for some 2-dilation-periodic function $h \in L^\infty(\mathbb{R})$. Write $\sigma = \sigma_E^F$ and $U_\sigma = U_E^F$. Then

$$U_\sigma^{-1} M_h U_\sigma = M_{h \circ \sigma^{-1}}. \quad (32)$$

So since the composition of a 2-dilation-periodic function with a 2-homogeneous function is 2-dilation-periodic, the proof is complete. \square

2.8.1. $\mathcal{C}_\psi(D, T)$ is Nonabelian

It can also be shown ([11, Theorem 5.2 (iii)]) that if E, F are wavelet sets with $E \neq F$ then U_E^F is not contained in the double commutant $\{\widehat{D}, \widehat{T}\}''$. So since U_E^F and $\{\widehat{D}, \widehat{T}\}'$ are both contained in the local commutant of $\mathcal{U}_{\widehat{D}, \widehat{T}}$ at $\widehat{\psi}_E$, this proves that $\mathcal{C}_{\widehat{\psi}_E}(\widehat{D}, \widehat{T})$ is nonabelian. In fact (see [11, Proposition 1.8]) this can be used to show that $\mathcal{C}_\psi(D, T)$ is nonabelian for every wavelet ψ . We suspected this, but we could not prove it until we discovered the “right” way of doing the needed computation using s -elementary wavelets.

The above shows that a pair (E, F) of wavelets sets (or, rather, their corresponding s -elementary wavelets) admits operator-theoretic interpolation if and only if $\text{Group}\{U_E^F\}$ is contained in the local commutant $\mathcal{C}_{\widehat{\psi}_E}(\widehat{D}, \widehat{T})$, since the requirement that U_E^F normalizes $\{\widehat{D}, \widehat{T}\}'$ is automatically satisfied. It is easy to see that this is equivalent to the condition that for each $n \in \mathbb{Z}$, σ^n is a 2π -congruence of E in the sense that $(\sigma^n(s) - s)/2\pi \in \mathbb{Z}$ for all $s \in E$, which in turn implies that $\sigma^n(E)$ is a wavelet set for all n . Here $\sigma = \sigma_E^F$. This property hold trivially if σ is *involutive* (i.e. $\sigma^2 = \text{identity}$).

2.8.2. The Coefficient Criterion

In cases where “torsion” is present, so $(\sigma_E^F)^k$ is the identity map for some finite integer k , the von Neumann algebra generated by $\{\widehat{D}, \widehat{T}\}'$ and $U := U_E^F$ has the simple form

$$\left\{ \sum_{n=0}^k M_{h_n} U^n : h_n \in L^\infty(\mathbb{R}) \text{ with } h_n(2s) = h_n(s), \quad s \in \mathbb{R} \right\},$$

and so each member of this “interpolated” family of wavelets has the form

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^k h_n(s) \chi_{\sigma^n(E)} \tag{33}$$

for 2-dilation periodic “coefficient” functions $\{h_n(s)\}$ which satisfy the necessary and sufficient condition that the operator

$$\sum_{n=0}^k M_{h_n} U^n \tag{34}$$

is unitary.

A standard computation shows that the map θ sending $\sum_0^k M_{h_n} U^n$ to the $k \times k$ function matrix (h_{ij}) given by

$$h_{ij} = h_{\alpha(i,j)} \circ \sigma^{-i+1} \tag{35}$$

where $\alpha(i, j) = (i + 1)$ modulo k , is a $*$ -isomorphism. This matricial algebra is the cross-product of $\{D, T\}'$ by the $*$ -automorphism $ad(U_E^F)$ corresponding to conjugation with U_E^F . For instance, if $k = 3$ then θ maps

$$M_{h_1} + M_{h_2}U_E^F + M_{h_3}(U_E^F)^2$$

to

$$\begin{pmatrix} h_1 & h_2 & h_3 \\ h_3 \circ \sigma^{-1} & h_1 \circ \sigma^{-1} & h_2 \circ \sigma^{-1} \\ h_2 \circ \sigma^{-2} & h_3 \circ \sigma^{-2} & h_1 \circ \sigma^{-2} \end{pmatrix}. \tag{36}$$

This shows that $\sum_0^k M_{h_n}U^n$ is a unitary operator iff the scalar matrix $(h_{ij})(s)$ is unitary for almost all $s \in \mathbb{R}$. Unitarity of this matrix-valued function is called the *Coefficient Criterion* in [11], and the functions h_i are called the interpolation coefficients. This leads to formulas for families of wavelets which are new to wavelet theory.

2.9. Interpolation Pairs of Wavelet Sets

For many interesting cases of note, the interpolation map σ_E^F will in fact be an *involution* of \mathbb{R} (i.e. $\sigma \circ \sigma = id$, where $\sigma := \sigma_E^F$, and where id denotes the identity map). So torsion *will* be present, as in the above section, and it will be present in an essentially simple form. The corresponding interpolation unitary will be a *symmetry* in this case (i.e. a selfadjoint unitary operator with square I).

It is curious to note that verifying a simple operator equation $U^2 = I$ directly by matricial computation can be extremely difficult. It is much more computationally feasible to verify an equation such as this by point-wise (a.e.) verifying explicitly the relation $\sigma \circ \sigma = id$ for the interpolation map. In [11] we gave a number of examples of interpolation pairs of wavelet sets. We give below a collection of examples that has not been previously published: Every pair sets from the Journe family is an interpolation pair.

2.10. Journe Family Interpolation Pairs

Consider the parameterized path of *generalized Journe* wavelet sets given in [11, Example 4.5(iii)]. We have

$$J_\beta = \left[-\frac{32\pi}{7}, -4\pi - 4\beta\right) \cup \left[-\pi + \beta, -\frac{4\pi}{7}\right) \cup \left[\frac{4\pi}{7}, \pi + \beta\right) \cup \left[4\pi + 4\beta, 4\pi + \frac{4\pi}{7}\right)$$

where the set of parameters β ranges $-\frac{\pi}{7} \leq \beta \leq \frac{\pi}{7}$.

Proposition 13: *Every pair $(J_{\beta_1}, J_{\beta_2})$ is an interpolation pair.*

Proof: Let $\beta_1, \beta_2 \in [-\frac{\pi}{7}, \frac{\pi}{7})$ with $\beta_1 < \beta_2$. Write $\sigma = \sigma_{J_{\beta_2}}^{J_{\beta_1}}$. We need to show that

$$\sigma^2(x) = x \tag{*}$$

for all $x \in \mathbb{R}$. Since σ is 2-homogeneous, it suffices to verify (*) only for $x \in J_{\beta_1}$. For $x \in J_{\beta_1} \cap J_{\beta_2}$ we have $\sigma(x) = x$, hence $\sigma^2(x) = x$. So we only need to check (*) for $x \in (J_{\beta_1} \setminus J_{\beta_2})$. We have

$$J_{\beta_1} \setminus J_{\beta_2} = [-\pi + \beta_1, -\pi + \beta_2) \cup [4\pi + 4\beta_1, 4\pi + 4\beta_2).$$

It is useful to also write

$$J_{\beta_2} \setminus J_{\beta_1} = [-4\pi + 4\beta_1, -4\pi + 4\beta_2) \cup [\pi + \beta_1, \pi + \beta_2).$$

On $[-\pi + \beta_1, -\pi + \beta_2)$ we have $\sigma(x) = x + 2\pi$, which lies in $[\pi + \beta_1, \pi + \beta_2)$. If we multiply this by 4, we obtain $4\sigma(x) \in [4\pi + 4\beta_1, 4\pi + 4\beta_2) \subset J_{\beta_1}$. And on $[4\pi + 4\beta_1, 4\pi + 4\beta_2)$ we clearly have $\sigma(x) = x - 8\pi$, which lies in $[-4\pi + 4\beta_1, -4\pi + 4\beta_2)$.

So for $x \in [-\pi + \beta_1, -\pi + \beta_2)$ we have

$$\sigma^2(x) = \sigma(\sigma(x)) = \frac{1}{4}\sigma(4\sigma(x)) = \frac{1}{4}[4\sigma(x) - 8\pi] = \sigma(x) - 2\pi = x + 2\pi - 2\pi = x.$$

On $[4\pi + 4\beta_1, 4\pi + 4\beta_2)$ we have $\sigma(x) = x - 8\pi$, which lies in $[-4\pi + 4\beta_1, -4\pi + 4\beta_2)$. So $\frac{1}{4}\sigma(x) \in [-\pi + \beta_1, -\pi + \beta_2)$. Hence

$$\sigma\left(\frac{1}{4}\sigma(x)\right) = \frac{1}{4}\sigma(x) + 2\pi$$

and thus

$$\sigma^2(x) = 4\sigma\left(\frac{1}{4}\sigma(x)\right) = 4\left[\frac{1}{4}\sigma(x) + 2\pi\right] = \sigma(x) + 8\pi = x - 8\pi + 8\pi = x$$

as required.

We have shown that for all $x \in J_{\beta_1}$ we have $\sigma^2(x) = x$. This proves that $(J_{\beta_1}, J_{\beta_2})$ is an interpolation pair. \square

3. Unitary Systems and Frames

In [33] we developed an operator-theoretic approach to discrete frame theory (i.e. frame sequences, as opposed to continuous frame transforms) on a separable Hilbert space. We then applied it to an investigation of frame vectors for unitary systems, frame wavelets and group representations. The starting-point idea, which is pretty simple-minded in fact, is to realize any frame sequence for a Hilbert space H as a compression of a Riesz basis for

a larger Hilbert space. In other words, a frame is a sequence of vectors in a Hilbert space which *dilates*, (in the operator-theoretic or geometric sense, as opposed to the function-theoretic sense of multiplication of the independent variable of a function by a dilation constant), or *extends*, to a (Riesz) basis for a larger space. From this idea much can be developed, and some new perspective can be given to certain concepts that have been used in engineering circles for many years. See section 3.2. below.

3.1. Basics on Frames

Let H be a separable complex Hilbert space. Let $B(H)$ denote the algebra of all bounded linear operators on H . Let \mathbb{N} denote the natural numbers, and \mathbb{Z} the integers. We will use \mathbb{J} to denote a generic countable (or finite) index set such as $\mathbb{Z}, \mathbb{N}, \mathbb{Z}^{(2)}, \mathbb{N} \cup \mathbb{N}$ etc.

A sequence $\{x_j: j \in \mathbb{N}\}$ of vectors in H is called a *frame* if there are constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_j |\langle x, x_j \rangle|^2 \leq B\|x\|^2$$

for all $x \in H$. The optimal constant (maximal for A and minimal for B) are called the *frame bounds*. The frame $\{x_j\}$ is called a *tight frame* if $A = B$, and is called *Parseval* if $A = B = 1$. (Originally, in [33] and a number of subsequent papers, the term *normalized tight frame* was used for this. However, this term had also been applied by Benedetto and Ficus [5] for another concept: a tight frame of unit vectors; what we now call a uniform tight frame, or spherical frame. So, after all parties involved, the name *Parseval* was adopted. It makes a lot of sense, because a Parseval frame is precisely a frame which satisfies Parseval's identity.) A sequence $\{x_j\}$ is defined to be a *Riesz basis* if it is a frame and is also a basis for H in the sense that for each $x \in H$ there is a *unique* sequence $\{\alpha_j\}$ in \mathbb{C} such that $x = \sum \alpha_j x_j$ with the convergence being in norm. We note that a Riesz basis is also defined to be basis which is obtained from an orthonormal basis by applying a bounded linear invertible operator. This is equivalent to the first definition. It should be noted that in Hilbert spaces the Riesz bases are precisely the bounded unconditional bases. We will say that frames $\{x_j: j \in \mathbb{J}\}$ and $\{y_j: j \in \mathbb{J}\}$ on Hilbert spaces H, K , respectively, are *unitarily equivalent* if there is a unitary $U: H \rightarrow K$ such that $Ux_j = y_j$ for all $j \in \mathbb{J}$. We will say that they are *similar* (or *isomorphic*) if there is a bounded linear invertible operator $T: H \rightarrow K$ such that $Tx_j = y_j$ for all $j \in \mathbb{J}$.

Example 14: Let $K = L^2(\mathbb{T})$ where \mathbb{T} is the unit circle and measure is normalized Lebesgue measure, and let $\{e^{ins} : n \in \mathbb{Z}\}$ be the standard orthonormal basis for $L^2(\mathbb{T})$. If $E \subseteq \mathbb{T}$ is any measurable subset then $\{e^{ins}|_E : n \in \mathbb{Z}\}$ is a Parseval frame for $L^2(E)$. This can be viewed as obtained from the single vector χ_E by applying all integral powers of the (unitary) multiplication operator $M_{e^{is}}$. It turns out that these are all (for different E) unitarily *inequivalent*. This is an example of a Parseval frame which is generated by the action of a unitary group on a single vector. This can be compared with the definition of a *frame wavelet*. (As one might expect, a single function ψ in $L^2(\mathbb{R})$ which generates a frame for $L^2(\mathbb{R})$ under the action of $U_{D,T}$ is called a frame-wavelet.)

3.2. Dilation of Frames: The Discrete Version of Naimark's Theorem

Now let $\{x_n\}_{n \in \mathbb{J}}$ be a Parseval frame and let $\theta: H \rightarrow K := l^2(\mathbb{J})$ be the usual *analysis operator* (this was called the *frame transform in [HL]*) defined by $\theta(x) := (\langle x, x_n \rangle)_{n \in \mathbb{J}}$. This is obviously an isometry. Let P be the orthogonal projection from K onto $\theta(H)$. Denote the standard orthonormal basis for $l^2(\mathbb{J})$ by $\{e_j : j \in \mathbb{J}\}$. For any $m \in \mathbb{J}$, we have

$$\begin{aligned} \langle \theta(x_m), Pe_n \rangle &= \langle P\theta(x_m), e_n \rangle = \langle \theta(x_m), e_n \rangle \\ &= \langle x_m, x_n \rangle = \langle \theta(x_m), \theta(x_m) \rangle. \end{aligned}$$

It follows easily that $\theta(x_n) = Pe_n$, $n \in \mathbb{J}$. Identifying H with $\theta(H)$, this shows indeed that every Parseval frame can be realized by compressing an orthonormal basis, as claimed earlier.

This can actually be viewed as a special case (probably the simplest possible special case) of an old theorem of Naimark concerning operator algebras and dilation of positive operator valued measures to projection valued measures. The connection between Naimark's theorem and the dilation result for Parseval frames, and that the latter can be viewed as a special case of the former, was pointed out to me by Chandler Davis and Dick Kadison in a conference (COSY-1999: The Canadian Operator Algebra Symposium, Prince Edward Island, May 1999).

3.3. Complements of Frames

It is useful to note that P will equal I iff $\{x_n\}$ is a basis. Indeed, if $P \neq I$, then choose $z \neq 0$, $z \in (I - P)K$, and write $z = \sum \alpha_n e_n$ for some sequence $\alpha_n \in \mathbb{C}$. Then $0 = Pz = \sum \alpha_n \theta(x_n)$, and not all the scalars α_n are zero.

Hence $\{x_n\}$ is not topologically linearly independent so cannot even be a Schauder basis. On the other hand if $P = I$ then $\{x_n\}$ is obviously an orthonormal basis.

Suppose $\{x_n\}_{n \in \mathbb{J}}$ is a Parseval frame for H , and let θ, P, K, e_n be as above. Let $M = (I - P)K$. Then $y_n := (I - P)e_n$ is a Parseval frame on M which is *complementary* to $\{x_n\}$ in the sense that the inner direct sum $\{x_n \oplus y_n : n \in \mathbb{J}\}$ is an orthonormal basis for the direct sum Hilbert space $H \oplus M$. Moreover there is uniqueness: The extension of a tight frame to an orthonormal basis described in the above paragraph is unique up to unitary equivalence. That is if N is another Hilbert space and $\{z_n\}$ is a tight frame for N such that $\{x_n \oplus z_n : n \in \mathbb{J}\}$ is an orthonormal basis for $H \oplus N$, then there is a unitary transformation U mapping M onto N such that $Uy_n = z_n$ for all n . In particular, $\dim M = \dim N$.

If $\{x_j\}$ is a Parseval frame, we will call any Parseval frame $\{z_j\}$ such that $\{x_j \oplus z_j\}$ is an orthonormal basis for the direct sum space, a *strong complement* to $\{x_j\}$. So every Parseval frame has a strong complement which is unique up to unitary equivalence. More generally, if $\{y_j\}$ is a general frame we will call any frame $\{w_j\}$ such that $\{y_j \oplus w_j\}$ is a Riesz basis for the direct sum space a *complementary frame* (or *complement*) to $\{x_j\}$.

The notion of strong complement has a natural generalization. Let $\{x_n\}_{n \in \mathbb{J}}$ and $\{y_n\}_{n \in \mathbb{J}}$ be Parseval frames in Hilbert spaces H, K , respectively, indexed by the same set \mathbb{J} . Call these two frames *strongly disjoint* if the (inner) direct sum $\{x_n \oplus y_n : n \in \mathbb{J}\}$ is a Parseval frame for the direct sum Hilbert space $H \oplus K$. It is not hard to see that this property of strong disjointness is equivalent to the property that the ranges of their analysis operators are orthogonal in $l^2(\mathbb{J})$. More generally, we call a k -tuple of Parseval frames $(\{z_{1n}\}_{n \in \mathbb{J}}, \dots, \{z_{kn}\}_{n \in \mathbb{J}})$ in Hilbert spaces H_1, \dots, H_k , respectively, a *strongly disjoint k -tuple* if $\{z_{1n} \oplus \dots \oplus z_{kn} : n \in \mathbb{J}\}$ is a Parseval frame for $H_1 \oplus \dots \oplus H_k$, and we call it a *complete strongly disjoint k -tuple* if $\{z_{1n} \oplus \dots \oplus z_{kn} : n \in \mathbb{J}\}$ is an orthonormal basis for $H_1 \oplus \dots \oplus H_k$. If $\theta_i : H_i \rightarrow l^2(\mathbb{J})$ is the frame transform, $1 \leq i \leq k$, then strong disjointness of a k -tuple is equivalent to mutual orthogonality of $\{\text{ran } \theta_i : 1 \leq i \leq k\}$, and complete strong disjointness is equivalent to the condition that $\bigoplus_{i=1}^k \text{ran } \theta_i = l^2(\mathbb{J})$.

There is a particularly simple intrinsic (i.e. non-geometric) characterization of strong disjointness which is potentially useful in applications: Let $\{x_n\}_{n \in \mathbb{J}}$ and $\{y_n\}_{n \in \mathbb{J}}$ be Parseval frames for Hilbert spaces H and K , re-

spectively. Then $\{x_n\}$ and $\{y_n\}$ are strongly disjoint if and only if one of the equations

$$\begin{aligned} \sum_{n \in \mathbb{J}} \langle x, x_n \rangle y_n = 0 \quad \text{for all } x \in H \\ \text{or} \quad \sum_{n \in \mathbb{J}} \langle y, y_n \rangle x_n = 0 \quad \text{for all } y \in K \end{aligned} \tag{37}$$

holds. Moreover, if one holds the other holds also.

3.4. Super-frames, Super-wavelets, and Multiplexing

Suppose that $\{x_n\}_{n \in \mathbb{J}}$ and $\{y_n\}_{n \in \mathbb{J}}$ are strongly disjoint Parseval frames for Hilbert spaces H and K , respectively. Then given any pair of vectors $x \in H$, $y \in K$, we have that

$$x = \sum_n \langle x, x_n \rangle x_n, \quad y = \sum_n \langle y, y_n \rangle y_n.$$

If we let $a_n = \langle x, x_n \rangle$ and $b_n = \langle y, y_n \rangle$, and then let $c_n = a_n + b_n$, we have

$$\sum_n a_n y_n = 0, \quad \sum_n b_n x_n = 0,$$

by (37) and therefore we have

$$x = \sum_n c_n x_n, \quad y = \sum_n c_n y_n. \tag{38}$$

This says that, by using one set of data $\{c_n\}$, we can recover two vectors x and y (they may even lie in different Hilbert spaces) by applying the respective inverse transforms (synthesis operators) corresponding to the two frame $\{x_n\}$ and $\{y_n\}$. The above argument obviously extends to the k -tuple case: If $\{f_{in} : n \in \mathbb{J}\}$, $i = 1, \dots, k$, is a strongly disjoint k -tuple of Parseval frames for Hilbert spaces H_1, \dots, H_k , and if (x_1, \dots, x_k) is an arbitrary k -tuple of vectors with $x_i \in H_i$, $1 \leq i \leq k$, then (38) generalizes to

$$x_i = \sum_{n \in \mathbb{J}} \langle x_i, f_{in} \rangle f_{in}$$

for each $1 \leq i \leq k$. So if we define a single “master” sequence of complex numbers $\{c_n : n \in \mathbb{J}\}$ by

$$c_n = \sum_{i=1}^k \langle x_i, f_{in} \rangle,$$

then the strong disjointness implies that for each *individual* i we have

$$x_i = \sum_{n \in \mathbb{J}} c_n f_{in}.$$

This simple observation might be useful in applications to data compression.

In [33] we called such an n -tuple of strongly disjoint (or simply just disjoint) frames a *super-frame*, because it (or rather its inner direct sum) is a frame for the superspace which is the direct sum of the individual Hilbert spaces for the frames. In connection with wavelet systems this observation lead us to the notion of *superwavelet*, which is a particular type of vector-valued wavelet. In operator-theoretic terms this is just a restatement of the fact outlined above that a strongly disjoint k -tuple of Parseval frames have frame-transforms which are isometries into the same space $l^2(\mathbb{J})$ which have mutually orthogonal ranges.

The notion of superframes and superwavelets, and many of their properties, were also discovered and investigated by Radu Balan [3] in his Ph.D. thesis, in work that was completely independent from ours.

3.5. Frame Vectors For Unitary Systems

Let \mathcal{U} be a unitary system on a Hilbert space H . Suppose $\mathcal{W}(\mathcal{U})$ is non-empty, and fix $\psi \in \mathcal{W}(\mathcal{U})$. Recall from Section 1 that if η is an arbitrary vector in H , then $\eta \in \mathcal{W}(\mathcal{U})$ if and only if there is a unitary V (which is unique if it exists) in the local commutant $C_\psi(\mathcal{U})$ such that $V\psi = \eta$. The following proposition shows that this idea generalizes to the theory of frames. Analogously to the notion of a wandering vector and a complete wandering vector, a vector $x \in H$ is called a *Parseval frame vector* (resp. *frame vector* with bounds a and b) for a unitary system \mathcal{U} if $\mathcal{U}x$ forms a tight frame (resp. frame with bounds a and b) for $\overline{\text{span}}(\mathcal{U}x)$. It is called a *complete Parseval frame vector* (resp. *complete frame vector* with bounds a and b) when $\mathcal{U}x$ is a Parseval frame (resp. frame with bounds a and b) for H .

Proposition 15: *Suppose that ψ is a complete wandering vector for a unitary system \mathcal{U} . Then*

- (i) *a vector η is a Parseval frame vector for \mathcal{U} if and only if there is a (unique) partial isometry $A \in C_\psi(\mathcal{U})$ such that $A\psi = \eta$.*
- (ii) *a vector η is a complete Parseval frame vector for \mathcal{U} if and only if there is a (unique) co-isometry $A \in C_\psi(\mathcal{U})$ such that $A\psi = \eta$.*

The above result does not tell the whole story. The reason is that many unitary systems do not have wandering vectors but do have frame vectors. For instance, this is the case in Example 14, where the unitary system is the group of multiplication operators $\mathcal{U} = \{M_{e^{ins}} : n \in \mathbb{Z}\}$ acting on $L^2(E)$. In the case of a unitary system such as the wavelet system $\mathcal{U}_{D,T}$ there exist *both* complete wandering vectors *and* nontrivial Parseval frame vectors, so the theory seems richer (however less tractable) and Proposition 15 is very relevant.

Much of Example 14 generalizes to the case of an arbitrary countable unitary group. There is a corresponding (geometric) dilation result.

Proposition 16: *Suppose that \mathcal{U} is a unitary group such that $\mathcal{W}(\mathcal{U})$ is non-empty. Then every complete Parseval frame vector must be a complete wandering vector.*

Theorem 17: *Suppose that \mathcal{U} is a unitary group on H and η is a complete Parseval frame vector for \mathcal{U} . Then there exists a Hilbert space $K \supseteq H$ and a unitary group \mathcal{G} on K such that \mathcal{G} has complete wandering vectors, H is an invariant subspace of \mathcal{G} such that $\mathcal{G}|_H = \mathcal{U}$, and the map $g \rightarrow g|_H$ is a group isomorphism from \mathcal{G} onto \mathcal{U} .*

The following is not hard, but it is very useful.

Proposition 18: *Suppose that \mathcal{U} is a unitary group which has a complete Parseval frame vector. Then the von Neumann algebra $w^*(\mathcal{U})$ generated by \mathcal{U} is finite.*

3.6. An Operator Model

The following is a corollary of Theorem 17. It shows that Example 14 can be viewed as a model for certain operators.

Corollary 19: *Let $T \in B(H)$ be a unitary operator and let $\eta \in H$ be a vector such that $\{T^n\eta : n \in \mathbb{Z}\}$ is a Parseval frame for H . Then there is a unique (modulo a null set) measurable set $E \subset \mathbb{T}$ such that $\{T^n\eta : n \in \mathbb{Z}\}$ and $\{e^{ins}|_E : n \in \mathbb{Z}\}$ are unitarily equivalent frames.*

3.7. Group Representations

These concepts generalize. For a unitary system \mathcal{U} on a Hilbert space H , a closed subspace M of H is called a *complete wandering subspace* for \mathcal{U} if $\text{span}\{UM : U \in \mathcal{U}\}$ is dense in H , and $UM \perp VM$ with $U \neq V$. Let

$\{e_i: i \in I\}$ be an orthonormal basis for M . Then M is a complete wandering subspace for \mathcal{U} if and only if $\{Ue_i: U \in \mathcal{U}, i \in I\}$ is an orthonormal basis for H . We call $\{e_i\}$ a *complete multi-wandering vector*. Analogously, an n -tuple (η_1, \dots, η_n) of non-zero vectors (here n can be ∞) is called *complete Parseval multi-frame vector* for \mathcal{U} if $\{U\eta_i: U \in \mathcal{U}, i = 1, \dots, n\}$ forms a complete Parseval frame for H . Let G be a group and let λ be the left regular representation of G on $l^2(G)$. Then $\{\lambda_g \times I_n: g \in G\}$ has a complete multi-wandering vector (f_1, \dots, f_n) , where $f_1 = (x_e, 0, \dots, 0), \dots, f_n = (0, 0, \dots, x_e)$. Let P be any projection in the commutant of $(\lambda \otimes I_n)(\mathcal{G})$. Then (Pf_1, \dots, Pf_n) is a complete Parseval multi-frame vector for the subrepresentation $(\lambda \otimes I_n)|_P$. It turns out that every representation with a complete Parseval multi-frame vector arises in this way. Item (i) of the following theorem is elementary and was mentioned earlier; it is included for completeness.

Theorem 20: Let G be a countable group and let π be a representation of G on a Hilbert space H . Let λ denote the left regular representation of G on $l^2(G)$. Then

- (i) if $\pi(G)$ has a complete wandering vector then π is unitarily equivalent to λ ,
- (ii) if $\pi(G)$ has a complete Parseval frame vector then π is unitarily equivalent to a subrepresentation of λ ,
- (iii) if $\pi(G)$ has a complete Parseval multi-frame vector $\{\psi_1, \psi_2, \dots, \psi_n\}$, for some $1 \leq n < \infty$, then π is unitarily equivalent to a subrepresentation of $\lambda \otimes I_n$.

4. Decompositions of Operators and Operator-Valued Frames

The material we present here is contained in two recent papers. The first [15] was authored by a [VIGRE/REU] team consisting of K. Dykema, D. Freeman, K. Kornelson, D. Larson, M. Ordower, and E. Weber, with the title *Ellipsoidal Tight Frames*. This article started as an undergraduate research project at Texas A&M in the summer of 2002, in which Dan Freeman was the student and the other five were faculty mentors. Freeman is now a graduate student at Texas A&M. The project began as a solution of a finite dimensional frame research problem, but developed into a rather technically deep theory concerning a class of frames on an infinite dimensional Hilbert space. The second paper [44], entitled *Rank-one decomposition of*

operators and construction of frames, is a joint article by K. Kornelson and D. Larson.

4.1. Ellipsoidal Frames

We will use the term *spherical frame* (or *uniform frame*) for a frame sequence which is *uniform* in the sense that all its vectors have the same norm. Spherical frames which are tight have been the focus of several articles by different researchers. Since frame theory is essentially geometric in nature, from a purely mathematical point of view it is natural to ask: Which other surfaces in a finite or infinite dimensional Hilbert space contain tight frames? (These problems can make darn good REU projects, in particular.) In the first article we considered ellipsoidal surfaces.

By an *ellipsoidal surface* we mean the image of the unit sphere S_1 in the underlying Hilbert space H under a bounded invertible operator A in $B(H)$, the set of all bounded linear operators on H . Let E_A denote the ellipsoidal surface $E_A := AS_1$. A frame contained in E_A is called an *ellipsoidal frame*, and if it is tight it is called an ellipsoidal tight frame (ETF) for that surface. We say that a frame bound K is *attainable* for E_A if there is an ETF for E_A with frame bound K .

Given an ellipsoidal surface $E := E_A$, we can assume $E = E_T$ where T is a positive invertible operator. Indeed, given an invertible operator A , let $A^* = U|A^*|$ be the polar decomposition, where $|A^*| = (AA^*)^{1/2}$. Then $A = |A^*|U^*$. By taking $T = |A^*|$, we see that $TS_1 = AS_1$. Moreover, it is easily seen that the positive operator T for which $E = E_T$ is unique.

The starting point for the work in the first paper was the following Proposition. For his REU project Freeman found an elementary calculus proof of this for the real case. Others have also independently found this result, including V. Paulsen, and P. Casazza and M. Leon.

Proposition 21: *Let E_A be an ellipsoidal surface on a finite dimensional real or complex Hilbert space H of dimension n . Then for any integer $k \geq n$, E_A contains a tight frame of length k , and every ETF on E_A of length k has frame bound $K = k [\text{trace}(T^{-2})]^{-1}$.*

We use the following standard definition: For an operator $B \in B(H)$, the *essential norm* of B is:

$$\|B\|_{ess} := \inf\{\|B - K\| : K \text{ is a compact operator in } B(H)\}$$

Our main frame theorem from the first paper is:

Theorem 22: Let E_A be an ellipsoidal surface in an infinite dimensional real or complex Hilbert space. Then for any constant $K > \|T^{-2}\|_{ess}^{-1}$, E_T contains a tight frame with frame bound K .

So, for fixed A , in finite dimensions the set of attainable ETF frame bounds is finite, whereas in infinite dimensions it is a continuum.

Problem. If the essential norm of A is replaced with the norm of A in the above theorem, or if the inequality is replaced with equality, then except for some special cases, and trivial cases, no theorems of any degree of generality are known concerning the set of attainable frame bounds for ETF's on E_A . It would be interesting to have a general analysis of the case where $A - I$ is compact. In this case, one would want to know necessary and sufficient conditions for existence of a tight frame on E_A with frame bound 1. In the special case $A = I$ then, of course, any orthonormal basis will do, and these are the only tight frames on E_A in this case. What happens in general when $\|A\|_{ess} = 1$ and A is a small perturbation of I ?

We use elementary tensor notation for a rank-one operator on H . Given $u, v, x \in H$, the operator $u \otimes v$ is defined by $(u \otimes v)x = \langle x, v \rangle u$ for $x \in H$. The operator $u \otimes u$ is a projection if and only if $\|u\| = 1$.

Let $\{x_j\}_j$ be a frame for H . The standard frame operator is defined by: $S w = \sum_j \langle w, x_j \rangle x_j = \sum_j (x_j \otimes x_j) w$. Thus $S = \sum_j x_j \otimes x_j$, where this series of positive rank-1 operators converges in the strong operator topology (i.e. the topology of pointwise convergence). In the special case where each x_j is a unit vector, S is the sum of the rank-1 projections $P_j = x_j \otimes x_j$.

For A a positive operator, we say that A has a *projection decomposition* if A can be expressed as the sum of a finite or infinite sequence of (not necessarily mutually orthogonal) self-adjoint projections, with convergence in the strong operator topology.

If x_j is a frame of unit vectors, then $S = \sum_j x_j \otimes x_j$ is a projection decomposition of the frame operator. This argument is trivially reversible, so a positive invertible operator S is the frame operator for a frame of unit vectors if and only if it admits a projection decomposition $S = \sum_j P_j$. If the projections in the decomposition are not of rank one, each projection can be further decomposed (orthogonally) into rank-1 projections, as needed, expressing $S = \sum_n x_n \otimes x_n$, and then the sequence $\{x_n\}$ is a frame of unit vectors with frame operator S .

In order to prove Theorem 22, we first proved Theorem 23 (below), using purely operator-theoretic techniques.

Theorem 23: Let A be a positive operator in $B(H)$ for H a real or complex

Hilbert space with infinite dimension, and suppose $\|A\|_{ess} > 1$. Then A has a projection decomposition.

Suppose, then, that $\{x_n\}$ is a frame of unit vectors with frame operator S . If we let $y_j = S^{-\frac{1}{2}}x_j$, then $\{y_j\}_j$ is a *Parseval* frame. So $\{y_j\}_j$ is an ellipsoidal tight frame for the ellipsoidal surface $E_{S^{-\frac{1}{2}}} = S^{-\frac{1}{2}}S_1$. This argument is reversible: Given a positive invertible operator T , let $S = T^{-2}$. Scale T if necessary so that $\|S\|_{ess} > 1$. Let $S = \sum_j x_j \otimes x_j$ be a projection decomposition of S . Then $\{Tx_j\}$ is an ETF for the ellipsoidal surface TS_1 . Consideration of frame bounds and scale factors then yields Theorem 22.

Most of our second paper concerned *weighted* projection decompositions of positive operators, and resultant theorems concerning frames. If T is a positive operator, and if $\{c_n\}$ is a sequence of positive scalars, then a weighted projection decomposition of T with weights $\{c_n\}$ is a decomposition $T = \sum_j P_j$ where the P_j are projections, and the series converges strongly. We have since adopted the term *targeted* to refer to such a decomposition, and generalizations thereof. By a *targeted decomposition* of T we mean any strongly convergent decomposition $T = \sum_n T_n$ where the T_n is a sequence of *simpler* positive operators with special prescribed properties. So a weighted decomposition is a targeted decomposition for which the scalar weights are the prescribed properties. And, of course, a projection decomposition is a special case of targeted decomposition.

After a sequence of Lemmas, building up from finite dimensions and employing spectral theory for operators, we arrived at the following theorem. We will not discuss the details here because of limited space. It is the *weighted* analogue of theorem 23.

Theorem 24: Let B be a positive operator in $B(H)$ for H with $\|B\|_{ess} > 1$. Let $\{c_i\}_{i=1}^{\infty}$ be any sequence of numbers with $0 < c_i \leq 1$ such that $\sum_i c_i = \infty$. Then there exists a sequence of rank-one projections $\{P_i\}_{i=1}^{\infty}$ such that $B = \sum_{i=1}^{\infty} c_i P_i$

4.2. A Problem in Operator Theory

We will discuss a problem in operator theory that was motivated by a problem in the theory of Modulation Spaces. We tried to obtain an actual "reformulation" of the modulation space problem in terms of operator theory, and it is well possible that such a reformulation can be found. At the least we (Chris Heil and myself) found the following operator theory problem,

whose solution could conceivably impact mathematics beyond operator theory. I find it rather fascinating. I need to note that we subsequently showed (in an unpublished jointly-written expository article) that the actual *modulation space* connection requires a modified and more sophisticated version of the problem we present below. I still feel, that the problem I will present here has some independent interest, and may serve as a "first step" in developing a theory that might have some usefulness. Thus, I hope that the reader will find it interesting.

Let H be an infinite dimensional separable Hilbert space. As usual, denote the Hilbert space norm on H by $\|\cdot\|$. If x and y are vectors in H , then $x \otimes y$ will denote the operator of rank one defined by $(x \otimes y)z = \langle z, y \rangle x$. The operator norm of $x \otimes y$ is then just the product of $\|x\|$ and $\|y\|$.

Fix an orthonormal basis $\{e_n\}_n$ for H . For each vector v in H , define

$$\|v\| = \sum_n |\langle v, e_n \rangle|$$

This may be $+\infty$.

Let L be the set of all vectors v in H for which $\|v\|$ is finite. Then L is a dense linear subspace of H , and is a Banach space in the "triple norm". It is of course isomorphic to ℓ^1

Let T be any positive trace-class operator in $B(H)$.

The usual eigenvector decomposition for T expresses T as a series converging in the strong operator topology of operators $h_n \otimes h_n$, where $\{h_n\}$ is an orthogonal sequence of eigenvectors of T . That is,

$$T = \sum_n h_n \otimes h_n$$

In this representation the eigenvalue corresponding to the eigenvector h_n is the square of the norm: $\|h_n\|^2$. The trace of T is then

$$\sum_n \|h_n\|^2$$

and since T is positive this is also the trace-class norm of T .

Let us say that T is of *Type A* with respect to the orthonormal basis $\{e_n\}$ if, for the eigenvectors $\{h_n\}$ as above, we have that $\sum_n \|h_n\|^2$ is finite. [Note that this is just the (somewhat unusual) formula displayed

above for the trace of T with the triple norm used in place of the usual Hilbert space norm of the vectors $\{h_n\}$.]

And let us say that T is of *Type B* with respect to the orthonormal basis $\{e_n\}$ if there is *some* sequence of vectors $\{v_n\}$ in H with $\sum_n \|v_n\|^2$ *finite* such that

$$T = \sum_n v_n \otimes h_n$$

where the convergence of this series is in the strong operator topology.

Problem: If T is of *Type B* with respect to an orthonormal basis $\{e_n\}$, then must it be of *Type A* with respect to $\{e_n\}$?

Note: If the answer to this problem is negative (as I suspect it is), then the following subproblem would be an interesting one.

Subproblem: Let $\{e_n\}$ be an orthonormal basis for H . Find a characterization of all positive trace class operators T that are of *Type B* with respect to $\{e_n\}$. In particular, is every positive trace class operator T of *Type B* with respect to $\{e_n\}$? My feeling is *no*. (See the next example.)

Example 25: Let x be any vector in H that is not in L , and let $T = x \otimes x$. Then T is trace class, in fact has rank one, but clearly T is clearly *not* of Type A. Can such a T be of type B? (I don't think it is necessarily of Type B for all such T , however.)

References

1. A. Aldroubi, D.R. Larson, W.-S. Tang, and E. Weber, *Geometric aspects of frame representations of abelian groups*, Trans. Amer. Math. Soc. **356** (2004), 4767–4786.
2. E.A. Azoff, E.J. Ionascu, D.R. Larson, and C.M. Pearcy, *Direct paths of wavelets*, Houston J. Math. **29** (2003), no. 3, 737–756.
3. R. Balan, *A study of Weyl-Heisenberg and wavelet frames*, Ph.D. thesis, Princeton University, 1998.
4. L. Baggett, H. Medina, and K. Merrill, *Generalized multi-resolution analyses and a construction procedure for all wavelet sets in R^n* , J. Fourier Anal. Appl. **5** (1999).
5. J. Benedetto and M. Fickus, *Finite normalized tight frames*, Adv. Comput. Math., **18** (2003), 357–385.
6. J.J. Benedetto and M. Leon, *The construction of single wavelets in D -dimensions*, J. Geom. Anal. **11** (2001), no. 1, 1–15.

7. P. Casazza, D. Han and D. Larson, *Frames for Banach spaces*, Contemp. Math., **247**, Amer. Math. Soc., Providence, RI, 1999.
8. O. Christensen, *An introduction to frames and Riesz bases*, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, MA, 2003.
9. X. Dai, *Norm principal bimodules of nest algebras*, J. Functional Analysis, **90** (1990), 369–390.
10. X. Dai, Y. Diao, Q. Gu and D. Han, *Wavelets with frame multiresolution analysis*, J. Fourier Analysis and Applications, **9** (2003), 39–48.
11. X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Mem. Amer. Math. Soc. **134** (1998).
12. X. Dai, D. Larson and D. Speegle, *Wavelet sets in R^n* , J. Fourier Anal. Appl. **3** (1997), no. 4, 451–456.
13. X. Dai, D. Larson and D. Speegle, *Wavelet sets in R^n - II*, Contemp. Math, **216** (1998), 15–40.
14. I. Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.
15. D.E. Dutkay, *The local trace function for super-wavelets*, Wavelets, Frames, and Operator Theory , Contemp. Math., vol. **345**, (2004), pp. 115–136.
16. K. Dykema, D. Freeman, K. Kornelson, D. Larson, M. Ordower, and E. Weber, *Ellipsoidal tight frames and projection decompositions of operators*, Illinois J. Math. **48** (2004), no. 2, 477–489.
17. X. Fang and X. Wang, *Construction of minimally-wupported frequencies wavelets*, J. Fourier Anal. Appl. **2** (1996), 315–327.
18. H. Feichtinger, *Atomic characterization of modulation spaces through Gabor type representations*, Rocky Mountain J. Math., **19** (1989), 113–126.
19. M. Frank and D.R. Larson, *Frames in Hilbert C^* -modules and C^* - algebras*, J. Operator Theory **48** (2002), no. 2, 273–314.
20. M. Frank and D.R. Larson, *A module frame concept for Hilbert C^* -modules*, Contemporary Mathematics, **247** (1999), 207–234.
21. M. Frank and D.R. Larson, *Frames for Hilbert C^* Modules*, SPIE Proceedings Vol. 4119, Wavelet Applications in Signal And Image Processing VIII, (2000), 325–336.
22. M. Frank, V. I. Paulsen and T. R. Tiballi , *Symmetric approximation of frame*, Trans. Amer. Math. Soc., **354** (2002), 777–793.
23. J.P. Gabardo, D. Han, and D. Larson, *Gabor frames and operator algebras*, Wavelet Applications in Signal and Image Processing, Proc. SPIE, vol. 4119, 2000, pp. 337–345.
24. J-P, Gabardo and D. Han, *Subspace Weyl-Heisenberg frames*, J. Fourier Analysis and Appl., **7**(2001), 419–433.
25. T.N.T. Goodman, S.L. Lee and W.S. Tang, *Wavelets in wandering subspaces*, Trans. Amer. Math. Soc., **338** (1993), 639–654.
26. Q. Gu, *On interpolation families of wavelet sets*, Proc. Amer. Math. Soc., **128** (2000), 2973–2979.
27. Q. Gu and D. Han, *On multiresolution analysis wavelets in R^n* , J. Fourier Analysis and Applications, **6**(2000), 437–448.
28. Q. Gu and D. Han, *Phases for dyadic orthonormal wavelets*, J. of Mathe-

- mathematical Physics, **43** (2002), no. 5, 2690–2706.
29. Q. Gu and D. Han, *Functional Gabor frame multipliers*, J. Geometric Analysis, **13** (2003), 467–478.
 30. D. Han, *Wandering vectors for irrational rotation unitary systems*, Trans. Amer. Math. Soc., **350** (1998), 309–320.
 31. D. Han, *Tight frame approximation for multi-frames and super-frames*, J. Approx. Theory, **129** (2004), 78–93.
 32. D. Han, J-P. Gabardo, and D.R. Larson, *Gabor frames and operator algebras*, Wavelet Applications in Signal and Image Processing, Proc. SPIE, 4119 (2000), 337–345.
 33. D. Han and D.R. Larson, *Frames, bases and group representations*, Memoirs American Math. Society, **697**, (2000).
 34. D. Han and D.R. Larson, *Wandering vector multipliers for unitary groups*, Trans. Amer. Math. Soc., **353** (2001), 3347–3370.
 35. D. Han and Y. Wang, *The existence of Gabor bases and frames*, Contemp. Math., **345** (2004), 183–192.
 36. C. Heil, P.E.T. Jorgensen, and D.R. Larson (eds.), *Wavelets, Frames and Operator Theory*, Contemp. Math., vol. 345, American Mathematical Society, Providence, RI, 2004, Papers from the Focused Research Group Workshop held at the University of Maryland, College Park, MD, January 15–21, 2003.
 37. E. Hernandez, X. Wang and G. Weiss, *Smoothing minimally supported frequency (MSF) wavelets: Part I.*, J. Fourier Anal. Appl. **2** (1996), 329–340.
 38. E. Hernandez and G. Weiss, *A First Course on Wavelets*, CRC Press, Inc., 1996.
 39. E. Ionascu, D. Larson and C. Percay, *On the unitary systems affiliated with orthonormal wavelet theory in n -dimensions*, J. Funct. Anal. **157** (1998), no. 2, 413–431.
 40. E. Ionascu, D. Larson and C. Percay, *On wavelet sets*, J. Fourier Analysis and Applications, **4** (1998), 711–721.
 41. R. Kadison and J. Ringrose, *Fundamentals of the Theory of Operator Algebras, Vol. I and II*, Academic Press, Inc. 1983 and 1985.
 42. K. Kornelson and D. Larson, *Rank-one decomposition of operators and construction of frames*, Wavelets, Frames, and Operator Theory, Contemp. Math, vol. 345, Amer. Math. Soc., 2004, pp. 203–214.
 43. D. R. Larson, *Von Neumann algebras and wavelets. Operator algebras and applications* (Samos, 1996), 267–312, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 495, Kluwer Acad. Publ., Dordrecht, 1997.
 44. D. R. Larson, *Frames and wavelets from an operator-theoretic point of view*, Operator algebras and operator theory (Shanghai, 1997), 201–218, Contemp. Math., 228, Amer. Math. Soc., Providence, RI, 1998.
 45. D.R. Larson, E. Schulz, D. Speegle and K. Taylor, *Explicit cross sections of singly generated group actions*, to appear.
 46. D.R. Larson, W-S. Tang, and E. Weber, *Riesz wavelets and multiresolution structures*, SPIE Proc. Vol. 4478, Wavelet Applications in Signal and Image

- Processing IX (2001), 254-262.
47. D.R. Larson, W.S. Tang, and E. Weber, *Multiwavelets associated with countable groups of unitary operators in Hilbert spaces*, Int. J. Pure Appl. Math. **6** (2003), no. 2, 123–144.
 48. G. Ólafsson and D. Speegle, *Wavelets, wavelet sets, and linear actions on \mathbb{R}^n* , Wavelets, frames and operator theory, Contemp. Math., vol. 345, Amer. Math. Soc., Providence, RI, 2004, pp. 253–281.
 49. D. Speegle, *On the existence of wavelets for non-expansive dilation matrices*, Collect. Math., **54** (2003), 163–179.
 50. D. Speegle, *The s -elementary wavelets are path-connected*, Proc. Amer. Math. Soc., **132** (2004), 2567–2575
 51. P. Wood, *Wavelets and Hilbert modules*, to appear in the Journal of Fourier Analysis and Applications (2004)
 52. Wutam Consortium, *Basic properties of wavelets* J. Fourier Analysis and Applications, **4** (1998), 575-594.