

# $P_1$ subalgebras of $M_n(\mathbb{C})$

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## Abstract

A linear subspace  $B$  of  $L(H)$  has property  $P_1$  if every element of its predual  $B_*$  has the form  $x + B_\perp$  with  $\text{rank}(x) \leq 1$ . We prove that if  $\dim H \leq 4$  and  $B$  is a unital operator subalgebra of  $L(H)$  which has property  $P_1$ , then  $\dim B \leq \dim H$ . We consider the question of whether this is true for arbitrary  $H$ .

## 1 Introduction

The duality between the full algebra  $L(H)$  of bounded linear operators on a Hilbert space  $H$  and its ideal  $L_*$  of trace class operators plays an important role in invariant subspace theory. Indeed, it is easy to use rank one operators in the preannihilator of an operator algebra  $B$  to construct non-trivial invariant subspaces for  $B$  and conversely (see [11]). In his proof [3] that subnormal operators are intransitive, S.Brown focused attention on a more subtle connection between rank one operators and invariant subspaces. He showed that certain linear subspaces  $B$  of  $L(H)$  have the following property: every element of its predual  $B_*$  has the form  $x + B_\perp$  with  $\text{rank}(x) \leq 1$ , where  $B_\perp = \{a \in L_* : \text{Tr}(ba) = 0, \forall b \in B\}$  is the preannihilator of  $B$ . This property was called  $P_1$  property by the third author([11]). Working independently, D.Hadwin and E.Nordgren [7] and the third author observed the connection between this property and “reflexivity”. Although neither property implies the other, if an algebra  $B$  has property  $P_1$  and is also reflexive ( $B = \text{AlgLat}(B)$ ) then so are all of its ultraweakly closed subalgebras.

Azoff obtained many results about linear subspaces of  $L(H)$  which have property  $P_1$ . Among them, he proved the following simple but beautiful result by using ideas from algebraic geometry. If  $\dim H = n \in \mathbb{N}$  and a linear space  $S \subset L(H) \cong M_n(\mathbb{C})$  has property  $P_1$ , then the dimension of  $S$  is no larger than  $2n - 1$ . Furthermore, there exists a subspace  $S \subset M_n(\mathbb{C})$  which has property  $P_1$  and  $\dim S = 2n - 1$ . For an expository account of these and related results, we refer to Azoff’s paper [1]. (Note that linear spaces with property

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$P_1$  are called elementary spaces in [1]. For this article the original term  $P_1$  seemed more suitable because we want to work with the more general property  $P_k$  in the same context.)

In this paper we consider the analogue of Azoff's result for the subcase of *unital operator subalgebras* in  $L(H) \equiv M_n(\mathbb{C})$  (an operator algebra is unital if it contains the identity operator of  $L(H)$ ). If  $B$  is the diagonal subalgebra of  $L(H)$ , it is easy to show that  $B$  has property  $P_1$  and  $\dim B = n$ . In section 5 we show that if  $n \leq 4$  and  $B \subset M_n(\mathbb{C})$  is a unital subalgebra which has property  $P_1$ , then  $\dim B \leq n$ . It is natural to conjecture that this is also true for arbitrary  $n$ . We make this formal:

**Question 1:** Suppose  $\dim H = n \in \mathbb{N}$  and  $B \subset L(H) \equiv M_n(\mathbb{C})$  is a unital operator algebra with property  $P_1$ . Must  $\dim B \leq n$ ?

Note that if the above conjecture is true, then we can deduce Azoff's result as a corollary. Indeed, if  $S \subset L(H) \equiv M_n(\mathbb{C})$  is a linear space with property  $P_1$ , then

$$B = \left\{ \begin{pmatrix} \lambda & s \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C}, s \in S \right\} \subset L(H^{(2)}) \equiv M_{2n}(\mathbb{C})$$

is a unital operator algebra with property  $P_1$  (see [9, 10, 1]). So  $\dim B \leq 2n$  implies  $\dim S \leq 2n - 1$ .

An algebra  $B \subset L(H)$  is called a  $P_1$  algebra if  $A$  has property  $P_1$ . An algebra  $B \subset L(H)$  is called a *maximal  $P_1$  algebra* if whenever  $A$  is a subalgebra of  $L(H)$  having property  $P_1$  and  $A \supset B$ , then  $A = B$ . We consider a subquestion of Question 1:

**Question 2:** Suppose  $\dim H = n \in \mathbb{N}$  and  $B \subset L(H) \equiv M_n(\mathbb{C})$  is a unital operator algebra. If  $B$  has property  $P_1$  and  $\dim B = n$ , is  $B$  a maximal  $P_1$  algebra?

In section 3 and section 4, we prove that if a unital  $P_1$  subalgebra  $B \subset M_n(\mathbb{C})$  is semi-simple or singly generated and  $\dim B = n$ , then  $B$  is a maximal  $P_1$  algebra.

In [11], the third author showed that if a weakly closed operator algebra  $B$  has property  $P_1$ , then  $B$  is 3-reflexive [2], i.e., its three-fold ampliation  $B^{(3)}$  is reflexive. (This result also holds for linear subspaces with the same proof). He raised the following problem: Suppose  $\dim H = n \in \mathbb{N}$  and  $B \subset L(H) \equiv M_n(\mathbb{C})$  is a unital operator algebra with property  $P_1$ . Is  $B$  2-reflexive? Note that this question also makes sense for linear subspaces. In [1], Azoff showed that the answer to the above question is affirmative for  $n = 3$  (for all linear subspaces of  $M_3(\mathbb{C})$  with property  $P_1$ ). Very little additional progress has been made on this problem since the mid 1980's. The purpose of the research project resulting in this article was to push further on this problem. In section 6 of this paper, we will show that the answer to the above question for unital algebras is also affirmative for  $n = 4$ . The proof requires a detailed analysis of several subcases undertaken in the preceding sections.

We would like to pose the following subquestion:

**Question 3:** Suppose  $\dim H = n \in \mathbb{N}$  and  $B \subset L(H) \equiv M_n(\mathbb{C})$  is a unital operator algebra with property  $P_1$  and  $\dim B = n$ . Is  $B$  2-reflexive?

Throughout this paper, we will use the following notation. If  $H$  is a Hilbert space and  $n$  is a positive integer, then  $H^{(n)}$  denotes the direct sum of  $n$  copies of  $H$ , i.e., the Hilbert

space  $H \oplus \cdots \oplus H$ . If  $a$  is an operator on  $H$ , then  $a^{(n)}$  denotes the direct sum of  $n$  copies of  $a$  (regarded as an operator on  $H^{(n)}$ ). However, we will use  $I_n$  instead of  $I^{(n)}$  to denote the identity operator on  $H^{(n)}$ . If  $B$  is a set of operators on  $H$ , then  $B^{(n)} = \{b^{(n)} : b \in B\}$ .

This paper focuses on problems concerning operator algebras and linear subspaces of operators in finite dimensions. All of our results and proofs are given for finite dimensions. However, many of the definitions are given in the mathematics literature for infinite (as well as finite) dimensions, where the Hilbert space is assumed to be separable. The Hahn-Banach Theorem and the Riesz Representation Theorem, the definitions of reflexive algebras and subspaces, the properties  $P_1$  and  $P_k$ , are all given in the literature for infinite dimensions, but we will only use them here in the context of finite dimensions. In cases where proofs of known results are given for sake of exposition, we will usually just give the proofs for finite dimensions. However, we will adopt the convention that if the statement of a result or definition in this article does not specify finite dimensions then the reference we cite actually gives the infinite dimensional proof, or, if no reference is cited then the proof we provide is in fact valid for infinite dimensions.

## 2 Preliminaries

Let  $H$  be a Hilbert space with  $\dim H = n$ . Then  $L(H) \equiv M_n(\mathbb{C})$ . Let  $\{e_i\}_{i=1}^n$  be an orthonormal basis of  $H$ . If  $a \in L(H) \equiv M_n(\mathbb{C})$  is an arbitrary operator, then the trace of  $a$  is defined as

$$\mathrm{Tr}(a) = \sum_{i=1}^n \langle ae_i, e_i \rangle.$$

It is easy to show that  $\mathrm{Tr}(a)$  does not depend on the choice of  $\{e_i\}_{i=1}^n$ . Moreover, the trace has the important property that  $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$  for all  $a, b \in L(H) \equiv M_n(\mathbb{C})$ . In this case, the space of trace class operators on  $H$ , denoted  $L_*$ , can be identified algebraically with  $M_n(\mathbb{C})$ , and is equipped with the trace class norm:

$$\|a\|_1 = \mathrm{Tr}((a^*a)^{1/2}).$$

Recall that the dual of a linear space is the space of all (continuous) linear functionals on the space. In the case of  $L_* = M_n(\mathbb{C})$ , every linear functional on  $L_*$  has the form  $a \mapsto \mathrm{Tr}(ab)$  for some  $b \in L(H) \equiv M_n(\mathbb{C})$ . In this way,  $L(H)$  is identified as the dual space of  $L_*$ , and  $L_*$  is called the predual of  $L(H)$ . If  $S \subset L(H)$  is a linear subspace, then as a linear space itself  $S$  can be identified as the dual of the quotient linear space  $L_*/S_\perp$ , where  $L_\perp = \{a \in L_* \mid \mathrm{Tr}(ba) = 0 \text{ for all } b \in S\}$  is the preannihilator of  $S$ . Here, as usual, the quotient space  $L_*/S_\perp$  means the set of all cosets of  $L_*$ ,  $\{x + S_\perp \mid x \in L_*\}$ . We also write  $x + S_\perp$  as  $[x]$ . We write  $S_* = L_*/S_\perp$ . The duality between  $S$  and  $S_*$  is that if  $[x] \in S_*$  for

some  $x \in L_*$ , and associate the linear functional on  $S$  given by

$$b \rightarrow \text{Tr}(bx), \quad \forall b \in S.$$

This is well-defined by the definition of  $S_\perp$ . In order to obtain  $S$  as exactly the dual of the space  $S_*$ , one needs to apply a version of the Hahn-Banach Theorem (cf [5]). We say a linear subspace  $S$  of  $L(H) \equiv M_n(\mathbb{C})$  has property  $P_1$  if every element of its predual  $B_*$  has the form  $x + B_\perp$  with  $\text{rank}(x) \leq 1$ .

Let  $B \subset L(H) \equiv M_n(\mathbb{C})$  be a unital operator subalgebra. If  $z \in L(H)$  is an invertible operator, elementary computations yield  $(zBz^{-1})_\perp = z^{-1}B_\perp z$  and  $(zBz^{-1})_* = z^{-1}B_* z$ , where the multiplication action of  $z$  on the quotient space  $B_*$  is given by

$$z^{-1}(x + B_\perp)z = z^{-1}xz + z^{-1}B_\perp z = z^{-1}xz + (zBz^{-1})_*.$$

From this it is easy to see that if  $B$  has property  $P_1$ , then so does  $zBz^{-1}$ . It is also true that  $B$  has property  $P_1$  if and only if its adjoint algebra  $B^* = \{b^* | b \in B\}$  has property  $P_1$ .

**Lemma 2.1.** [11] *An algebra  $B$  has property  $P_1$  if and only if every element  $b^* \in B^*$  has the form  $x + B_\perp$  with  $\text{rank}(x) \leq 1$ .*

*Proof.* “only if” is trivial. Suppose every element  $b^* \in B^*$  has the form  $x + B_\perp$  with  $\text{rank}(x) \leq 1$ . Note that for each  $b \in B$  and each  $b_\perp \in B_\perp$ ,  $\text{Tr}(bb_\perp) = 0$ . This implies that  $L(H) = B^* \oplus B_\perp$  with respect to the inner product  $\langle x, y \rangle = \text{Tr}(y^*x)$ . So for each  $a \in L(H)$ ,  $a = b^* + b_\perp$  for some  $b^* \in B^*$  and  $b_\perp \in B_\perp$ . Therefore,  $a = x + B_\perp$  with  $\text{rank}(x) \leq 1$  by the assumption of the lemma.  $\square$

**Lemma 2.2.** *Let  $B$  be a subalgebra of  $L(H)$ . If  $B$  has property  $P_1$  and  $p \in B$  is a projection, then  $pBp \subset L(pH)$  also has property  $P_1$ .*

*Proof.* Suppose  $z \in B_\perp$  and  $b \in B$ . Then  $\text{Tr}(pbppzp) = \text{Tr}(pbpz) = 0$ . So  $pzp \in (pBp)_\perp$ . For each  $a \in L(H)$ , there exists a  $b_\perp \in B_\perp$  such that the rank of  $a + b_\perp$  is at most 1. So the rank of  $pap + pb_\perp b = p(a + b_\perp)p$  is at most 1. This proves the lemma.  $\square$

Recall that a vector  $\xi \in \mathcal{H}$  is a separating vector of  $B$  if  $b\xi = 0$  for some  $b \in B$  then  $b = 0$ . The following result is the finite-dimensional special case of Proposition 1.2 of [6].

**Theorem 2.3.** *If  $B$  is a subalgebra of  $L(H)$ , with  $H$  finite dimensional, such that either  $B$  or  $B^*$  has a separating vector, then  $B$  has property  $P_1$ .*

Property  $P_k$ , a generalization of property  $P_1$ , was also introduced by the third author in [11]. Recall that an algebra  $B$  has *property  $P_k$*  if every element of its predual  $B_*$  has the form  $x + B_\perp$  with  $\text{rank}(x) \leq k$ .

**Lemma 2.4.** [11] *Let  $B$  be a subalgebra of  $L(H)$ . Then  $B$  has property  $P_k$  if and only if  $B^{(k)} = \{b^{(k)} | b \in B\} \subset L(H^{(k)})$  has property  $P_1$ .*

*Proof.* “ $\Rightarrow$ ”. By Lemma 2.1, we need to show that each operator  $(b^*)^{(k)}$ ,  $b \in B$ , can be written as  $f + B_\perp$  with  $\text{rank}(f) \leq 1$ . Note that

$$B_\perp^{(k)} = \{(x_{ij})_{k \times k} | x_{11} + \cdots + x_{kk} \in B_\perp\} \supset \{(x_{ij})_{k \times k} | x_{11} \cdots, x_{kk} \in B_\perp\}.$$

By the assumption,  $B$  has property  $P_k$ . So there exists a  $b_\perp \in B_\perp$  such that the rank of  $b^* + b_\perp$  is at most  $k$ . We can write  $b^* + b_\perp = \xi_1 \otimes \eta_1 + \cdots + \xi_k \otimes \eta_k$ , where  $\xi_i \otimes \eta_i$  is the rank one operator defined by  $\xi_i \otimes \eta_i(\xi) = \langle \xi, \eta_i \rangle \xi_i$ . Let  $z_{ii} = k\xi_i \otimes \eta_i - \sum_{1 \leq r \leq k} \xi_r \otimes \eta_r$ ,  $1 \leq i \leq k$ , and let

$$z = \begin{pmatrix} z_{11} & k\xi_2 \otimes \eta_2 & \cdots & k\xi_k \otimes \eta_k \\ k\xi_1 \otimes \eta_1 & z_{22} & \cdots & k\xi_k \otimes \eta_k \\ \cdots & \cdots & \cdots & \cdots \\ k\xi_1 \otimes \eta_1 & k\xi_2 \otimes \eta_2 & \cdots & z_{kk} \end{pmatrix}.$$

Then  $z \in B_\perp^{(k)}$  and

$$(b^*)^{(k)} + (b_\perp)^{(k)} + z = k \begin{pmatrix} \xi_1 \otimes \eta_1 & \xi_2 \otimes \eta_2 & \cdots & \xi_k \otimes \eta_k \\ \xi_1 \otimes \eta_1 & \xi_2 \otimes \eta_2 & \cdots & \xi_k \otimes \eta_k \\ \cdots & \cdots & \cdots & \cdots \\ \xi_1 \otimes \eta_1 & \xi_2 \otimes \eta_2 & \cdots & \xi_k \otimes \eta_k \end{pmatrix}$$

is a rank 1 matrix.

“ $\Leftarrow$ ”. By the assumption, for each  $a \in L(H)$  there exists  $z \in B_\perp^{(k)}$  such that the rank of  $a^{(k)} + z$  is at most 1. Write  $z = (z_{ij})_{k \times k}$ . Then  $z_{11} + \cdots + z_{kk} \in B_\perp$  and the rank of  $a + z_{ii}$  is at most 1. So the rank of

$$a + \frac{1}{k}(z_{11} + \cdots + z_{kk}) = \frac{1}{k}((a + z_{11}) + \cdots + (a + z_{kk}))$$

is at most  $k$ . □

**Corollary 2.5.** *If  $B$  is a subalgebra of  $L(H)$  and  $\dim H = k$ , then  $B^{(k)} \subset L(H^{(k)})$  has property  $P_1$ .*

### 3 Semi-simple maximal $P_1$ algebras

Suppose  $B$  is a subalgebra of  $M_n(\mathbb{C})$  which has property  $P_1$ . Recall that  $B$  is a maximal  $P_1$  algebra of  $M_n(\mathbb{C})$  if whenever  $A$  is a subalgebra of  $M_n(\mathbb{C})$  having property  $P_1$  and  $A \supseteq B$ , then  $A = B$ . The main result of this section is the following theorem.

**Theorem 3.1.** *Let  $B \subseteq M_n(\mathbb{C})$  be a unital semi-simple algebra. If  $B$  has property  $P_1$ , then  $\dim B \leq n$ . Furthermore, if  $\dim B = n$ , then  $B$  is a maximal  $P_1$  algebra.*

To prove this theorem, we will need prove the following lemmas.

**Lemma 3.2.** *Let  $B \subseteq L(H) = M_n(\mathbb{C})$  be a semi-simple algebra. If  $B$  has property  $P_1$ , then  $\dim B \leq n$ .*

*Proof.* We will use induction on  $n$ . The case  $n = 1$  is clear. Suppose this is true for  $n \leq k$  and let  $B \subseteq M_{k+1}(\mathbb{C})$  be a semi-simple algebra. We need to show  $\dim B \leq k + 1$ . Suppose  $B$  has a non-trivial central projection,  $p$ ,  $0 < p < 1$ . Then,  $B = pBp \oplus (1-p)B(1-p)$ . By Lemma 2.1,  $pBp \subset L(pH)$  and  $(1-p)B(1-p) \subset L((1-p)H)$  are both semi-simple algebras with property  $P_1$ . By the assumption of induction  $\dim pBp \leq \dim(pH)$  and  $\dim(1-p)B(1-p) \leq \dim(1-p)H$ . Therefore,  $\dim B = \dim(pBp) + \dim((1-p)B(1-p)) \leq \dim pH + \dim(1-p)H = \dim H = k + 1$ . Suppose  $B$  does not have a nontrivial central projection. Then,  $B \cong M_r(\mathbb{C})$ . Since  $B$  has  $P_1$ ,  $r^2 \leq n + 1$  by Lemma 2.4. So  $r \leq n + 1$ .  $\square$

**Lemma 3.3.** *Suppose  $0 \neq a \in M_n(\mathbb{C})$ . Then there exists finite elements  $b_1, \dots, b_k, c_1, \dots, c_k$ , such that  $\sum_{i=1}^k b_i a c_i = I_n$ .*

*Proof.* Note that  $M_n(\mathbb{C})aM_n(\mathbb{C})$  is a two sided ideal of  $M_n(\mathbb{C})$  and  $M_n(\mathbb{C})aM_n(\mathbb{C}) \neq 0$ . Since  $M_n(\mathbb{C})$  is a simple algebra,  $M_n(\mathbb{C})aM_n(\mathbb{C}) = M_n(\mathbb{C})$ , which implies the lemma.  $\square$

The following well known lemma will be very helpful.

**Lemma 3.4.** *There are finitely many unitary matrices  $u_1, u_2, \dots, u_k \in M_n(\mathbb{C})$  such that  $\frac{1}{k} \sum_{i=1}^k u_i a u_i^* = \frac{\text{Tr}(a)}{n} I_n$  for all  $a \in M_n(\mathbb{C})$ .*

The following lemma is a special case of Lemma 3.6. However, we include its proof to illustrate our idea.

**Lemma 3.5.** *Suppose  $B$  is a unital subalgebra of  $M_4(\mathbb{C})$  and  $B \cong M_2(\mathbb{C})$ , then  $B$  is a maximal  $P_1$  algebra.*

*Proof.* We may write  $M_4(\mathbb{C})$  as  $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$  and assume  $B = M_2(\mathbb{C}) \otimes I_2$ . Note that with respect to the matrix units of  $I_2 \otimes M_2(\mathbb{C})$ , each element of  $B = M_2(\mathbb{C}) \otimes I_2$  has the following form  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ ,  $a \in M_2(\mathbb{C})$ . By Corollary 2.5,  $B$  has property  $P_1$ . Assume  $B \subsetneq R \subseteq M_4(\mathbb{C})$  and  $R$  is an algebra with property  $P_1$ . We can write  $R = R_1 + J$ , where  $R_1 \supset B$  is the semi-simple part and  $J$  is the radical of  $R$ . Since  $R$  has property  $P_1$ ,  $R_1$  has property  $P_1$ . By Lemma 3.2,  $\dim R_1 \leq 4$ . Since  $\dim B = 4$ , we have  $R_1 = B$ .

Suppose  $0 \neq x = (x_{ij})_{1 \leq i, j \leq 2} \in J$  with respect to the matrix units  $I_2 \otimes M_2(\mathbb{C})$ . Without loss of generality, we may assume  $x_{11} \neq 0$ . By Lemma 3.3, there exists a finite elements  $b_1, \dots, b_k, c_1, \dots, c_k \in M_2(\mathbb{C})$ , such that

$$\sum_{i=1}^k b_i x_{11} c_i = I_2. \quad (1)$$

Let  $y = (y_{ij})_{1 \leq i, j \leq 2} = \sum_{i=1}^k (b_i \otimes I_2) x (c_i \otimes I_2) \in J$ . By (1), we have  $y_{11} = I_2$ . Choose unitary matrices  $u_1, \dots, u_k$  as in Lemma 3.4. Let  $z = (z_{ij}) = \sum_{i=1}^k (u_i \otimes I_2) y (u_i^* \otimes I_2) \in J$ . Then,  $z_{11} = I_2$  and  $z_{ij} = \lambda_{ij} I_2$  for some  $\lambda_{ij} \in \mathbb{C}, 1 \leq i, j \leq 2$ . So,  $z \in I_2 \otimes M_2(\mathbb{C})$ . Since  $z \in J, z^2 = 0$ , as elements in the radical are nilpotent. By the Jordan Canonical theorem, there exists an invertible matrix  $w \in I_2 \otimes M_2(\mathbb{C})$  such that  $wz w^{-1} = I_2 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ .

Replacing  $R$  by  $wRw^{-1}$ , we may assume that  $R$  contains  $B$  and  $I_2 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Furthermore, we may assume that  $R$  is the algebra generated by  $M_2(\mathbb{C}) \otimes I_2$  and  $I_2 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Then

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in M_2(\mathbb{C}) \right\}.$$

Simple computation shows that  $R$  does not have property  $P_1$ . This is a contradiction. Therefore  $J = 0$  and  $R = B$ .  $\square$

**Lemma 3.6.** *Let  $B$  be a unital subalgebra of  $M_{n^2}(\mathbb{C})$  such that  $B \cong M_n(\mathbb{C})$ . Then  $B$  is a maximal  $P_1$  algebra.*

*Proof.* We may write  $M_{n^2}(\mathbb{C})$  as  $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$  and assume  $B = M_n(\mathbb{C}) \otimes I_n$ . Note that with respect to the matrix units of  $I_n \otimes M_n(\mathbb{C})$ , each element of  $B = M_n(\mathbb{C}) \otimes I_n$  has the following form  $\begin{pmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{pmatrix}, a \in M_n(\mathbb{C})$ . By Corollary 2.5,  $B$  has property  $P_1$ . Assume  $B \subsetneq R \subseteq M_{n^2}(\mathbb{C})$  and  $R$  is an algebra with property  $P_1$ . We can write  $R = R_1 + J$ , where  $R_1 \supset B$  is the semi-simple part and  $J$  is the radical of  $R$ . Since  $R$  has property  $P_1$ ,  $R_1$  has property  $P_1$ . By Lemma 3.2,  $\dim R_1 \leq n^2$ . Since  $\dim B = n^2$ , we have  $R_1 = B$ .

Suppose  $0 \neq x = (x_{ij})_{1 \leq i, j \leq n} \in J$  with respect to the matrix units  $I_n \otimes M_n(\mathbb{C})$ . Without loss of generality, we may assume  $x_{11} \neq 0$ . By Lemma 3.3, there exists a finite elements  $b_1, \dots, b_k, c_1, \dots, c_k \in M_n(\mathbb{C})$ , such that

$$\sum_{i=1}^k b_i x_{11} c_i = I_n. \quad (2)$$

Let  $y = (y_{ij})_{1 \leq i, j \leq n} = \sum_{i=1}^k (b_i \otimes I_n) x (c_i \otimes I_n) \in J$ . By (2), we have  $y_{11} = I_n$ . Choose unitary matrices  $u_1, \dots, u_k$  as in Lemma 3.4. Let  $z = (z_{ij}) = \sum_{i=1}^k (u_i \otimes I_n) y (u_i^* \otimes I_n) \in J$ . Then,  $z_{11} = I_n$  and  $z_{ij} = \lambda_{ij} I_n$  for some  $\lambda_{ij} \in \mathbb{C}, 1 \leq i, j \leq n$ . So,  $z \in I_n \otimes M_n(\mathbb{C})$ .

Since  $z \in J$ ,  $z^n = 0$ , as elements in the radical are nilpotent. By the Jordan Canonical theorem, there exists an invertible matrix  $w \in I_n \otimes M_n(\mathbb{C})$  such that  $0 \neq wzw^{-1} = \bigoplus_{i=1}^k z_i \in I_n \otimes M_n(\mathbb{C})$  and each  $z_i$  is a Jordan block with diagonal 0. Replacing  $R$  by  $wRw^{-1}$ , we may assume  $R$  contains  $B$  and  $wzw^{-1} \in I_n \otimes M_n(\mathbb{C})$ .

Suppose  $r = \max\{\text{rank}z_i : 1 \leq i, \leq k\}$ . We may assume  $\text{rank}z_1 = \dots = \text{rank}z_s = r$  and  $\text{rank}z_i < r$  for all  $s < i \leq k$ . Then  $z^{r-1} = I_n \otimes ((\bigoplus_{i=1}^s z_i^{r-1}) \oplus 0)$ . Note that  $z_i^{r-1} = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \vdots & & & \\ 0 & \dots & 0 & 0 \end{pmatrix}$ . We may assume  $R$  is the algebra generated by  $M_n(\mathbb{C}) \otimes I_n$  and  $z^{r-1}$ .

Without loss of generality, we assume  $r = 2$ , and  $s = \frac{n}{2}$ . The general case can be proved similarly. Then

$$R = \left\{ \begin{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \end{pmatrix}_{s \times s} : a, b \in M_n(\mathbb{C}) \right\}.$$

Simple computations show that

$$R_{\perp} = \left\{ \begin{pmatrix} \begin{pmatrix} x_1 & * \\ y_1 & x_2 \end{pmatrix} & & * \\ & \ddots & \\ * & & \begin{pmatrix} x_{n-1} & * \\ y_s & x_n \end{pmatrix} \end{pmatrix}_{s \times s} : x_i, y_i \in M_n(\mathbb{C}), \sum_{i=1}^n x_i = \sum_{i=1}^s y_i = 0 \right\}.$$

Let  $m = \begin{pmatrix} 0_n & 0_n \\ I_n & 0_n \end{pmatrix}$ . Since  $R$  has property  $P_1$ , we can write  $m^{(s)} = x + R_{\perp}$  such that the rank of  $x$  is at most 1. This implies that  $I_n + y_1, I_n + y_2, \dots, I_n + y_s$  are all rank-1 matrices for some  $y_1, \dots, y_s \in M_n(\mathbb{C})$  with  $y_1 + \dots + y_s = 0$ . Therefore, the rank of  $I_n + y_1 + I_n + y_2 + \dots + I_n + y_s = sI_n$  is at most  $s = \frac{n}{2} < n$ . This is a contradiction. So  $J = 0$  and  $R = B$ .  $\square$

The following is a key lemma to prove Theorem 3.1, which has an independent interest.

**Lemma 3.7.** *Let  $\lambda \neq 0$  be a complex number, and let  $y_1, y_2, \dots, y_n \in M_n(\mathbb{C})$  satisfy  $y_1 + y_2 + \dots + y_n = 0$ . Suppose  $\eta_1, \eta_2, \dots, \eta_n \in \mathbb{C}^n$  are linearly dependent vectors, and*

$$t = \begin{pmatrix} \lambda & * & * & * & \dots & * \\ \eta_1 & I_n + y_1 & * & * & \dots & * \\ \eta_2 & * & I_n + y_2 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \eta_n & * & * & * & \dots & I_n + y_n \end{pmatrix}.$$

Then  $\text{rank } t > 1$ .

*Proof.* We may assume that  $\eta_1, \dots, \eta_{k-1}, k \leq n$ , are linearly independent vectors, and each  $\eta_j, k \leq j \leq n$ , can be written as a linear combination of  $\eta_1, \dots, \eta_{k-1}$ . Write  $\eta_i = \begin{pmatrix} \sigma_{i1} \\ \vdots \\ \sigma_{in} \end{pmatrix}$ .

We may assume that the  $(k-1) \times (k-1)$  matrix  $(\sigma_{i,j})_{(k-1) \times (k-1)}$  is invertible. Using row reduction, we can transform  $t$  to a new matrix

$$\begin{pmatrix} \lambda & * & * & * & \cdots & * \\ \eta'_1 & I_n + y'_1 & * & * & \cdots & * \\ \eta'_2 & * & I_n + y'_2 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \eta'_n & * & * & * & \cdots & I_n + y'_n \end{pmatrix}$$

such that the  $k$ -th row of each  $\eta'_j$  is 0 for  $1 \leq j \leq n$ , and  $y'_1 + \dots + y'_n = 0$ . So the  $(jk+1, 1)$ -th entry of  $t'$  is zero for all  $1 \leq j \leq n$ .

Suppose  $t$  is a rank 1 matrix. Then  $t'$  is also a rank 1 matrix. By the assumption,  $\lambda \neq 0$ . This implies that each entry of the  $(jk+1)$ -th row of  $t'$  is zero for all  $1 \leq j \leq n$ . In particular, the  $(k, k)$ -th entry of  $I_n + y'_j$  is 0 for all  $1 \leq j \leq n$ . Therefore, the  $(k, k)$ -th of  $I_n + y'_1 + I_n + y'_2 + \dots + I_n + y'_n = nI_n$  is zero. This is a contradiction. So  $\text{rank } t > 1$ .  $\square$

The following lemma is a special case of Lemma 3.10. However, we include its proof to illustrate our idea.

**Lemma 3.8.** *Suppose  $\dim H = 5$  and*

$$B = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : \lambda \in \mathbb{C}, a \in M_2(\mathbb{C}) \right\} \subset L(H) = M_5(\mathbb{C}).$$

*Then,  $B$  is a maximal  $P_1$  algebra.*

*Proof.* Since  $B$  has a separating vector,  $B$  has property  $P_1$  by Theorem 2.3. Suppose  $B \subset R \subseteq M_5(\mathbb{C})$  and  $R$  has property  $P_1$ . We can write  $R = R_1 + J$ , where  $R_1 \supset B$  is the semi-simple part and  $J$  is the radical part. By Lemma 3.2,  $B = R_1$ .

Suppose  $0 \neq x \in J$ . Let  $p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  and  $q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_2 & 0_2 \\ 0 & 0_2 & I_2 \end{pmatrix}$ . Then  $qBq \subseteq qRq \subset B(PH) = M_4(\mathbb{C})$ . By Lemma 3.5,  $qBq = qRq$ . This implies we may assume  $0 \neq x = \begin{pmatrix} 0 & \xi^T & \eta^T \\ 0 & 0_2 & 0_2 \\ 0 & 0_2 & 0_2 \end{pmatrix}$ ,

where  $\xi, \eta \in \mathbb{C}^2$ .

Case 1:  $\xi$  and  $\eta$  are linearly independent vectors. Note that

$$x \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & \xi^T a & \eta^T a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R.$$

Since  $\xi$  and  $\eta$  are linearly independent, and  $a \in M_2(\mathbb{C})$  is arbitrary, this implies that

$$R \supseteq \left\{ \begin{pmatrix} \lambda & \xi^T & \eta^T \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : \lambda \in \mathbb{C}, \xi, \eta \in \mathbb{C}^2, a \in M_2(\mathbb{C}) \right\}.$$

Simple computation shows that

$$R_{\perp} \subseteq \left\{ \begin{pmatrix} 0 & * & * \\ 0 & y_1 & * \\ 0 & * & y_2 \end{pmatrix} : y_1, y_2 \in M_2(\mathbb{C}), y_1 + y_2 = 0 \right\}.$$

Since  $R$  has property  $P_1$ , we can write  $I_5 = x + R_{\perp}$  such that the rank of  $x$  is at most 1. This gives us a rank 1 matrix  $x$  of the form  $R_{\perp} = \begin{pmatrix} 1 & * & * \\ 0 & y_1 + I_2 & * \\ 0 & * & y_2 + I_2 \end{pmatrix}$ , where  $y_1 + y_2 = 0$ . This contradicts Lemma 3.7.

Case 2:  $\xi$  and  $\eta$  are linearly dependent. Without loss of generality, assume  $\eta = t\xi$ . So  $x = \begin{pmatrix} 0 & \xi^T & t\xi^T \\ 0 & 0_2 & 0_2 \\ 0 & 0_2 & 0_2 \end{pmatrix}$  and  $x \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & \xi^T a & t\xi^T a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . Since  $\xi \neq 0$ , and  $a \in M_2(\mathbb{C})$  is arbitrary, this implies that

$$R \supset \left\{ \begin{pmatrix} \lambda & \xi^T & t\xi^T \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : \lambda \in \mathbb{C}, \xi \in \mathbb{C}^2, a \in M_2(\mathbb{C}) \right\}.$$

Simple computation shows that

$$R_{\perp} \subset \left\{ \begin{pmatrix} 0 & * & * \\ \eta_1 & y_1 & * \\ \eta_2 & * & y_2 \end{pmatrix} : y_1, y_2 \in M_2(\mathbb{C}) : y_1 + y_2 = 0, \eta_1, \eta_2 \in \mathbb{C}^2, \eta_1 + t\eta_2 = 0 \right\}. \quad (3)$$

Since  $R$  has property  $P_1$ , we can write  $I_5 = x + R_{\perp}$  such that the rank of  $x$  is at most 1. This gives us a rank 1 matrix  $x$  of the form  $R_{\perp} = \begin{pmatrix} 1 & * & * \\ \eta_1 & y_1 + I_2 & * \\ \eta_2 & * & y_2 + I_2 \end{pmatrix}$ , where  $\eta_1 + t\eta_2 = 0$  and  $y_1 + y_2 = 0$ . This contradicts Lemma 3.7.  $\square$

**Lemma 3.9.** *Suppose  $\{z_{ij}\}_{1 \leq i \leq s, 1 \leq j \leq r} \subseteq M_{sr}(\mathbb{C})$  and  $\{c_{ji}\}_{1 \leq i \leq s, 1 \leq j \leq r} \subseteq M_{rs}(\mathbb{C})$  such that*

$$\sum_{i=1}^s \sum_{j=1}^r z_{ij} a c_{ji} b = 0, \forall a \in M_r(\mathbb{C}), \forall b \in M_s(\mathbb{C}).$$

*If  $c_{ji} \neq 0$  for some  $1 \leq i \leq s, 1 \leq j \leq r$ , then  $z_{ij}$  are linearly dependent.*

*Proof.* We may assume  $c_{11} \neq 0$  and the  $(1, 1)$  entry of  $c_{11}$  is 1. Replacing  $c_{ji}$  by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} c_{ji} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

we may assume  $c_{ji} = \lambda_{ij} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ , where  $\lambda_{11} = 1$ .

Let  $z_{ij}^k$  be the  $k$ -th column of  $z_{ij}$ . Simple computation shows that  $\sum_{i=1}^s \sum_{j=1}^r z_{ij} c_{ji} = 0$  is equivalent to  $\sum_{i=1}^s \sum_{j=1}^r \lambda_{ij} z_{ij}^1 = 0$ . Let  $a = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ . Simple computation shows that  $\sum_{i=1}^s \sum_{j=1}^r z_{ij} a c_{ji} = 0$  is equivalent to  $\sum_{i=1}^s \sum_{j=1}^r \lambda_{ij} z_{ij}^2 = 0$ . Choosing  $a$  appropriately, we have  $\sum_{i=1}^s \sum_{j=1}^r \lambda_{ij} z_{ij}^k = 0$  for all  $1 \leq k \leq n$ . This implies  $\sum_{i=1}^s \sum_{j=1}^r \lambda_{ij} z_{ij} = 0$ .  $\square$

**Lemma 3.10.** *Suppose  $\dim H = (r^2 + s^2)$  and*

$$B = \{a^{(r)} \oplus b^{(s)} : a \in M_r(\mathbb{C}), b \in M_s(\mathbb{C})\} \subset L(H) = M_{(r^2+s^2)}(\mathbb{C}).$$

*Then  $B$  is a maximal  $P_1$  algebra.*

*Proof.* Since  $B$  has a separating vector,  $B$  has property  $P_1$  by Theorem 2.3. Suppose  $B \subseteq R \subseteq M_{(r^2+s^2)}(\mathbb{C})$  and  $R$  has property  $P_1$ . We can write  $R = R_1 + J$ , where  $R_1 \supset B$  is the semi-simple part and  $J$  is the radical part. By Lemma 3.2,  $B = R_1$ .

Suppose  $0 \neq x \in J$ . Let  $p = I_r^{(r)} \oplus 0$  and  $q = 0 \oplus I_s^{(s)}$ . Then,  $pBp \subseteq pRp \subseteq B(pH)$  and  $pRp$  has property  $P_1$ . By Lemma 3.6,  $pRp = pBp$ . Similarly,  $qRq = qBq$ . So we may assume  $0 \neq x = \begin{pmatrix} 0_r^{(r)} & c \\ 0 & 0_s^{(s)} \end{pmatrix}$ . Write  $c = (c_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}$ . Note that  $c \neq 0$ .

Suppose

$$z = \begin{pmatrix} x_1 & * & \cdots & * & * & * & \cdots & * \\ * & x_2 & \cdots & * & * & * & \cdots & * \\ & & \ddots & & & & \ddots & \\ * & * & \cdots & x_r & * & * & \cdots & * \\ z_{11} & z_{12} & \cdots & z_{1r} & y_1 & * & \cdots & * \\ z_{21} & z_{22} & \cdots & z_{2r} & * & y_2 & \cdots & * \\ & & \ddots & & & & \ddots & \\ z_{s1} & z_{s2} & \cdots & z_{sr} & * & * & \cdots & y_s \end{pmatrix} \in R_{\perp}.$$

Since  $R_{\perp} \subset B_{\perp}$ ,  $x_1 + x_2 + \cdots + x_r = 0_r$  and  $y_1 + y_2 + \cdots + y_s = 0_s$ . Note that  $x(a^{(r)} \oplus b^{(s)}) = \begin{pmatrix} 0_r^{(r)} & cb^{(s)} \\ 0 & 0_s^{(s)} \end{pmatrix}$ . Since  $x \in R_{\perp}$  and  $x(a^{(r)} \oplus b^{(s)}) \in R$ , we have

$$\text{Tr} \left( \begin{pmatrix} z_{11} & \cdots & z_{1r} \\ \vdots & & \vdots \\ z_{s1} & \cdots & z_{sr} \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1s} \\ \vdots & & \vdots \\ c_{r1} & \cdots & c_{rs} \end{pmatrix} \begin{pmatrix} b & & \\ & \ddots & \\ & & b \end{pmatrix} \right) = 0.$$

Simple computation shows that  $\text{Tr}\left(\sum_{i=1}^s \sum_{j=1}^r z_{ij} c_{ji} b\right) = 0$ . Since  $b \in M_s(\mathbb{C})$  is an arbitrary matrix,  $\sum_{i=1}^s \sum_{j=1}^r z_{ij} c_{ji} = 0$ .

Note that

$$(a^{(r)} \oplus 0)x(0 \oplus b^{(s)}) = \begin{pmatrix} 0_r^{(r)} & a^{(r)} c b^{(s)} \\ 0 & 0_s^{(s)} \end{pmatrix} = \begin{pmatrix} 0_r^{(r)} & (ac_{ij}b)_{1 \leq i \leq r, 1 \leq j \leq s} \\ 0 & 0_s^{(s)} \end{pmatrix}$$

By similar arguments as above, we have  $\sum_{i=1}^s \sum_{j=1}^r z_{ij} a c_{ji} b = 0$  for all  $a \in M_r(\mathbb{C})$  and  $b \in M_s(\mathbb{C})$ . By Lemma 3.9, this implies that  $\{z_{ij}\}_{1 \leq i \leq s, 1 \leq j \leq r}$  are linearly dependent matrices.

Since  $R$  has property  $P_1$ ,  $I_{r^2+s^2} = x + R_{\perp}$  for some  $x$  such that the rank of  $x$  is at most 1. So  $x$  is a matrix of the form

$$\begin{pmatrix} I_r + x_1 & * & \cdots & * & * & * & \cdots & * \\ * & I_r + x_2 & \cdots & * & * & * & \cdots & * \\ & & \ddots & & & & \ddots & \\ * & * & \cdots & I_r + x_r & * & * & \cdots & * \\ z_{11} & z_{12} & \cdots & z_{1r} & I_s + y_1 & * & \cdots & * \\ z_{21} & z_{22} & \cdots & z_{2r} & * & I_s + y_2 & \cdots & * \\ & & \ddots & & & & \ddots & \\ z_{s1} & z_{s2} & \cdots & z_{sr} & * & * & \cdots & I_s + y_s \end{pmatrix}$$

Since  $x$  is a rank 1 matrix,  $(z_{ij})_{1 \leq i \leq s, 1 \leq j \leq r}$  are rank 1 matrices. So there are  $\xi_1, \dots, \xi_s \in \mathbb{C}^s, \eta_1, \dots, \eta_r \in \mathbb{C}^r$  such that  $z_{ij} = \xi_i \otimes \eta_j$  for  $1 \leq i \leq s$  and  $1 \leq j \leq r$ . Since  $\{z_{ij}\}_{1 \leq i \leq s, 1 \leq j \leq r}$  are linearly dependent matrices, either  $\{\xi_i\}_{i=1}^s$  are linearly dependent or  $\{\eta_j\}_{j=1}^r$  are linearly dependent. Without loss of generality, assume  $\{\xi_i\}_{i=1}^s$  are linearly dependent. Now,  $x$  is a matrix of the form

$$\begin{pmatrix} I_r + x_1 & * & \cdots & * & * & * & \cdots & * \\ * & I_r + x_2 & \cdots & * & * & * & \cdots & * \\ & & \ddots & & & & \ddots & \\ * & * & \cdots & I_r + x_r & * & * & \cdots & * \\ \xi_1 \otimes \eta_1 & \xi_1 \otimes \eta_2 & \cdots & \xi_1 \otimes \eta_r & I_s + y_1 & * & \cdots & * \\ \xi_2 \otimes \eta_1 & \xi_2 \otimes \eta_2 & \cdots & \xi_2 \otimes \eta_r & * & I_s + y_2 & \cdots & * \\ & & \ddots & & & & \ddots & \\ \xi_s \otimes \eta_1 & \xi_s \otimes \eta_1 & \cdots & \xi_s \otimes \eta_r & * & * & \cdots & I_s + y_s \end{pmatrix}$$

Since  $x_1 + \dots + x_r = 0$ , one entry of  $I_r + x_i$  is not zero for some  $1 \leq i \leq r$ . We may assume

the  $(1, 1)$  entry of  $I_r + x_1$  is  $\lambda \neq 0$ . Let  $\eta_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{pmatrix}$ . Then the matrix

$$\begin{pmatrix} \lambda & * & \cdots & * \\ \alpha_1 \xi_1 & I_s + y_1 & \cdots & * \\ \vdots & & \ddots & \\ \alpha_1 \xi_s & * & \cdots & I_s + y_s \end{pmatrix}$$

has rank 1 since it is a submatrix of  $x$ . This contradicts Lemma 3.7. So  $R = B$ .  $\square$

*Proof of Theorem 3.1.* By Lemma 3.2, if  $B$  has  $P_1$ , then  $\dim B \leq n$ . Assume  $B$  has property  $P_1$ , and  $\dim B = n$ . We claim  $B = \bigoplus_{i=1}^r M_{n_i}(\mathbb{C})^{(n_i)}$  and  $n = \sum_{i=1}^r n_i^2$ . We will proceed by induction on  $n$ . If  $n = 1$ , this is clear. Assume our claim is true for  $n \leq k$ . Let  $B \subseteq M_{k+1}(\mathbb{C})$  be a semi-simple  $P_1$  algebra and  $\dim B = k + 1$ . Suppose  $B$  has a nontrivial central projection  $p$ ,  $0 < p < 1$ . Then,  $B = pBp \oplus (1-p)B(1-p)$ . By Lemma 2.1,  $pBp \subseteq B(pH)$  and  $(1-p)B(1-p) \subseteq B((1-p)H)$  are both semi-simple algebras with property  $P_1$ . By Lemma 3.2,  $\dim(pBp) = \dim(pH)$  and  $\dim((1-p)B(1-p)) = \dim((1-p)H)$ . By induction,  $pBp = \bigoplus_{i=1}^{r_1} M_{n_i}(\mathbb{C})^{(n_i)}$ ,  $(1-p)B(1-p) = \bigoplus_{i=1}^{r_2} M_{m_i}(\mathbb{C})^{(m_i)}$ , and  $\sum_{i=1}^{r_1} n_i^2 + \sum_{i=1}^{r_2} m_i^2 = k + 1$ . Suppose  $B$  does not have a nontrivial central projection. Then  $B = M_r(\mathbb{C}) \subseteq M_{n+1}(\mathbb{C})$  and  $\dim B = r^2 = n + 1$  by Lemma 2.4.

Suppose  $B \subsetneq R \subseteq M_k(\mathbb{C}) \in L(H)$  and  $R$  is an algebra with property  $P_1$ . Let  $0 \neq x \in R \setminus B$ . Note that  $B = \bigoplus_{i=1}^r M_{n_i}(\mathbb{C})^{(n_i)}$ . Let  $p_i$  be the projection of  $B$  that corresponds to the summand  $M_{n_i}(\mathbb{C})^{(n_i)}$ . Then, we have  $p_i B p_i \subseteq p_i R p_i \subseteq L(p_i H)$  and  $p_i R p_i$  has property  $P_1$ . By Lemma 3.6,  $p_i R p_i = p_i B p_i$ . So we may assume

$$0 \neq x = \begin{pmatrix} 0_{n_1}^{(n_1)} & x_{12} & x_{13} & \cdots & x_{1n_r} \\ & 0_{n_2}^{(n_2)} & x_{23} & \cdots & x_{2n_r} \\ & & \ddots & & \vdots \\ & & & 0_{n_{r-1}}^{(n_{r-1})} & x_{r-1r} \\ 0 & & & & 0_{n_r}^{(n_r)} \end{pmatrix}.$$

We may assume that  $x_{12} \neq 0$ . Then

$$(p_1 + p_2)x(p_1 + p_2) \in (p_1 + p_2)R(p_1 + p_2) \setminus (p_1 + p_2)B(p_1 + p_2).$$

By Lemma 2.1,  $(p_1 + p_2)R(p_1 + p_2)$  has property  $P_1$ . By Lemma 3.10,  $(p_1 + p_2)B(p_1 + p_2) = M_{n_1}(\mathbb{C})^{(n_1)} \oplus M_{n_2}(\mathbb{C})^{(n_2)}$  is a maximal  $P_1$  algebra. This is a contradiction. So  $B$  is a maximal  $P_1$  algebra.  $\square$

## 4 Singly generated maximal $P_1$ algebras

In this section, we prove the following result.

**Theorem 4.1.** *Suppose  $B$  is a singly generated unital subalgebra of  $M_n(\mathbb{C})$  and  $\dim B = n$ . Then  $B$  is a maximal  $P_1$  algebra.*

To prove Theorem 4.1, we need several lemmas. Let  $J_n$  be the  $n \times n$  Jordan block.

**Lemma 4.2.** *Let  $B$  be the unital subalgebra of  $M_n(\mathbb{C})$  generated by the Jordan block  $J_n$ . If  $N \supset B$  is a subalgebra of the uptriangular algebra of  $M_n(\mathbb{C})$  and  $N$  has property  $P_1$ , then  $N = B$ .*

*Proof.* Suppose  $N \supsetneq B$  is a subalgebra of the uptriangular algebra and  $N$  has property  $P_1$ . Note that

$$B = \left\{ \sum_{k=0}^{n-1} \lambda_k (J_n)^k : \lambda_0, \dots, \lambda_{n-1} \in \mathbb{C} \right\}.$$

**A special case.** Suppose  $N$  contains an operator  $x$  of the following form

$$x = \begin{pmatrix} 0 & \cdots & 0 & \lambda & 0 \\ & 0 & \cdots & 0 & \eta \\ & & 0 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 0 \end{pmatrix}, \quad (4)$$

where  $\lambda \neq \eta$ . Then  $N$  contains the algebra generated by  $B$  and  $x$ . Therefore,

$$N \supset \left\{ \left( \begin{array}{ccccc} \lambda_1 & \cdots & \lambda_{n-2} & \alpha & \gamma \\ & \lambda_1 & \cdots & \lambda_{n-2} & \beta \\ & & \lambda_1 & \cdots & \lambda_{n-2} \\ & & & \ddots & \vdots \\ & & & & \lambda_1 \end{array} \right) : \lambda_1, \dots, \lambda_{n-2}, \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

Simple computation shows that

$$N_{\perp} \subset \left\{ \left( \begin{array}{ccccc} * & \cdots & * & 0 & 0 \\ & * & \cdots & * & 0 \\ & & * & \cdots & * \\ & & & \ddots & \vdots \\ & & & & * \end{array} \right) \right\}.$$

It is easy to see that the operator  $(J_n)^{n-2}$  can not be written as a sum of a rank one operator and an operator in  $N_\perp$ . This contradicts to the assumption that  $N$  has property  $P_1$ .

**The general case.** Suppose  $z \in N \setminus B$ . By the assumption of the lemma,  $z = (z_{i,j})_{n \times n}$  is an uptriangular matrix. Since  $z \notin B$ , we may assume that  $z_{j,j+k-1} \neq z_{j+r,j+r+k-1}$  for some positive integers  $j, k, r$ , and  $z_{s,t} = 0$  for  $t < s + k - 1$ . Without loss of generality, we assume that  $z_{1,k} \neq z_{2,1+k}$  and  $1 \leq k \leq n - 1$ . If  $k = n - 1$ , then this implies that  $N$  contains an  $x$  as in (4). If  $k < n - 2$ , then  $(J_n)^{k+1}z$  (or consider  $z(J_n)^{k+1}$  if  $z_{n-1,n-1} \neq z_{n,n}$ ) is a matrix in  $N$ . If we write  $(J_n)^{k+1}z = (y_{ij})_{n \times n}$ . Then  $y_{1,k+1} \neq y_{2,k+2}$  and  $y_{s,t} = 0$  for  $t < s + k$ . Repeating the above arguments, we can see that  $N$  contains an  $x$  as in (4). This completes the proof.  $\square$

**Lemma 4.3.** *Let  $B$  be the unital subalgebra of  $M_n(\mathbb{C})$  generated by the Jordan block  $J_n$ . Then  $B$  is a maximal  $P_1$  algebra.*

*Proof.* Suppose  $N \supset B$  is a subalgebra of  $M_n(\mathbb{C})$  and  $N$  has property  $P_1$ . By Wedderburn's Theorem,

$$N = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C}) \oplus J,$$

where  $J$  is the radical of  $N$ .

**Case 1.**  $n_1 = \cdots = n_s = 1$ . Then  $N$  is triangularizable, i.e., there exists a unitary matrix  $u \in M_n(\mathbb{C})$  such that  $uNu^*$  is contained in the algebra of uptriangular matrices (see Proposition 2.5 of [4]). Since  $J_n \in B \subset N$ ,  $uJ_nu^*$  is a strictly uptriangular matrix. Simple computation shows that  $u$  has to be a diagonal matrix. Therefore,  $N = u^*(uNu^*)u$  is contained in the algebra of uptriangular matrices. Since  $N$  has property  $P_1$ ,  $N = B$  by Lemma 4.2.

**Case 2.** Suppose  $n_i \geq 2$  for some  $i$ ,  $1 \leq i \leq s$ . Choose a nonzero partial isometry  $v \in M_{n_i}(\mathbb{C})$  such that  $v^2 = 0$ . Then either  $v \notin B$  or  $v^* \notin B$  since  $B$  does not contain any nontrivial projections. We may assume that  $v \notin B$ . Consider the subalgebra  $\tilde{N}$  generated by  $v$  and  $B$ . An element of  $\tilde{N}$  can be written as  $b_1vb_2v \cdots vb_n$ , where  $b_i \in J$  for  $2 \leq i \leq n - 1$ ,  $b_1 = 1$  or  $b_1 \in J$ ,  $b_n = 1$  or  $b_n \in J$ . By Lemma 2.1 of [4],  $\tilde{N} = \mathbb{C}1 \oplus \tilde{J}$ , where  $\tilde{J}$  is the radical part of  $\tilde{N}$  such that  $v \in \tilde{J}$ . Note that  $\tilde{N}$  also has property  $P_1$ . By Case 1,  $\tilde{N} = B$ . So  $v \in B$ . This is a contradiction.  $\square$

**Lemma 4.4.** *Let  $B_i \subset M_{n_i}(\mathbb{C})$  be the unital subalgebra generated by the Jordan block  $J_{n_i}$  for  $i = 1, 2$ . Then  $B = B_1 \oplus B_2$  is a maximal  $P_1$  subalgebra of  $M_{n_1+n_2}(\mathbb{C})$ .*

*Proof.* Suppose  $B \subsetneq N \subset M_{n_1+n_2}(\mathbb{C})$  and  $N$  has property  $P_1$ . Let  $p_i$  be the central projections of  $B$  corresponding to  $B_i$ . Then  $B_1 \subset p_1Np_1 \subset M_{n_1}(\mathbb{C})$  and  $p_1Np_1$  has property  $P_1$ . By Lemma 4.3,  $p_1Np_1 = B_1$ . Similarly,  $p_2Np_2 = B_2$ . Suppose  $x \in N \setminus B$ .

Then we may assume that  $0 \neq x = p_1 x p_2$ . With respect to matrix units of  $M_{n_1}(\mathbb{C})$  and  $M_{n_2}(\mathbb{C})$ , we can write  $x$  as

$$x = \begin{pmatrix} 0 & (x_{ij})_{n_1 \times n_2} \\ 0 & 0 \end{pmatrix},$$

where  $(x_{ij})_{n_1 \times n_2}$  is a nonzero matrix. Multiplying on the left by a suitable matrix of  $B$ , we may assume that  $x_{ij} = 0$  for all  $i \geq 2$  (which can be easily seen for the case  $n_2 = 1$ , other cases are similar). Multiplying on the right by another suitable matrix of  $B$ , we may further assume that  $x_{1,n_2} = 1$  and  $x_{1,j} = 0$  for  $1 \leq j \leq n_2 - 1$ . So we may assume that

$$x = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}_{n_1 \times n_2} \\ 0 & 0_{n_2 \times n_2} \end{pmatrix}.$$

Let  $\tilde{N}$  be the algebra generated by  $B$  and  $x$  above. Then

$$\tilde{N} = \left\{ \begin{pmatrix} \begin{pmatrix} \lambda_1 & \cdots & \lambda_{n_1} \\ & \ddots & \vdots \\ 0 & & \lambda_1 \end{pmatrix} & \begin{pmatrix} 0 & \cdots & \alpha \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}_{n_1 \times n_2} \\ 0 & \begin{pmatrix} \eta_1 & \cdots & \eta_{n_2} \\ & \ddots & \vdots \\ 0 & & \eta_1 \end{pmatrix} \end{pmatrix} : \lambda_i, \eta_j, \alpha \in \mathbb{C} \right\}.$$

Simple computation shows that

$$\tilde{N}_\perp \subset \left\{ \begin{pmatrix} \begin{pmatrix} * & \cdots & 0 \\ & \ddots & \vdots \\ * & & * \end{pmatrix} & \begin{pmatrix} * & \cdots & 0 \\ * & \cdots & * \\ \cdots & \cdots & \cdots \\ * & \cdots & * \end{pmatrix} \\ * & \begin{pmatrix} * & \cdots & 0 \\ & \ddots & \vdots \\ * & & * \end{pmatrix} \end{pmatrix} \right\}.$$

Let

$$y = \begin{pmatrix} \begin{pmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} & & \\ & 0_{n_1 \times n_2} & \\ & & \begin{pmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} \end{pmatrix}.$$

It is easy to see that the operator  $y$  can not be writtens a sum of a rank one operator and an operator in  $\tilde{N}_\perp$ . This contradicts to the fact that  $\tilde{N}$  has property  $P_1$ .  $\square$

*Proof of Theorem 4.1.* Suppose  $B$  is generated by a matrix  $T$ . By the Jordan Canonical Theorem, we may assume that  $T = \oplus_{i=1}^r (\lambda_i + J_{n_i})$  and  $\sum_{i=1}^r n_i = n$ . Note that  $\dim(B) = n$  if and only if  $\lambda_i \neq \lambda_j$  for  $i \neq j$ , and if and only if  $B = \oplus_{i=1}^r B_i$ , where each  $B_i$  is the subalgebra of  $M_{n_i}(\mathbb{C})$  generated by the Jordan block  $J_{n_i}$ .

Suppose  $B \subsetneq N \subset M_n(\mathbb{C})$  and  $N$  has property  $P_1$ . Let  $p_i$  be the central projection of  $B$  corresponding to  $B_i$ . Then  $B_i \subset p_i N p_i \subset M_{n_i}(\mathbb{C})$  and  $p_i N p_i$  has property  $P_1$ . By Lemma 4.3,  $B_i = p_i N p_i$ . Since  $B \neq N$ , there is an element  $0 \neq x \in N$  such that  $x = p_i x p_j$  for some  $i \neq j$ . Without loss of generality, we may assume that  $0 \neq x = p_1 x p_2$ . Now we have  $B_1 \oplus B_2 \subsetneq (p_1 + p_2)N(p_1 + p_2) \subseteq M_{n_1+n_2}(\mathbb{C})$  and  $(p_1 + p_2)N(p_1 + p_2)$  also has property  $P_1$ . On the other hand, by Lemma 4.4,  $B_1 \oplus B_2 = (p_1 + p_2)N(p_1 + p_2)$ . This is a contradiction.  $\square$

## 5 $P_1$ algebras in $M_n(\mathbb{C})$ , $n \leq 4$

Let  $B$  be a subalgebra of  $M_n(\mathbb{C})$ . Then  $B = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C}) \oplus J$ , where  $J$  is the radical part of  $B$ . If  $n_1, \dots, n_s = 1$ , then  $B$  is uptrianglizable, i.e., there exists a unitary matrix  $u$  such that  $uBu^*$  is a subalgebra of the uptriangular algebra of  $M_n(\mathbb{C})$  (see [4], Proposition 2.5 or [8], Corollary A, p.17). The following lemma will be useful.

**Lemma 5.1.** [Azoff] *Let  $S$  be a subspace of  $L(H)$  and consider the subalgebras of  $L(H^{(2)})$  defined by*

$$B = \left\{ \begin{pmatrix} \lambda e & a \\ 0 & \lambda e \end{pmatrix} : \lambda \in \mathbb{C}, a \in S \right\}, \quad C = \left\{ \begin{pmatrix} \lambda e & a \\ 0 & \mu e \end{pmatrix} : \lambda, \mu \in \mathbb{C}, a \in S \right\}.$$

1.  $B$  has property  $P_1$  if and only if  $S$  has property  $P_1$ .
2.  $C$  has property  $P_1$  if and only if  $S$  has property  $P_1$  and is intransitive.

**Proposition 5.2.** *Let  $B$  be a unital subalgebra of  $M_2(\mathbb{C})$  with property  $P_1$ . Then  $B$  is unitarily equivalent to one of the following three subalgebras:*

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \eta \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}.$$

*Proof.* It is easy to verify that the above algebras have property  $P_1$ . Suppose  $B$  has property  $P_1$ . Then the semi-simple part of  $B$  must be abelian. Conjugating by a unitary matrix, we may assume that  $B$  is a subalgebra of the algebra of uptriangular matrices. Note that the algebra of uptriangular matrices does not have property  $P_1$ . So  $B$  must be one of the algebras listed in the lemma.  $\square$

**Proposition 5.3.** *Let  $B$  be a unital subalgebra of  $M_3(\mathbb{C})$  with property  $P_1$ . Then either  $B$  or  $B^*$  has a separating vector. Therefore,  $\dim B \leq 3$ . Furthermore, if  $\dim B = 3$ , then  $B$  is similarly conjugate to one of the following algebras*

$$\begin{aligned} A_1 &= \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}, & A_2 &= \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}, \\ A_3 &= \left\{ \begin{pmatrix} \lambda_1 & \lambda_3 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}, & A_4 &= \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}, \\ A_5 &= \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}, & A_6 &= \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 0 & \lambda_1 & \lambda_3 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}. \end{aligned}$$

*Proof.* Suppose  $B$  has property  $P_1$ . Then the semi-simple part of  $B$  must be abelian. Conjugating by a unitary matrix, we may assume that  $B$  is a subalgebra of the algebra of uptriangular matrices. We consider the following cases.

**Case 1.** Suppose the semi-simple part of  $B$  is  $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ . Then  $B = A_1$  by Theorem 3.1.

**Case 2.** Suppose the semi-simple part of  $B$  is  $\mathbb{C} \oplus \mathbb{C}$ . We may assume that the semi-simple

part of  $B$  consists of matrices  $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$ . We consider two subcases.

**Subcase 2.1.** Suppose  $B$  is contained in the following algebra

$$B_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_1 & \lambda_4 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

Simple computation shows that  $B_1$  does not have property  $P_1$  (the identity matrix can not be written as  $x + (B_1)_\perp$  such that the rank of  $x$  is at most 1). So  $B$  is a proper subalgebra of  $B_1$ . This implies that there exist  $\alpha, \beta$  such that

$$B_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_3\alpha \\ 0 & \lambda_1 & \lambda_3\beta \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

If  $\alpha \neq 0$ , let

$$s = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Simple computation shows that  $sA_2s^{-1} = B$ , i.e.,  $s^{-1}Bs = A_2$ . If  $\alpha = 0, \beta \neq 0$ , let

$$s = \begin{pmatrix} 0 & 1 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then  $sA_2s^{-1} = B$ , i.e.,  $s^{-1}Bs = A_2$ . If  $\alpha = \beta = 0$ , then clearly  $B$  has a separating vector.

**Subcase 2.2.** Suppose  $B$  is not contained in  $B_1$ . Since  $B$  is an algebra,  $B$  contains  $A_3$ . It

is easy to see that  $A_3$  is the algebra generated by the matrix  $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  and  $\dim A_3 = 3$ .

So  $B = A_3$  by Theorem 4.1.

**Case 3.** Suppose the semi-simple part of  $B$  is  $\mathbb{C}$ . Then  $B$  is contained in the following algebra

$$B_3 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & \lambda_4 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that  $B_3$  does not have property  $P_1$ . So  $B$  is a proper subalgebra of  $B_3$ . We consider the following subcases.

**Subcase 3.1** Suppose  $B$  contains an element

$$b = \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}$$

such that  $\alpha \neq 0$  and  $\beta \neq 0$ . Conjugating by an invertible uptriangular matrix, we may assume that  $b = J_3$  is the Jordan block. So  $B$  contains  $A_4$ . By Theorem 4.1,  $B = A_4$ .

**Subcase 3.2** Suppose  $B$  does not contain an element  $b$  as in subcase 3.1. Then  $B \subseteq A_5$  or  $B \subseteq A_6$ . Note that  $A_5^*$  has a separating vector and  $A_6$  has a separating vector. So both  $A_5$  and  $A_6$  have property  $P_1$ .  $\square$

**Lemma 5.4.** *Let*

$$B = \left\{ \left( \begin{array}{cccc} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{array} \right) : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\} \subset M_4(\mathbb{C}).$$

*Then  $B$  is a maximal  $P_1$  algebra.*

*Proof.* Note that  $B^*$  has a separating vector. So  $B$  has property  $P_1$ . Suppose  $A \supsetneq B$  is a  $P_1$  algebra. Suppose  $A$  contains a matrix

$$a_1 = \begin{pmatrix} 0 & \alpha & * & * \\ 0 & 0 & \beta & * \\ 0 & 0 & \lambda_1 & \gamma \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

such that  $\gamma \neq 0$ . Since  $B \subset A$ , we may assume that  $\alpha \neq 0$  and  $\beta \neq 0$ . Conjugating by an upper-triangular invertible matrix, we may assume that  $A$  contains the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So  $A$  is the algebra generated by the Jordan block by Theorem 4.1 and  $\dim A = 4$ . However,  $\dim B = 4$  and  $B \subsetneq A$ . This is a contradiction.

Therefore,  $A$  is contained in

$$\left\{ \left( \begin{array}{cccc} \lambda_1 & * & * & * \\ 0 & \lambda_1 & * & * \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{array} \right) : \lambda_1 \in \mathbb{C} \right\}.$$

Since  $A$  is an algebra containing  $B$  and  $A \neq B$ , we may assume that  $A$  contains a matrix of the following form

$$a_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix},$$

where either  $s \neq 0$  or  $t \neq 0$ . Furthermore, we can assume that  $s = 1$  and  $t \neq 0$ . Let  $A_1$  be the algebra generated by  $B$  and  $a_2$ . Then

$$A_1 = \left\{ \left( \begin{array}{cccc} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & \lambda_2 + \lambda_5 & t\lambda_5 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{array} \right) : \lambda_1, \dots, \lambda_5 \in \mathbb{C} \right\}.$$

Simple computation shows that the predual space of  $A_1$  is

$$\left\{ \left( \begin{array}{cccc} \eta_1 & * & * & * \\ t\eta_5 & \eta_2 & * & * \\ 0 & -t\eta_5 & \eta_3 & 0 \\ 0 & \eta_5 & 0 & \eta_4 \end{array} \right) : \eta_1, \dots, \eta_4 \in \mathbb{C}, \eta_1 + \eta_2 + \eta_3 + \eta_4 = 0 \right\}.$$

It is easy to show that the following matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -t & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

can not be written as  $x + (A_1)_\perp$  such that the rank of  $x$  is at most 1. This is a contradiction. So  $B$  is a maximal  $P_1$  algebra. □

**Proposition 5.5.** *Let  $B$  be a unital subalgebra of  $M_4(\mathbb{C})$  with property  $P_1$ . Then  $B$  satisfies one the following conditions:*

1.  $B$  has a separating vector;
2.  $B^*$  has a separating vector;
3.  $B$  is similarly conjugate to an algebra of the form

$$\left\{ \left( \begin{array}{cc} \lambda I_2 & s \\ 0 & \eta I_2 \end{array} \right) : \lambda, \eta \in \mathbb{C}, s \in S \right\},$$

where  $S$  is a subspace of  $M_2(\mathbb{C})$  with dimension 2.

In particular,  $\dim B \leq 4$ .

*Proof.* Suppose  $B$  has property  $P_1$ . Then the semi-simple part of  $B$  must be  $M_2(\mathbb{C})$  or abelian. If the semi-simple part of  $B$  is  $M_2(\mathbb{C})$ , then  $B = M_2(\mathbb{C})^{(2)}$  by Theorem 3.1. So  $B$  has a separating vector. Suppose the semi-simple part of  $B$  is abelian. Conjugating by a unitary matrix, we may assume that  $B$  is a subalgebra of the algebra of uptriangular matrices. We consider the following cases.

**Case 1.** Suppose the semi-simple part of  $B$  is  $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ . Then

$$B = \left\{ \left( \begin{array}{cccc} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{array} \right) : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}$$

by Theorem 3.1. So  $B$  has a separating vector.

**Case 2.** Suppose the semi-simple part of  $B$  is  $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ . We may assume that the

semi-simple part of  $B$  consists of matrices  $\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$ . Let

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 2.1,  $(e_2 + e_3)B(e_2 + e_3) \subset M_2(\mathbb{C})$  has property  $P_1$ . By Theorem 3.1 and the assumption of case 2,  $(e_2 + e_3)B(e_2 + e_3) = \left\{ \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix} : \lambda_2, \lambda_3 \in \mathbb{C} \right\}$ . We consider two subcases.

**Subcase 2.1.** Suppose  $B$  is contained in the following algebra

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & \lambda_6 \\ 0 & \lambda_1 & \lambda_5 & \lambda_7 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_7 \in \mathbb{C} \right\}.$$

By Lemma 2.1,  $(e_1 + e_2)B(e_1 + e_2) \subset M_3(\mathbb{C})$  has property  $P_1$ . Note that

$$(e_1 + e_2)B(e_1 + e_2) \subseteq \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 \\ 0 & \lambda_1 & \lambda_5 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \dots, \lambda_5 \in \mathbb{C} \right\}.$$

By the proof of Subcase 2.1 of Proposition 5.3, there exists an invertible matrix

$$s = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}$$

such that

$$s^{-1}[(e_1 + e_2)B(e_1 + e_2)]s \subseteq \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

Conjugating by  $(s \oplus 1)^{-1} \in M_4(\mathbb{C})$ , we may assume that  $B$  is contained in the following algebra

$$B_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & \lambda_5 \\ 0 & \lambda_1 & 0 & \lambda_6 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_6 \in \mathbb{C} \right\}.$$

It is easy to see that  $B_1$  is similarly conjugate to the following algebra

$$\left\{ \left( \begin{array}{cccc} \lambda_1 & 0 & \lambda_5 & 0 \\ 0 & \lambda_1 & \lambda_6 & \lambda_4 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{array} \right) : \lambda_1, \dots, \lambda_6 \in \mathbb{C} \right\}.$$

So we may assume that

$$B_1 = \left\{ \left( \begin{array}{cccc} \lambda_1 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & \lambda_5 & \lambda_6 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{array} \right) : \lambda_1, \dots, \lambda_6 \in \mathbb{C} \right\}.$$

Repeating the above arguments, we may assume that  $B$  is contained in the following algebra

$$B_2 = \left\{ \left( \begin{array}{cccc} \lambda_1 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & 0 & \lambda_5 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{array} \right) : \lambda_1, \dots, \lambda_5 \in \mathbb{C} \right\}.$$

Simple computation shows that  $B_2$  does not have property  $P_1$  (the identity matrix can not be written as  $x + (B_2)_\perp$  such that the rank of  $x$  is at most 1). So  $B$  is a proper subalgebra of  $B_2$ . Therefore, there exist  $\alpha, \beta$  such that

$$B = \left\{ \left( \begin{array}{cccc} \lambda_1 & 0 & \lambda_4\alpha & 0 \\ 0 & \lambda_1 & 0 & \lambda_4\beta \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{array} \right) : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

If  $\alpha = \beta = 0$ , then clearly  $B$  has a separating vector.

If  $\alpha \neq 0$  and  $\beta \neq 0$ , let

$$t = \begin{pmatrix} \alpha^{-1} & 0 & 0 & 0 \\ 0 & \beta^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Simple computation shows that

$$tBt^{-1} = \left\{ \left( \begin{array}{cccc} \lambda_1 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & 0 & \lambda_4 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{array} \right) : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

So  $B$  has a separating vector.

If  $\alpha \neq 0, \beta = 0$  or  $\alpha = 0, \beta \neq 0$ , then  $B$  is similarly conjugate to the following algebra

$$\left\{ \left( \begin{array}{cccc} \lambda_1 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{array} \right) : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

So  $B$  has a separating vector.

**Subcase 2.2.** Suppose  $B$  is not contained in  $B_1$ . Since  $B$  is an algebra,  $B$  contains the following algebra

$$B_3 = \left\{ \left( \begin{array}{cccc} \lambda_1 & \lambda_4 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{array} \right) : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that  $B_3$  is the algebra generated by the matrix  $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$  and

$\dim B_3 = 4$ . So  $B = B_3$  by Theorem 4.1 and  $B$  has a separating vector.

**Case 3.** Suppose the semi-simple part of  $B$  is  $\mathbb{C} \oplus \mathbb{C}$ .

**Subcase 3.1.** Suppose  $B$  contains the following subalgebra

$$\left\{ \left( \begin{array}{cccc} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{array} \right) : \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

Let

$$f_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 2.1,  $f_i B f_i \subset M_2(\mathbb{C})$  has property  $P_1$ . By Proposition 5.2,  $f_i B f_i = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\}$

or  $f_i B f_i = \left\{ \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}$ . We consider the following subsubcases.

**Subsubcase 3.1.1.** Suppose  $f_1 B f_1 = f_2 B f_2 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\}$ . This implies that

$$B \subset \left\{ \begin{pmatrix} \lambda I_2 & * \\ 0 & \eta I_2 \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}.$$

By Lemma 5.1,

$$B = \left\{ \begin{pmatrix} \lambda I_2 & S \\ 0 & \eta I_2 \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\},$$

where  $S$  has property  $P_1$  and is intransitive. By [[2],Table 5A, page 34],  $S$  is equivalent to one of the following spaces: zero space, or

$$\begin{aligned} & \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & 0 \end{pmatrix} : \zeta \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} : \zeta \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \zeta & \xi \\ 0 & 0 \end{pmatrix} : \zeta, \xi \in \mathbb{C} \right\}, \\ & \left\{ \begin{pmatrix} \zeta & 0 \\ \xi & 0 \end{pmatrix} : \zeta, \xi \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \xi \end{pmatrix} : \zeta, \xi \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \zeta & \xi \\ 0 & \zeta \end{pmatrix} : \zeta, \xi \in \mathbb{C} \right\}. \end{aligned}$$

Note that in the last four cases neither  $B$  nor  $B^*$  has a separating vector.

**Subsubcase 3.1.2.** Suppose  $f_1 B f_1 = f_2 B f_2 = \left\{ \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}$ . This implies that  $B$  contains the following subalgebra

$$B_4 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that  $B_4$  is the algebra generated by the matrix  $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$  and

$\dim B_4 = 4$ . So  $B = B_4$  by Theorem 4.1 and  $B$  has a separating vector.

**Subsubcase 3.1.3.** Suppose  $f_1 B f_1 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\}$  and  $f_2 B f_2 = \left\{ \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}$ .

If  $\dim B > 3$ , then  $B$  contains a nonzero matrix  $b = \begin{pmatrix} 0_2 & a \\ 0_2 & 0_2 \end{pmatrix}$ . Let  $B_5$  be the subalgebra generated by  $f_1 B f_1$ ,  $f_2 B f_2$  and  $b$ . Then  $\dim B_5 = 4$  and  $B_5$  is the algebra generated the matrix

$$\begin{pmatrix} 0_2 & a \\ 0_2 & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

So  $B = B_5$  by Theorem 4.1 and

$$B = \left\{ \begin{pmatrix} \lambda_1 I_2 & \lambda_4 a \\ 0_2 & \begin{pmatrix} \lambda_2 & \lambda_3 \\ 0 & \lambda_2 \end{pmatrix} \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\},$$

where  $a$  is a  $2 \times 2$  matrix. Let

$$t = \begin{pmatrix} b & 0 \\ 0_2 & I_2 \end{pmatrix}.$$

Then

$$tBt^{-1} = \left\{ \begin{pmatrix} \lambda_1 I_2 & \lambda_4 ba \\ 0_2 & \begin{pmatrix} \lambda_2 & \lambda_3 \\ 0 & \lambda_2 \end{pmatrix} \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

So we can choose  $b$  appropriately such that  $ba = 0_2$ , or  $ba = I_2$ , or

$$ba = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad ba = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad ba = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad ba = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

In each case,  $B$  has a separating vector.

**Subcase 3.2.** Suppose  $B$  contains the following subalgebra

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

Let

$$p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 2.1,  $pBp \subset M_3(\mathbb{C})$  has property  $P_1$ . By Lemma 5.2,

$$pBp = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}$$

or

$$pBp = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

We consider the following subcases.

**Subsubcase 3.2.1.** Suppose  $pBp = \left\{ \begin{pmatrix} \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_2 & \lambda_3 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C} \right\}$ . Then  $B$  contains the following subalgebra

$$B_6 = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & 0 & \lambda_2 & \lambda_3 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that  $B_6$  is the algebra generated by the matrix  $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$  and

$\dim B_6 = 4$ . So  $B = B_6$  by Theorem 4.1 and  $B$  has a separating vector.

**Subsubcase 3.2.2.** Suppose  $pBp = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\}$ . If  $\dim B > 3$ ,

then  $B$  contains a nonzero matrix  $b = \begin{pmatrix} 0 & a \\ 0 & 0_3 \end{pmatrix}$ . Let  $B_7$  be the subalgebra generated by  $(1-p)B(1-p)$ ,  $pBp$  and  $b$ . Then  $\dim B_7 = 4$  and  $B_7$  is the algebra generated the matrix

$$\begin{pmatrix} 0 & a \\ 0 & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}.$$

So  $B = B_7$  by Theorem 4.1 and

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_4 a \\ 0 & \begin{pmatrix} \lambda_2 & 0 & \lambda_3 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

Conjugating by an appropriate invertible matrix

$$t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & * & * \\ 0 & 0 & \eta & * \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

we have

$$tBt^{-1} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 & \lambda_3 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\},$$

or

$$tBt^{-1} = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 & 0 \\ 0 & \lambda_2 & 0 & \lambda_3 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\},$$

or

$${}^t B t^{-1} = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & \lambda_2 \\ 0 & \lambda_2 & 0 & \lambda_3 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

In each case,  $B^*$  has a separating vector.

**Case 4.** Suppose the semi-simple part of  $B$  is  $\mathbb{C}$ . Consider matrices in  $B$  with the following form

$$b = \begin{pmatrix} 0 & \alpha & * & * \\ 0 & 0 & \beta & * \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Subcase 4.1.**  $B$  contains a matrix  $b$  with  $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$ . Conjugating by an upper-triangular invertible matrix, we may assume that  $B$  contains the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So  $B$  is the algebra generated by the Jordan block by Theorem 4.1. Note that  $B$  has a separating vector.

**Subcase 4.2.**  $B$  does not contain a matrix  $B$  as in subcase 4.1 and  $B$  contains a matrix  $B$  with two elements of  $\alpha, \beta, \gamma$  nonzero. We may assume that  $\alpha \neq 0$  and  $\beta \neq 0$ . Conjugating by an upper-triangular invertible matrix, we may assume that  $B$  contains the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and therefore,

$$B \supseteq \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

By the assumption of subcase 4.2, we have

$$B \subset \left\{ \begin{pmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_1 & * & * \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1 \in \mathbb{C} \right\}. \quad (5)$$

**Subsubcase 4.2.1** Suppose the  $(2,4)$ -entry of every matrix in  $B$  is zero. Then  $B$  is contained in the following algebra

$$B_8 \subset \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_4 & \lambda_5 \\ 0 & \lambda_1 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_5 \in \mathbb{C} \right\}.$$

Simple computation shows that  $B_8$  does not have property  $P_1$ . So  $B$  is a proper subalgebra of  $B_8$ . By (5), there exist  $\alpha, \beta$  such that

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\alpha \\ 0 & \lambda_1 & \lambda_2 + \lambda_4\beta & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

If  $\alpha = 0$  and  $\beta \neq 0$ , then  $B$  does not have property  $P_1$ . So we may assume that  $\alpha \neq 0$ . It is easy to see that  $B^*$  has a separating vector.

**Subsubcase 4.2.2** Suppose the  $(2,4)$ -entry of a matrix in  $B$  is not zero. By (5),  $B$  contains an element

$$b = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \beta & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where  $\gamma \neq 0$ . Since  $B$  is an algebra,  $B$  contains

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} b = \begin{pmatrix} 0 & 0 & \beta & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By (5),  $B$  contains

$$\begin{pmatrix} 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since  $B$  is an algebra,  $B$  contains the following subalgebra

$$B_9 \subseteq \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

By Lemma 5.4,  $B_9$  is a maximal  $P_1$  algebra. Hence,  $B = B_9$  and  $B^*$  has a separating vector.

**Subcase 4.3.**  $B$  does not contain a matrix  $B$  as in subcase 4.1, subcase 4.2, and  $B$  contains a matrix  $B$  with one element of  $\alpha, \beta, \gamma$  nonzero. We may assume that  $\alpha \neq 0$ . Conjugating by an upper-triangular invertible matrix, we may assume that  $B$  contains the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By the assumption of subcase 4.3,  $B$  is contained in the following algebra

$$B_{10} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & 0 & \lambda_5 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_5 \in \mathbb{C} \right\}.$$

Simple computation shows that  $B_{10}$  does not have property  $P_1$ . So  $B$  is a proper subalgebra of  $B_{10}$ . We consider the following subsubcases.

**Subsubcase 4.3.1.** If the  $(1, 3)$  entry of each element of  $B$  is zero. Then  $B$  is contained in the following algebra

$$B_{11} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & \lambda_3 \\ 0 & \lambda_1 & 0 & \lambda_4 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

Simple computation shows that  $B_{11}$  does not have property  $P_1$ . So there exist  $\alpha, \beta$  such that

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & \lambda_3 \alpha \\ 0 & \lambda_1 & 0 & \lambda_3 \beta \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

If  $\beta = 0$ , then  $B^*$  has a separating vector. If  $\beta \neq 0$ , then  $B$  has a separating vector.

**Subsubcase 4.3.2.** If the  $(2, 4)$  entry of each element of  $B$  is zero. Then  $B$  is contained in the following algebra

$$B_{12} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

Note that  $B_{12}^*$  has a separating vector and hence  $B^*$  has a separating vector.

**Subsubcase 4.3.3** Suppose  $B$  contains an element

$$b = \begin{pmatrix} 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where  $\alpha \neq 0$  and  $\gamma \neq 0$ . Let

$$t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha^{-1} & -\frac{\beta}{\alpha\gamma} \\ 0 & 0 & 0 & \gamma^{-1} \end{pmatrix}.$$

Then

$$t^{-1}bt = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Conjugating by  $t^{-1}$  if necessary, we may assume that  $\alpha = \gamma = 1$  and  $\beta = 0$ . Since  $B$  is a proper subalgebra of  $B_{10}$ ,  $B$  is the algebra,

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & 0 & \lambda_3 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that  $B^*$  has a separating vector.

**Subcase 4.4.**  $B$  does not contain a matrix  $B$  as in subcase 4.1, subcase 4.2, subcase 4.3.

Then

$$B \subset \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & 0 & \lambda_4 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

Combining Lemma 5.1, [[2],Table 5A, page 34], and similar arguments as in subsubcase 3.1.1,

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

or

$$B = \left\{ \left( \begin{array}{cccc} \lambda_1 & 0 & 0 & \lambda_2 \\ 0 & \lambda_1 & 0 & \lambda_3 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{array} \right) : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

or

$$B = \left\{ \left( \begin{array}{cccc} \lambda_1 & 0 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & 0 & \lambda_2 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{array} \right) : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

or

$$B = \left\{ \left( \begin{array}{cccc} \lambda_1 & 0 & \lambda_2 & 0 \\ 0 & \lambda_1 & 0 & \lambda_3 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{array} \right) : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

It is easy to show that in each case either  $B$  or  $B^*$  has a separating vector.  $\square$

## 6 2-reflexivity and property $P_1$

Let  $H$  be a Hilbert space. The usual notation  $\text{Lat}(B)$  will denote the lattice of invariant subspaces (or projections) for a subset  $B \subseteq L(H)$ , and  $\text{Alg}(L)$  will denote the algebra of bounded linear operators leaving invariant every member of a family  $L$  of subspaces (or projections). An algebra  $B$  is called reflexive if  $B = \text{AlgLat}(B)$ . An algebra  $B$  is called  $n$ -reflexive if the  $n$ -fold inflation  $B^{(n)} = \{b^{(n)} : b \in B\}$ , acting on  $\mathcal{H}^{(n)}$ , is reflexive (see [1]). In [11], the third author proved the following result: An algebra  $B$  is  $n$ -reflexive if and only if  $B_\perp$ , the annihilator of  $B$ , is the trace class norm closed linear span of operators of rank  $\leq n$ . In [11], the third author also showed the following connection between  $n$ -reflexivity and  $P_1$  property: If an algebra  $B$  has property  $P_1$ , then  $B$  is 3-fold reflexive. (This result also holds for linear subspaces with the same proof). He raised the following problem: Suppose  $\dim H = n \in \mathbb{N}$  and  $B \subset L(H) \cong M_n(\mathbb{C})$  is a unital operator algebra with property  $P_1$ . Is  $B$  2-reflexive? Note that this question also makes sense for linear subspaces. In [1], Azoff showed that the answer to the above question is affirmative for  $n = 3$  (for all linear subspaces of  $M_3(\mathbb{C})$  with property  $P_1$ ). In this section, we prove the following result.

**Proposition 6.1.** *If  $\dim H = 4$  and  $B \subset L(H) \cong M_4(\mathbb{C})$  is a unital operator algebra with property  $P_1$ , then  $B$  is 2-reflexive.*

*Proof.* By Proposition 5.5, either  $B$  or  $B^*$  has a separating vector or  $B$  is similarly conjugate to an algebra of the form

$$\left\{ \begin{pmatrix} \lambda I_2 & s \\ 0 & \eta I_2 \end{pmatrix} : \lambda, \eta \in \mathbb{C}, s \in S \right\},$$

where  $S$  is a subspace of  $M_2(\mathbb{C})$  with dimension 2. If  $B$  has a separating vector or  $B^*$  has a separating vector, then  $B$  is 2-reflexive follows from the proofs of Corollary 7 of [11] and Proposition 1.2 of [6]. If  $B$  is similarly conjugate to an algebra of the form

$$\left\{ \begin{pmatrix} \lambda I_2 & s \\ 0 & \eta I_2 \end{pmatrix} : \lambda, \eta \in \mathbb{C}, s \in S \right\},$$

where  $S$  is a subspace of  $M_2(\mathbb{C})$  with dimension 2, then  $B$  is 2-reflexive follows from Proposition 1 of [10].

□

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