

ANNIHILATORS OF OPERATOR ALGEBRAS

David R. Larson

The structure of the predual of an ultraweakly closed operator algebra can be very revealing of internal structural properties of the algebra. This relationship has been most important in the theory of von Neumann algebras, and has recently become significant in the study of more general ultraweakly closed algebras.

The purpose of this note is to show that, analogously, knowledge of the annihilator of such an algebra can be revealing of external structural properties of the algebra and seems to lend perspective to certain open questions. In particular, the notions of reflexivity, n -reflexivity, and the Arveson distance estimate have natural formulations in this setting.

Let H be separable Hilbert space. The usual notation $\text{Lat}(A)$ will denote the lattice of invariant subspaces (or projections) for a subset $A \subset L(H)$, and $\text{Alg}(L)$ will denote the algebra of bounded linear operators leaving invariant every member of a family L of subspaces (or projections). A is reflexive if $A = \text{Alg Lat}(A)$ and L is reflexive if $L = \text{Lat Alg}(L)$. An operator T is said to be reflexive if the weakly closed algebra generated by T and I , denoted $W(T)$, is reflexive. T is said to be n -reflexive if the n -fold inflation $T^{(n)} = T \otimes T \otimes \dots \otimes T$, acting on $H^{(n)} = H \otimes \dots \otimes H$, is reflexive, and an algebra A is n -reflexive if $A^{(n)} = \{T^{(n)} : T \in A\}$ is reflexive. If $\{A_\lambda\}$ is a family of algebras then $\bigcap_\lambda \text{Lat}(A_\lambda) = \text{Lat}(\bigcup_\lambda A_\lambda)$, and if $\{L_\lambda\}$ is a family of lattices then $\bigcap_\lambda \text{Alg}(L_\lambda) = \text{Alg}(\bigcup_\lambda L_\lambda)$. So arbitrary intersections of reflexive algebras (lattices) are reflexive. The same is not true of joins.

Let L_* denote the trace class operators in $L(H)$. Then $L(H)$ is identified with $(L_*)'$ via the pairing $(f, T) = \text{Tr}(Tf)$, $f \in L_*$.

$\mathcal{L}(H)$. Let K denote the compact operators in $L(H)$. Then the same pairing identifies L_* with $(K)^*$. Let $\|f\|_1$ denote the trace class norm of f in L_* . By $\text{annih}(A)$ we mean the set of all trace-class operators f such that $\text{Tr}(Af)=0, \forall A \in A$. The ultraweak (σ -weak) topology on $L(H)$ is the w^* -topology under the identification $L(H) = (L_*)^*$. If A is σ -weakly closed then A is identified as the dual of the quotient $L_*/\text{annih} A$. We write $\text{predual}(A) = L_*/\text{annih}(A)$. If $\{S_\lambda : \lambda \in \Lambda\}$ is a family of σ -weakly closed linear subspaces of $L(H)$ then $\text{annih}(\overline{\cup S_\lambda}) = \overline{\text{span}(\cup \text{annih}(S_\lambda))}$, where closure denotes $\|\cdot\|_1$ closure, and $\text{annih}(\overline{\cup S_\lambda}) = \overline{\text{annih}(S_\lambda)}$.

If $x, y \in H$ we denote the rank-1 operator $u \rightarrow (u, x)y$ by $x \otimes y$. If A is an algebra containing I and $\text{PcLat}(A)$ with $P \neq 0, I$ let x, y be nonzero vectors in P^1H, PH , respectively. If $f = x \otimes y$ then $f \in \text{annih}(A)$ since $\text{Tr}(Af) = (Ay, x) = 0, \forall A \in A$. Conversely, if $f = x \otimes y$ is a rank-1 operator in $\text{annih}(A)$ then $(Ay, x) = 0, \forall A \in A$, so $[Ay]$ is a nontrivial invariant subspace for A . Thus A has a nontrivial invariant subspace iff $\text{annih}(A)$ contains a rank-1 operator. More can be said in this direction.

LEMMA 1. If A is an arbitrary operator algebra containing I then $\text{annih}(A)$ and $\text{annih}(\text{AlgLat } A)$ contain the same rank-1 operators. If f is rank-1, then $f \in \text{annih}(A)$ if and only if $f = P f P^1$ for some $P \in \text{PcLat}(A)$.

PROOF. Suppose $f = x \otimes y \in \text{annih}(A)$. Then for $A \in A$ we have $0 = \text{Tr}(Af) = (Ay, x)$. Let P be the projection onto the subspace $[Ay]$. If $B \in \text{AlgLat } A$ then $BPH \subseteq PH$, and Px , so $\text{Tr}(Bf) = (By, x) = 0$. So $f \in \text{annih}(\text{AlgLat } A)$.

For the last sentence, if f is a trace class operator with $f = P f P^1$ for $P \in \text{PcLat}(A)$ then $f \in \text{annih}(A)$ trivially. The converse follows from above. \blacksquare

LEMMA 2. Let A be a σ -weakly closed algebra containing I . Then A is reflexive if and only if $\text{annih}(A)$ is the $\|\cdot\|_1$ -closed linear span of rank-1 operators.

PROOF. We have $\text{AlgLat}(A) \supseteq A$ so $\text{annih}(\text{AlgLat } A) \subseteq \text{annih}(A)$. If $\text{annih}(A)$ is generated by rank-1 operators this last inclusion is

an equality by Lemma 1, so $\text{AlgLat } A = A$.

Conversely, suppose A is reflexive and let $L = \text{Lat}(A)$. For each $P \in \mathcal{L}$ let $L_P = (0, P, I)$. Then $A = \overline{\text{span}\{ \text{Alg}(L_P) : P \in \mathcal{L} \}}$. It is easily verified that $\text{annih}(\text{Alg}(L_P)) = PL_P P^1$ which is the closed linear span of its rank-1 elements since L_P is. Since $\text{annih}(A)$ is the $\|\cdot\|_1$ -closed span of $\cup \{ \text{annih}(\text{Alg}(L_P)) : P \in \mathcal{L} \}$ it also is generated by rank-1 elements. \blacksquare

While n -reflexivity for $n > 1$ does not imply the existence of a single nontrivial invariant subspace, it remains an interesting structural property that has a natural characterization in terms of the annihilator.

LEMMA 3. Let A be a σ -weakly closed linear subspace of $L(H)$, and let $1 \leq n < \infty$. Then $\text{annih}(A^{(n)})$ is the set of all trace class operators $f = (f_{ij})_{i,j=1}^n$ in $L(H^{(n)})$ such that $\sum_{i=1}^n f_{ii} \in \text{annih}(A)$.

PROOF. For $A \in L(H)$ we have $\text{Tr}(A^{(n)} f) = \sum_{i=1}^n \text{Tr}(A f_{ii})$. The conclusion follows. \blacksquare

THEOREM 4. Let A be a σ -weakly closed subalgebra of $L(H)$ containing I and let $1 \leq n < \infty$. Then A is n -reflexive if and only if $\text{annih}(A)$ is the trace class norm closed linear span of operators of rank $\leq n$.

PROOF. Let S be the trace-class norm closed linear span of operators in $\text{annih}(A^{(n)})$ of rank ≤ 1 . We will show that $S = \text{annih}(A^{(n)})$, thus proving reflexivity. Let $f = (f_{ij})$ be an element of $\text{annih}(A^{(n)})$, so $f \in S$. Every zero-diagonal operator matrix with trace class entries is clearly in S , thus we reduce to the case in which f is diagonal. We use the notation $f = \text{diag}(f_{11}, f_{22}, \dots, f_{nn})$.

First suppose that each f_{ii} has rank ≤ 1 . Write $f_{ii} = x_i \otimes y_i$ for vectors $x_i, y_i \in H$. Let $q_{ij} = x_j \otimes y_i, 1 \leq i, j \leq n$ and let $g = (q_{ij})$. Then $g \in \text{annih}(A^{(n)})$, and the range of g is the span of the vector $y_1 \otimes y_2 \otimes \dots \otimes y_n$ so g has rank ≤ 1 . By a standard argument f is a convex combination of operators unitarily equivalent to g having the same diagonal. These operators are in $\text{annih}(A^{(n)})$, which proves $f \in S$.

For f_{ii} of arbitrary rank proceed as follows. For $1 \leq i \leq n-1$

let h_1 be the diagonal matrix with i, i -element equal to $f_{11} + f_{22} + \dots + f_{ii}$, $i+1, i+1$ -element equal to $-(f_{11} + f_{22} + \dots + f_{ii})$, and all other elements 0. Let h be the diagonal matrix with n, n -element $f_{11} + \dots + f_{nn}$ and all other elements 0. Then $h_1 \in \text{cannih}(A^{(n)})$, and $f = h_1 + \dots + h_{n-1} + h$. From Lemma 3 it is clear that $h_1 \in S$, $1 \leq i \leq n-1$. We must show $h \in S$.

Since $f_{11} + \dots + f_{nn} \in \text{cannih}(A)$, by hypothesis this can be approximated by finite sums of operators in $\text{annih}(A)$ of rank $\leq n$. Thus it will suffice to show that every operator of the form $\text{diag}(0, 0, \dots, 0, r)$ with $r \in \text{annih}(A)$, $\text{rank}(r) \leq n$, is in S . Write $r = r_1 + \dots + r_n$ where $\text{rank}(r_1) \leq 1$, r_1 not necessarily in $\text{annih}(A)$. Now write $\bar{f}_{11} = r_1$ and $\bar{f} = \text{diag}(\bar{f}_{11}, \bar{f}_{22}, \dots, \bar{f}_{nn})$. Then $\bar{f} \in \text{cannih}(A^{(n)})$, and $\bar{f} \in S$ since $\text{rank}(\bar{f}_{11}) \leq 1$. Now repeat the argument in the previous paragraph replacing f with \bar{f} and f_{11} with \bar{f}_{11} , obtaining $\bar{h} = \bar{h}_1 + \bar{h}_2 + \dots + \bar{h}_{n-1} + \bar{h}$ with $\bar{h}_1 \in S$ and $\bar{h} = \text{diag}(0, 0, \dots, r)$. Hence $h \in S$. Thus $A^{(n)}$ is reflexive.

For the converse, assume $A^{(n)}$ is reflexive. Let $r \in \text{cannih}(A)$ and let $R = \text{diag}(0, 0, \dots, 0, r)$. We have $R = \lim_{\mathcal{L}} R_\ell$ where R_ℓ is a finite sum of operator in $\text{annih}(A^{(n)})$ of rank ≤ 1 .

Write $R_\ell = \sum_k ER_{\ell}^k$. Now let $R_{\ell ij}^k$ denote the i, j -matrix element of R_ℓ^k , and let R_{ij} denote the i, j -matrix element of R . We have $\lim_{\mathcal{L}} R_{\ell nn} = r$ and $\lim_{\mathcal{L}} R_{\ell ii} = 0$, $1 \leq i \leq n-1$, so $\lim_{\mathcal{L}} \sum_k ER_{\ell ii} = r$. Also, $R_{\ell ii} = \sum_k ER_{\ell ii}^k$, so $r = \lim_{\mathcal{L}} \sum_k \sum_i ER_{\ell ii}^k$. Each sum $\sum_{i=1}^n R_{\ell ii}^k$ is in $\text{annih}(A)$ and has rank $\leq n$. \blacksquare

COROLLARY 5. *If an algebra A is n -reflexive then it is m -reflexive for all $m \geq n$.*

Let A be σ -weakly closed operator algebra containing I . For $1 \leq n < \infty$ note that $\text{AlgLat}(A^{(n)})$ is contained in $(L(H))^{(n)}$ since the latter is reflexive and let $A_n = \{A \in L(H) : A^{(n)} \in \text{AlgLat}(A^{(n)})\}$. Then A_n is the smallest n -reflexive algebra containing A . We say that A_n is the n -reflexive algebra generated by A . If A is an algebra containing I and $1 \leq n < \infty$ let E_n be the $\|\cdot\|_1$ -closed span of the

operators in $\text{annih}(A)$ of rank $\leq n$. While it may not seem obvious that the polar of E_n in $L(H)$ is an algebra, this is indeed the case and we have $A_n = (E_n)^\perp$.

COROLLARY 6. *The annihilator of A_n is the trace-class norm closed linear span of the operators in $\text{annih}(A)$ of rank $\leq n$.*

PROOF. By (2), (3) and (4), $\text{annih}(A_n)$ is the $\|\cdot\|_1$ -closed linear span of operators $\sum_{i=1}^n f_{ii}$ where the operator matrix (f_{ij}) is a rank-1 operator in the annihilator of $\text{AlgLat}(A^{(n)})$. Each $\sum f_{ii}$ has rank $\leq n$ and is contained in $\text{annih}(A)$ since $A \in A_n$. Conversely, if $g \in \text{annih}(A)$ has rank $\leq n$ write $g = g_1 + \dots + g_n$ where $\text{rank}(g_i) \leq 1$, g_i not necessarily in $\text{annih}(A)$. Then $G = \text{diag}(g_1, g_2, \dots, g_n) \in \text{annih}(A^{(n)})$. As in the proof of Theorem 4 there exists a rank-1 operator with the same diagonal as G hence in $\text{annih}(A^{(n)})$. By (2) this is also in $\text{annih}(\text{AlgLat}(A^{(n)}))$. Thus $g = \sum E_{g_i} \in \text{annih}(A_n)$. \blacksquare

REMARK. It is easily proven from duality considerations that a σ -weakly closed linear subspace of $L(H)$ is weakly closed if and only if its annihilator is the $\|\cdot\|_1$ -closed linear span of finite rank operators. If A is a weakly closed algebra with I and $T \in L(H)$ it is well known that $\text{dist}(T, A) = \lim_{\mathcal{L}} \text{dist}(T, \text{AlgLat}(A^{(n)})) = \lim_{\mathcal{L}} \text{dist}(T, A_n)$. This can be seen intuitively from Corollary 6 since $\text{annih}(A)$ is the closed span of finite rank elements and the distance from T to any σ -weakly closed algebra B is $\sup\{|\text{Tr}(Tf)| : f \in \text{cannih} B, \|f\|_1 \leq 1\}$. For any algebra A containing I the weak closure of A is A_n .

These annihilator considerations seem to point out the special role of the weakly closed algebras in the general theory of ultraweakly closed algebras. In particular it can become of interest to know when, for a given operator T , the σ -weakly closed algebra generated by T and I is in fact weakly closed. Questions of this nature do not seem to have been pursued in the literature. The following simple application of Theorem 4 suggests that these questions could be interesting.

COROLLARY 7. *Let A be a weakly closed algebra containing I . Suppose every element of $L_n / \text{annih}(A)$ has the form $f + \text{annih}(A)$ with $\text{rank}(f) \leq 1$. Then A is 3-reflexive.*

PROOF. A is weakly closed so $\text{annih}(A)$ is the closed span of finite rank elements. Let g be a finite rank element of $\text{annih}(A)$ and write $g = g_1 + \dots + g_m$ where g_1 is rank-1, not necessarily in $\text{annih}(A)$. By hypothesis there exist h_1, \dots, h_{m-1} with $\text{rank}(h_i) \leq 1$ such that each term in the decomposition $g = (g_1 + g_2 - h_1) + (h_1 + g_3 - h_2) + \dots + (h_{m-2} + g_m - h_{m-1}) + h_{m-1}$ is in $\text{annih}(A)$. Then A is 3-reflexive by Theorem 4. ■

Let us say that an algebra $A \subset L(H)$ has property P_1 if it has the predual property in Corollary 6: every element of $L_*/\text{annih}(A)$ has the form $f + \text{annih}(A)$ with $\text{rank}(f) \leq 1$. Corollary 7 says that a weakly closed algebra containing I with property P_1 is 3-reflexive. It was shown by Brown, Chevreau and Pearcy in [2] that completely nonunitary contractions with "rich spectrum" not only have nontrivial invariant subspaces, but they showed that such an operator T which could not be eliminated as having an invariant subspace by preliminary means had the property that the σ -weakly closed algebra generated by $\{T, I\}$ has the predual property P_1 .

QUESTION 1. If T is a completely nonunitary contraction with "rich spectrum" is the σ -weakly closed algebra generated by $\{T, I\}$ closed? If so T would be 3-reflexive. This would be interesting structurally, although we note that this property above would not to our knowledge directly imply the existence of an invariant subspace. Rich spectrum yielded much stronger properties that were used in [2] to conclude that T was not transitive.

A strengthening of Q1 would be:

QUESTION 2. Is a σ -weakly closed algebra with property P_1 necessarily weakly closed?

QUESTION 3. Does a weakly closed algebra with property P_1 (hence 3-reflexive) necessarily have a nontrivial invariant subspace? Equivalently, does $\text{annih}(A)$ contain rank-1 operators?

REMARK. Corollary 7 has a natural generalization which we present for completeness. Let us say that an algebra A has property P_n if every element of $L_*/\text{annih}(A)$ has the form $f + \text{annih}(A)$ with $\text{rank}(f) \leq n$.

COROLLARY 8. Let A be a weakly closed algebra containing I which has property P_n . Then A is 3n-reflexive.

REMARK. It is in actuality somewhat misleading to refer to P_n as a predual property since it depends on the representation of a predual of a σ -weakly closed algebra as $L_*/\text{annih}(A)$ and is not independent of the particular representation of A as a σ -weakly closed algebra on Hilbert space.

QUESTION 4. Let A be a σ -weakly closed algebra containing I and let F_1 be the set of elements of $L_*/\text{annih}(A)$ which have the form $f + \text{annih}(A)$ with $\text{rank}(f) \leq 1$. Is F_1 necessarily closed in the quotient norm of $L_*/\text{annih}(A)$? We feel that the answer is likely negative without strong conditions imposed on A . Can one find reasonable general conditions? The motivation for this question comes from [2]. If it were known independently that F_1 is closed if A is the σ -weakly closed algebra generated by $\{T, I\}$ and T is a completely nonunitary contraction with rich spectrum, then the proof in [2] could have been simplified. The beautiful argument in [2] shows that F_1 is closed by in fact proving that it is all of $L_*/\text{annih}(A)$.

REMARK. It is known that a weakly closed algebra containing I with property P_1 need not be reflexive, even in finite dimensions. However we do have the following:

LEMMA 9. Let A be a reflexive algebra with property P_1 . Then every σ -weakly closed subalgebra containing I is also reflexive.

PROOF. Let B be a σ -weakly closed subalgebra of A , and let $\bar{B} = \text{AlgLat } B$. Then $\bar{B} \subset A$ since A is reflexive. If $B \neq \bar{B}$ there exists an element of $L_*/\text{annih}(A)$ that annihilates B but does not annihilate \bar{B} . This has the form $f + \text{annih}(A)$ with $\text{rank}(f) = 1$. But then $f \text{annih}(B)$ so by Lemma 1 $f \text{annih}(\bar{B})$, a contradiction. ■

REMARK. The above proof easily extends to show that if A is an n -reflexive algebra with property P_n then every σ -weakly closed subalgebra containing I is n -reflexive.

The annihilator of an algebra which is a commutant has a very simple characterization.

LEMMA 10. Let A be a subset of $L(H)$. Then the annihilator of the commutant A' is the trace-class norm closed span of $\{Af - fA : A \in A, f \in L_1\}$.

PROOF. If $A \in A$, $f \in L_1$, let $g = Af - fA$. Then for $B \in A'$, $\text{Tr}(Bg) = \text{Tr}((BA - AB)f) = 0$ so $g \in \text{annih}(A')$. Conversely, let B be an operator for which $\text{Tr}(B(Af - fA)) = 0$, $A \in A$, $f \in L_1$. Then $\text{Tr}((BA - AB)f) = 0$, $f \in L_1$, so $BA - AB = 0$. \square

COROLLARY 11. Every commutant is a 2-reflexive algebra.

PROOF. The above lemma shows that the annihilator of a commutant is generated by rank ≤ 2 operators. \square

REMARK. The above shows that if there exists a transitive operator then there exists a nontrivial 2-reflexive transitive algebra: namely its commutant. More generally, if there exists a transitive operator algebra with nontrivial commutant then there exists a nontrivial 2-reflexive transitive algebra: its double commutant.

QUESTION 5. Can it be shown that the existence of any nontrivial weakly closed transitive algebra would imply the existence of a nontrivial 2-reflexive transitive algebra? Note that it would imply the existence of a nontrivial n -reflexive transitive algebra for some finite n . Would the existence of a transitive operator imply the existence of a 2-reflexive transitive operator?

REMARK. Corollary 11 shows that while reflexive operators and algebras are rather special the larger class of 2-reflexive operators and algebras is significantly richer. Let $W(A)$ denote the weakly closed algebra generated by (A, I) . If $W(A) = (A)''$ then A is 2-reflexive. In particular, the Volterra operator is 2-reflexive, although it is unicellular so it is far from being reflexive.

QUESTION 6. Which weighted shifts are 2-reflexive? Is there a reasonable classification of compact 2-reflexive operators?

REMARK. If T is an operator the set $\{Tf - fT : f \in L_1\}$ is a linear space, so the question of whether T has a nontrivial hyperinva-

riant subspace translates to the question of whether the $\|\cdot\|_1$ -closure of this space contains a rank-1 operator.

The present investigation was motivated in part by work of W. Arveson [1] and by independent work of C. Lance [7] in which a formula for the distance from an arbitrary operator to a given nest algebra is obtained in terms of projections in the nest. This was extended by several authors [3,4,5,6]. The basic question remains: for which reflexive algebras A (if not for all) is there a constant K such that for every $T \in L(H)$ we have $\text{dist}(T, A) \leq K \cdot \sup\{\|P^1 T P\| : P \in \text{Lat } A\}$? We show that this question has a natural formulation in terms of the annihilator of A which seems to illuminate the question and raises some related questions. After obtaining this it became known to us that this formulation had been obtained by E. Christensen [4] for the case in which A is a von Neumann algebra.

PROPOSITION 12. Let A be a reflexive algebra. Let $B_1 = \{f \in \text{annih } A : \|f\|_1 \leq 1\}$, and let C_1 denote the $\|\cdot\|_1$ -closed convex hull of the operators in B_1 of rank ≤ 1 . Then there exists a constant K such that for all $T \in L(H)$ we have $\text{dist}(T, A) \leq K \cdot \sup\{\|P^1 T P\| : P \in \text{Lat } A\}$ if and only if C_1 has nonempty relative interior in $\text{annih}(A)$. In this case the smallest constant K for which this estimate holds is $1/R$ where R is the largest radius such that $\{f \in \text{annih } A : \|f\|_1 \leq R\} \subset C_1$.

PROOF. We may assume $A \neq L(H)$. Suppose C_1 has nonempty relative interior E . Then since C_1 is closed and balanced it contains a ball of maximal radius $R > 0$ with center 0 in $\text{annih}(A)$. Let $k = 1/R$. Let $T \in L(H)$ and let $d = \text{dist}(T, A)$. We have $d = \sup\{\|\text{Tr}(Tf)\| : f \in B_1\}$ from the duality, and since $kC_1 \supset B_1$ we have

$$d \leq k \sup\{\|\text{Tr}(Tf)\| : f \in C_1\} = k \sup\{\|\text{Tr}(Tf)\| : f \text{ of rank } \leq 1\}.$$

So by Lemma 1 we have

$$\begin{aligned} d &\leq k \sup\{\|\text{Tr}(T P P^1)\| : P \text{ of rank } \leq 1, \|P\|_1 \leq 1, P \in \text{Lat } A\} \\ &= k \sup\{\|\text{Tr}(P^1 T P)\| : P \text{ of rank } \leq 1, \|P\|_1 \leq 1, P \in \text{Lat } A\} \\ &= k \sup\{\|P^1 T P\| : P \in \text{Lat } A\}, \end{aligned}$$

as desired.

Conversely, suppose a distance estimate holds. The set of constants for which an estimate holds is closed so contains a minimal element K . Let $r=1/K$. Suppose there exists $f \in \text{annih}(A)$ with $\|f\|_1 \leq r$ and $f \in C_1$. By a standard separation theorem there exists an element ϕ in the dual of $\text{annih}(A)$ of norm 1 such that $\text{Re}\phi(q) < \text{Re}\phi(f)$, $g \in C_1$. Since $0 \in C_1$ we must have $\text{Re}\phi(f) > 0$. Since C_1 is invariant under multiplication by scalars of modulus 1 we have that

$$|\phi(g)| < \text{Re}\phi(f) \leq |\phi(f)|, \quad g \in C_1.$$

Since $\text{annih}(A)$ is a closed subspace of the Banach space L , its dual is identified isometrically with $L(H)/A$. If $T \in L(H)$ the norm of its image in this quotient is $\text{dist}(T, A)$. Thus there exists $T \in L(H)$ with $\text{dist}(T, A) = 1$ such that

$$|\text{Tr}(Tg)| < |\text{Tr}(Tf)|, \quad g \in C_1.$$

Choose $P_n \in L = \text{Lat}(A)$ such that $\lim \|P_n^1 T P_n\| \geq r \text{dist}(T, A) = r$, and choose unit vectors $u_n \in P_n^1 H$, $v_n \in P_n H$, such that $(T v_n, u_n) = \|P_n^1 T P_n\|$. Let $g_n = u_n \otimes v_n$. Then $g_n \in \text{annih}(A)$, $\|g_n\|_1 = 1$, so $g_n \in C_1$, and $|\text{Tr}(Tg_n)| = |(T v_n, u_n)| = \|P_n^1 T P_n\|$. Thus $\sup \{|\text{Tr}(Tg)| : g \in C_1\} \geq r$. Hence $|\text{Tr}(Tf)| > r$, a contradiction since $f \in \text{annih}(A)$, $\|f\|_1 \leq r$, and $\text{dist}(T, A) = 1$. So $C_1 \supset \{f \in \text{annih}(A) : \|f\|_1 \leq r\}$. Thus C_1 has nonempty relative interior. The above shows that $K \leq 1/R$ and also $R \geq 1/K$; hence $K = 1/R$. ■

REMARKS. The ideas in the proof of Proposition 12 were implicit in the proofs in [1] and especially in [7] for nest algebras. These showed essentially that every norm 1 trace class operator in the annihilator of a nest algebra could be approximated in norm by convex combinations of rank-1 operators of norm ≤ 1 from the annihilator.

In finite dimensions every reflexive algebra satisfies an Arveson distance estimate. The constant need not be 1. In infinite dimensions, it is known [4,6] that a von Neumann algebra satisfies an Arveson distance estimate if and only if every deri-

vation from it into $L(H)$ is inner. As pointed out in [5], if every member of a class of reflexive algebras closed under countable direct sums satisfies an Arveson distance estimate then there must be a universal constant for the class. No example is known of a reflexive algebra which fails to satisfy a distance estimate. However, the class of algebras for which a distance estimate is known to exist is presently rather limited. E. Christensen has shown that it contains "most" von Neumann algebras. Distance estimates have proven useful when they have been known to exist. From Proposition 12 it seems that problems concerning distance estimates have the appearance of extreme point problems. Rank-1 operators of norm 1 in the unit ball of the annihilator are extreme points, so in the case of a reflexive algebra the unit ball of the annihilator seems rich in extreme points. These are not in general the only ones as pointed out by finite dimensional examples for which the distance constant is not 1. (That the constant need not be 1 for even a von Neumann algebra in finite dimensions was first pointed out by H.D. Choi.)

QUESTION 7. Does the $\|\cdot\|_1$ -closed convex hull of the set of extreme points in the unit ball of the annihilator of a reflexive operator algebra necessarily have nonempty relative interior?

This would be most interesting in the case of a von Neumann algebra or of a CSL algebra. A positive resolution would not necessarily imply an Arveson distance estimate, but if a suitable characterization of the extreme points were obtained other estimates might arise.

We thank F. Gilfeather and A. Hopenwasser for useful conversations concerning the subject matter presented in this note.

REFERENCES

1. Arveson, W.: Interpolation problems in nest algebras, *J. Functional Analysis* 20(1975), 208-233.
2. Brown, S.; Chevreau, B.; Pearcy, C.: Contractions with rich spectrum have invariant subspaces, *J. Operator Theory* 1(1979), 123-136.
3. Christensen, E.: Extensions of derivations, *J. Functional Analysis* 27(1978), 234-247.

4. Christensen, E.: Extensions of derivations. II, preprint.
5. Davidson, K.: Commutative subspace lattices, *Indiana University Math.J.* 27(1978), 479-490.
6. Gilfeather, F.; Larson, D.: Nest-subalgebras of von Neumann algebras: commutants modulo compacts and distance estimates, *J.Operator Theory* 7(1982), 279-302.
7. Lance, E.C.: Cohomology and perturbations of nest algebras, *Proc.London Math.Soc.* (3) 43(1981), 334-356.

D.LARSON
Department of Mathematics and Statistics,
University of Nebraska,
Lincoln, Nebraska 68588,
U.S.A.

Supported in part by a grant from the N.S.F.

Added in proof: Since submission of this article we have learned that C.Apostol and J.Langsam have independently answered Question 1 affirmatively.