

# Wavelet Sets in $\mathbb{R}^n$

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**ABSTRACT.** A congruency theorem is proven for an ordered pair of groups of homeomorphisms of a metric space satisfying an abstract dilation-translation relationship. A corollary is the existence of wavelet sets, and hence of single-function wavelets, for arbitrary expansive matrix dilations on  $L^2(\mathbb{R}^n)$ . Moreover, for any expansive matrix dilation, it is proven that there are sufficiently many wavelet sets to generate the Borel structure of  $\mathbb{R}^n$ .

A dyadic orthonormal (or orthogonal) wavelet is a function  $\psi \in L^2(\mathbb{R})$ , (Lebesgue measure), with the property that the set

$$\{2^{\frac{n}{2}}\psi(2^n t - l): n, l \in \mathbb{Z}\}$$

is an orthonormal basis for  $L^2(\mathbb{R})$  (see [1, 2]). For certain measurable sets,  $E$ , the normalized characteristic function  $\frac{1}{\sqrt{2\pi}}\chi_E$  is the Fourier transform of such a wavelet. There are several characterizations of such sets (see [3] chapt. 4, and independently [5]). In [3] they are called wavelet sets. In [5, 6, 7] they are the support sets of MSF (minimally supported frequency) wavelets.

Dilation factors on  $\mathbb{R}$  other than 2 have been studied in the literature, and analogous wavelet sets corresponding to all dilations  $> 1$  are known to exist ([3], Example 4.5, part 10). Matrix dilations (for real expansive matrices) on  $\mathbb{R}^n$  have also been considered in the literature, usually for a “multi-” notion of wavelet. The translations involved are those along the coordinate axes. The purpose of this article is to prove a general-principle type of result that shows, as a corollary, that analogous wavelet sets exist (and are plentiful) for all such dilations. In particular, “single-function” wavelets always exist. This appears to be new. Theorem 1 seems to belong to the mathematics behind wavelet theory. For this reason we prove it in a more abstract setting than needed for our wavelet results. Essentially, it is a dual-dynamical system congruency principle. The general proof is no more difficult than that for  $\mathbb{R}^n$ .

We point out that the wavelets we obtain, which are analogs of Shannon’s wavelet, need not satisfy the regularity properties often desired (see [8]) in applications.

Let  $X$  be a metric space, and let  $m$  be a  $\sigma$ -finite nonatomic Borel measure on  $X$  for which the measure of every open set is positive and for which bounded sets have finite measure. Let  $\mathcal{T}$  and  $\mathcal{D}$  be countable groups of homeomorphisms of  $X$  which map bounded sets to bounded sets and which are absolutely continuous in the sense that they map  $m$ -null sets to  $m$ -null sets. A countable group  $\mathcal{G}$  of absolutely continuous Borel isomorphisms of  $X$  determines an equivalence relation on the family  $\mathcal{B}$  of Borel sets of  $X$  in a natural way:  $E$  and  $F$  are  $\mathcal{G}$ -congruent (written  $E \sim_{\mathcal{G}} F$ ) if

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there are measurable partitions  $\{E_g: g \in \mathcal{G}\}$  and  $\{F_g: g \in \mathcal{G}\}$  of  $E$  and  $F$ , respectively, such that  $F_g = g(E_g)$  for each  $g \in \mathcal{G}$ , modulo  $m$ -null sets.

If  $r > 0$  and  $y \in X$ , we write  $B_r(y) := \{x \in X: \|x - y\| < r\}$ , and abbreviate  $B_r := B_r(0)$ .

We will say that  $(\mathcal{D}, \mathcal{T})$  is an *abstract dilation-translation pair* if (1) for each bounded set  $E$  and each open set  $F$  there are elements  $\delta \in \mathcal{D}$  and  $\tau \in \mathcal{T}$  such that  $\tau(E) \subseteq \delta(F)$ , and (2) there is a fixed point  $\theta$  for  $\mathcal{D}$  in  $X$  which has the property that if  $N$  is any nhood of  $\theta$  and  $E$  is any bounded set, there is an element  $\delta \in \mathcal{D}$  such that  $\delta(E) \subseteq N$ .

**Theorem 1.**

Let  $X, \mathcal{B}, m, \mathcal{D}, \mathcal{T}$  be as above, with  $(\mathcal{D}, \mathcal{T})$  an abstract dilation-translation pair, and with  $\theta$  the  $\mathcal{D}$ -fixed point as above. Let  $E$  and  $F$  be bounded measurable sets in  $X$  such that  $E$  contains a nhood of  $\theta$ , and  $F$  has nonempty interior and is bounded away from  $\theta$ . Then there is a measurable set  $G \subseteq X$ , contained in  $\bigcup_{\delta \in \mathcal{D}} \delta(F)$ , which is both  $\mathcal{D}$ -congruent to  $F$  and  $\mathcal{T}$ -congruent to  $E$ .

**Proof.** We will use the term “ $\mathcal{D}$ -dilate” to denote the image of a set  $\Omega$  under an element of  $\mathcal{D}$ , and “ $\mathcal{T}$ -translate” for the image of  $\Omega$  under an element of  $\mathcal{T}$ .

We will construct a disjoint family  $\{G_{ij}: i \in \mathbb{N}, j \in \{1, 2\}\}$  of measurable sets whose  $\mathcal{D}$ -dilates form a partition  $\{F_{ij}\}$  of  $F$  and whose  $\mathcal{T}$ -translates form a partition  $\{E_{ij}\}$  of  $E$ , modulo  $m$ -null sets. Then  $G = \bigcup_{i,j} G_{ij}$  will clearly satisfy our requirements. The  $i^{\text{th}}$  induction step will consist of constructing  $G_{i1}$  and  $G_{i2}$ .

Let  $\{\alpha_i\}$  and  $\{\beta_i\}$  be sequences of positive constants decreasing to 0. Let  $N_1 \subset E$  be a ball centered at  $\theta$  with radius  $< \alpha_1$  such that  $m(E \setminus N_1) > 0$ . Let  $E_{11} = E \setminus N_1$ .

Observe that we may choose  $\delta_1 \in \mathcal{D}$ ,  $\tau_1 \in \mathcal{T}$ , so that  $(\delta_1^{-1} \circ \tau_1)(E_{11})$  is a subset of  $F$  whose relative complement in  $F$  has a nonempty interior. This is possible because since the interior of  $F$  is nonempty, there is a  $\delta_1$ -dilate of  $F$  which contains a ball large enough to contain some  $\tau_1$ -translate of  $E$  with ample room left over. Now set  $F_{11} := (\delta_1^{-1} \circ \tau_1)(E_{11})$ . (In this context, clearly we may choose  $\delta_1$  and  $\tau_1$  such that, in addition, the  $\tau_1$ -translate of  $E$  is disjoint from any prescribed bounded set — a fact that will be useful in the second and subsequent steps.)

Let  $G_{11} := \tau_1(E_{11}) = \delta_1(F_{11})$ . Since  $\delta_1$  is a homeomorphism of  $X$  which fixes  $\theta$ ,  $G_{11}$  is bounded away from  $\theta$  since  $F_{11}$  is. Let  $F_{12}$  be a measurable subset of  $F$  of positive measure, disjoint from  $F_{11}$ , such that the difference  $F \setminus (F_{11} \cup F_{12})$  has a nonempty interior and measure  $< \beta_1$ . Choose  $\gamma_1 \in \mathcal{D}$  such that  $\gamma_1(F_{12})$  is contained in  $N_1$  and is disjoint from  $G_{11}$ . Set  $E_{12} := \gamma_1(F_{12})$ , and set  $G_{12} := E_{12}$ . The first step is complete.

For the second step, note that since  $F$  is bounded away from  $\theta$ ,  $N_1 \setminus E_{12}$  contains a ball  $N_2$  centered at  $\theta$  with radius  $< \alpha_2$  such that  $N_1 \setminus (E_{12} \cup N_2)$  has positive measure. Let

$$E_{21} := N_1 \setminus (E_{12} \cup N_2) = E \setminus (E_{11} \cup E_{12} \cup N_2).$$

Choose  $\delta_2 \in \mathcal{D}$ ,  $\tau_2 \in \mathcal{T}$ , using similar reasoning to that used above, such that  $(\delta_2^{-1} \circ \tau_2)(E_{21})$  is a subset of  $F \setminus (F_{11} \cup F_{12})$  whose relative complement in  $F \setminus (F_{11} \cup F_{12})$  has a nonempty interior, and for which  $\tau_2(E_{21})$  is disjoint from  $G_{11}$  and  $G_{12}$ . Let  $F_{21} := (\delta_2^{-1} \circ \tau_2)(E_{21})$ , and let  $G_{21} := \tau_2(E_{21})$ .

Choose a measurable subset  $F_{22} \subset F$  of positive measure disjoint from  $F_{11}$ ,  $F_{12}$ ,  $F_{21}$  such that  $F \setminus (F_{11} \cup F_{12} \cup F_{21} \cup F_{22})$  has a nonempty interior and measure  $< \beta_2$ . Noting that  $G_{11}$ ,  $G_{12}$ ,  $G_{21}$  are bounded away from  $\theta$ , choose  $\gamma_2 \in \mathcal{D}$  such that  $\gamma_2(F_{22})$  is contained in  $N_2$  and is disjoint from  $G_{11}$ ,  $G_{12}$ ,  $G_{21}$ . Set  $E_{22} := \gamma_2(F_{22})$ , and let  $G_{22} := E_{22}$ .

Now proceed inductively, obtaining disjointed families of sets of positive measure  $\{E_{ij}\}$  in  $E$ ,  $\{F_{ij}\}$  in  $F$ , and  $\{G_{ij}\}$ , such that

$$\begin{aligned} \tau_i^{-1}(G_{i1}) &= E_{i1}, G_{i2} = E_{i2}, \delta_i^{-1}(G_{i1}) = F_{i1}, \\ \gamma_i^{-1}(G_{i2}) &= F_{i2}, \text{ for } i = 1, 2, \dots \text{ and } j = 1, 2. \end{aligned}$$

We have  $E \setminus (\cup E_{ij}) = \{\theta\}$ , a null set, since  $\alpha_i \rightarrow 0$ , and  $F \setminus (\cup F_{ij})$  is a null set since  $\beta_i \rightarrow 0$ . Let  $G = \cup G_{ij}$ . Since  $\delta_i, \gamma_i \in \mathcal{D}$  we have  $G_{ij} \in F$  for all  $i, j$ . So  $G \subseteq \bigcup_{\delta \in \mathcal{D}} \delta(F)$ . The proof is complete.  $\square$

**Remark.** Suppose  $K$  is any bounded set that is bounded away from  $\theta$  (i.e.,  $K$  is contained in an annulus centered at  $\theta$ ). Then the set  $G$  in Theorem 1 can be taken *disjoint* from  $K$ . This follows immediately from the way the sets  $G_{ij}$  in the proof are constructed. Moreover, for each  $n$  a disjoint  $n$ -tuple can be constructed, all of which satisfy the properties of  $G$  and  $K$  above. To see this, mimic the proof of Theorem 1, at each step constructing  $G_{ij}^1, \dots, G_{ij}^n$  simultaneously, making sure that they are disjoint from each other and also from all of the previous  $G_{lk}^h$  that have been constructed to that point. This construction can easily be modified to yield an infinite pairwise disjoint family  $\{G^k\}_{k=1}^\infty$ .  $\square$

We will now relate Theorem 1 to wavelets.

Let  $1 \leq m < \infty$ , and let  $A$  be an  $n \times n$  real matrix which is *expansive* (equivalently, all eigenvalues have modulus  $> 1$  (see [9])). By a dilation- $A$  orthonormal wavelet we mean a function  $\psi \in L^2(\mathbb{R}^n)$  such that

$$(*) \quad \{|\det(A)|^{\frac{n}{2}} \psi(A^n t - (l_1, l_2, \dots, l_n)^t) : n, l \in \mathbb{Z}\},$$

where  $t = (t_1, \dots, t_n)^t$ , is an orthonormal basis for  $L^2(\mathbb{R}^n; m)$ . (Here  $m$  is product Lebesgue measure, and the superscript “ $t$ ” means transpose.)

It is useful to introduce dilation and translation unitary operators. If  $A \in M_n(\mathbb{R})$  is invertible (so in particular if  $A$  is expansive), then the operator defined by

$$(D_A f)(t) = |\det A|^{\frac{1}{2}} f(At),$$

$f \in L^2(\mathbb{R}^n)$ ,  $t \in \mathbb{R}^n$ , is unitary. For  $1 \leq i \leq n$ , let  $T_i$  be the unitary operator determined by translation by 1 in the  $i^{\text{th}}$  coordinate direction. The set  $(*)$  is then

$$\{D_A^k T_1^{l_1} \dots T_n^{l_n} \psi : k, l_i \in \mathbb{Z}\}.$$

The term orthogonal wavelet has been extended in the literature to include a “multi” notion, which is an orthonormal  $p$ -tuple  $(f_1, \dots, f_p)$  of functions in  $L^2(\mathbb{R}^n)$ , each of which separately generates an incomplete orthonormal set under the system of unitaries, and which together form an o.n. basis.

Let  $\mathcal{F}$  be the Fourier-Plancherel transform on  $L^2(\mathbb{R})$ , normalized so it is a unitary transformation. For  $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ ,

$$\mathcal{F}(f)(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t) dt$$

and

$$\mathcal{F}^{-1}(g)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ist} g(s) ds.$$

On  $L^2(\mathbb{R}^n)$  the Fourier transform is

$$(\mathcal{F}f)(s) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(s \circ t)} f(t) dm,$$

where  $s \circ t$  denotes the real inner product. Write  $\hat{f} = \mathcal{F}f$ , and for  $A \in B(\mathbb{R}^n)$  write  $\hat{A} := \mathcal{F}A\mathcal{F}^{-1}$ . We have  $\hat{D}_A = D_{(A^t)^{-1}} (= D_{A^t}^{-1} = D_{A^t}^*)$ , where  $A^t$  is the transpose of  $A$ , and  $\hat{T}_j = M_{e^{-is_j}}$ , the multiplication operator on  $\mathbb{R}^n$  with symbol  $f(s_1, \dots, s_n) = e^{-is_j}$ .

By a *dilation- $A$  wavelet set* we will mean a measurable subset of  $\mathbb{R}^n$  (necessarily of finite measure) for which the inverse Fourier transform of  $(m(E))^{-\frac{1}{2}}\chi_E$  is a dilation- $A$  orthonormal wavelet.

We will say that measurable subsets  $H$  and  $K$  of  $\mathbb{R}^n$  are  *$A$ -dilation congruent* if there exist measurable partitions  $\{H_l\}$  of  $H$  and  $\{K_l\}$  of  $K$  such that  $K_l = A^l H_l, l \in \mathbb{Z}$ , modulo Lebesgue null-sets. Write  $H \sim_{\delta_A} K$ . We will also say that  $E, F$  are  *$2\pi$ -translation congruent* (write this  $E \sim_{\tau_{2\pi}} F$ ) if there exist measurable partitions  $\{E_l: l = (l_1, \dots, l_n) \in \mathbb{Z}^n\}$  of  $E$  and  $\{F_l: l \in \mathbb{Z}^n\}$  of  $F$  such that  $F_l = E_l + 2\pi l, l \in \mathbb{Z}^n$ , modulo null sets. If  $W$  is a measurable subset of  $\mathbb{R}^n$  which is  $2\pi$ -translation congruent to the  $n$ -cube  $E = [-\pi, \pi) \times \dots \times [-\pi, \pi)$ , it is clear from the exponential form of  $\widehat{T}_j$  that  $\{\widehat{T}_1^{l_1} \widehat{T}_2^{l_2} \dots \widehat{T}_n^{l_n} \cdot (m(W))^{-\frac{1}{2}}\chi_W: (l_1, \dots, l_n) \in \mathbb{Z}^n\}$  is an o.n. basis for  $L^2(W)$ .

If  $A$  is a strict dilation, so  $\|A^{-1}\| < 1$ , then  $AB_1 \supseteq B_{\|A^{-1}\|^{-1}}$ . It follows that if  $F = AB_1 \setminus B_1$ , then  $\{A^k F: k \in \mathbb{Z}\}$  is a partition of  $\mathbb{R}^n \setminus \{0\}$ . If  $A$  is expansive, then  $A$  is *similar* to a strict dilation. Therefore,  $A = TCT^{-1}$  for  $T$  a real invertible  $n \times n$  matrix, and with  $\|C^{-1}\| < 1$ . If  $F_A = T(CB_1 \setminus B_1)$ , then  $\{A^k F_A\}_{k \in \mathbb{Z}}$  is a partition of  $\mathbb{R}^n \setminus \{0\}$ . Therefore, an expansive matrix has a measurable complete *wandering set*  $F_A \subset \mathbb{R}^n$ . It follows that  $L^2(F_A)$ , considered as a subspace of  $L^2(\mathbb{R}^n)$ , is a complete wandering subspace for  $D_A$ . That is,  $L^2(\mathbb{R}^n)$  is the direct sum decomposition of the subspaces  $\{D_A^k L^2(F_A)\}_{k \in \mathbb{Z}}$ . Moreover, it is clear that any measurable set  $F'$  with  $F' \sim_{\delta_A} F_A$  has this same property.

**Corollary 1.**

Let  $1 \leq n < \infty$  and let  $A \in M_n(\mathbb{R})$  be expansive. There exist dilation- $A$  wavelet sets.

**Proof.** Let  $\mathcal{A}$  be the group of homeomorphisms of  $\mathbb{R}^n$  generated by the map  $x \rightarrow A^l x$ . Let  $\mathcal{T}$  be the group of homeomorphisms of  $\mathbb{R}^n$  generated by the translations in each of the coordinate directions by the integral multiples of  $2\pi$ . Then  $A$ -dilation-congruency means  $\mathcal{A}$ -congruency and  $2\pi$ -translation-congruency means  $\mathcal{T}$ -congruency. Moreover, it is clear that  $(\mathcal{A}, \mathcal{T})$  is an abstract dilation-translation pair on  $\mathbb{R}^n$  in the sense of Theorem 1, with  $\theta = 0$ .

By Theorem 1, a measurable set  $W$  exists with  $W \sim_{\delta_{A^l}} F_{A^l}$  and  $W \sim_{\tau_{2\pi}} E$ , where  $E = [-\pi, \pi)^n$ . As remarked above, the set  $\{\widehat{T}_1^{l_1} \widehat{T}_2^{l_2} \dots \widehat{T}_n^{l_n} \widehat{\psi}_W: l_j \in \mathbb{Z}, 1 \leq j \leq n\}$  is an o.n. basis for  $L^2(W)$ , (with  $\widehat{\psi}_W = (m(W))^{-\frac{1}{2}}\chi_W$ ). Therefore, since  $L^2(W)$ , regarded as a subspace of  $L^2(\mathbb{R}^n)$ , is wandering for  $\widehat{D} := \widehat{D}_A = D_{A^l}^{-1}$ , the set

$$\{\widehat{D}^k \widehat{T}_1^{l_1} \dots \widehat{T}_n^{l_n} \widehat{\psi}: k \in \mathbb{Z}, l_j \in \mathbb{Z}, 1 \leq j \leq n\}$$

is an o.n. basis for  $L^2(\mathbb{R}^n)$ , and  $W$  is a wavelet set for  $A$ . (Moreover, by Remark 5 it follows that there is a countably infinite pairwise disjoint family of such sets.)  $\square$

A *Hardy* dyadic orthonormal wavelet is a function  $\psi \in L^2(\mathbb{R})$  for which  $\{2^{\frac{n}{2}}\psi(2^n t - \ell): n, \ell \in \mathbb{Z}\}$  is an o.n. basis for the Hardy space of  $L^2$ -functions  $f$  whose Fourier transform  $\widehat{f}$  has support contained in  $[0, \infty)$ . An example is  $\widehat{\psi} = (2\pi)^{-\frac{1}{2}}\chi_{[2\pi, 4\pi)}$ . Therefore,  $[2\pi, 4\pi)$  is a Hardy wavelet set. This idea can be generalized.

**Corollary 2.**

Let  $A \in M_n(\mathbb{R})$  be expansive, and let  $M \subseteq \mathbb{R}^n$  be a measurable set of positive measure that is stable under  $A^l$  in the sense that  $A^l M = M$ . Suppose  $M \cap F_{A^l}$  has a nonempty interior. Then there exist measurable sets  $W \subset M$  with the property that, if  $\widehat{\psi}_W := (2\pi)^{-\frac{n}{2}}\chi_W$ , then

$$\{D_A^k T_1^{l_1} \dots T_n^{l_n} \psi_W: k, l_i \in \mathbb{Z}\}$$

is an orthonormal basis for  $\mathcal{F}^{-1}(L^2(M))$ .

(Wavelets of this type were studied in [4] for the dyadic,  $n = 1$  case, where they were called *subspace wavelets*. The concept is that they are wavelets for proper subspaces of  $L^2(\mathbb{R})$ .)

**Proof.** Apply Theorem 1, with  $F = M \cap F_{A^l}$  and  $E = [-\pi, \pi) \times \cdots \times [-\pi, \pi)$ , obtaining  $W$  with  $W \sim_{\tau_{2\pi}} E$  and  $W \sim_{\delta_{A^l}} F$ . Since  $M$  is  $A^l$ -stable,  $W \subset M$ . Also,  $\{(A^l)^k W : k \in \mathbb{Z}\}$  is a measurable partition of  $M$ . Therefore, an argument similar to that before shows that  $\{\widehat{D}_A^k \widehat{T}_1^{l_1} \cdots \widehat{T}_n^{l_n} \widehat{\psi}_W : k, l_i \in \mathbb{Z}\}$  is an o.n. basis for  $L^2(M)$ .  $\square$

The following result points out that the set of wavelet sets for any dilation is large. We will call an orthonormal wavelet for a dilation-factor  $a > 1$ ,  $a \in \mathbb{R}$ , an  $a$ -adic orthonormal wavelet.

**Corollary 3.**

*Let  $A \in M_n(\mathbb{R})$  be expansive. Every measurable subset of  $\mathbb{R}^n$  is a countable union of intersections of pairs of dilation- $A$  wavelet sets. The family of Borel dilation- $A$  wavelet sets generates the Borel structure of  $\mathbb{R}^n$ .*

**Proof.** We first prove the  $a$ -adic case. Let  $a > 1$  be arbitrary. Let  $d(\cdot)$  denote the projection map from  $\mathbb{R} \setminus \{0\}$  onto  $F = [-a, -1) \cup [1, a)$  determined by  $a$ -dilation, and let  $t(\cdot)$  denote the projection map from  $\mathbb{R}$  onto  $E = [-\pi, \pi)$  determined by  $2\pi$ -translation. That is, for  $x \in \mathbb{R} \setminus \{0\}$ ,  $d(x)$  is the unique  $a$ -dilate of  $x$  contained in  $F$ , and for  $x \in \mathbb{R}$ ,  $t(x)$  is the unique  $2\pi$ -translate of  $x$  contained in  $E$ . Note that  $E \sim_{\tau_{2\pi}} [0, 2\pi) \sim_{\tau_{2\pi}} ([-2\pi, \pi) \cup [\pi, 2\pi))$ . Suppose  $K$  is a measurable set in  $\mathbb{R} \setminus \{0\}$  for which the restrictions  $d|_K$  and  $t|_K$  are one-to-one. Let  $E_0 = E \setminus t(K)$  and  $F_0 = F \setminus d(K)$ . If  $E_0$  contains a nhood of 0 and  $F_0$  has a nonempty interior then by Theorem 1 and Remark 5 there are disjoint measurable sets  $G_1, G_2$  with  $G_i \sim_{\tau_{2\pi}} E_0$  and  $G_i \sim_{\delta_a} F_0$ ,  $i = 1, 2$ . (By the construction in the proof of Theorem 1 (and Remark 5) if  $K$  is Borel, then these can be taken Borel.) Let  $W_i = K \cup G_i$ . Then  $W_i \sim_{\tau_{2\pi}} E$  and  $W_i \sim_{\delta_a} F$ . So each  $W_i$  is an  $a$ -adic wavelet set. We have  $K = W_1 \cap W_2$ . We will show that each measurable set  $G \subseteq \mathbb{R}$  has a measurable partition  $\{G_j\}_j$  where each  $G_j$  has the property of  $K$ .

Observe that if  $K$  has the property in the above paragraph, i.e.,  $d(\cdot)$  and  $t(\cdot)$  are 1-1,  $E_0$  contains a nhood of 0 and  $F_0$  has nonempty interior then every subset of  $K$  also has the property.

Suppose  $0 < \alpha < \beta$ , and let  $J = [\alpha, \beta]$ . If  $\beta - \alpha < 2\pi$  then  $t|_J$  is 1-1, and if  $\beta < a\alpha$  then  $d|_J$  is 1-1. If, in addition,  $J$  contains no integral multiple of  $2\pi$ , then  $J$  satisfies the property of  $K$  above. Let  $\mathcal{J}_+$  be the set of all intervals  $[\alpha, \beta]$  with  $0 < \alpha < \beta$ ,  $\beta < \min\{a\alpha, \alpha + 2\pi\}$ ,  $[\alpha, \beta] \cap 2\pi\mathbb{Z} = \emptyset$ ,  $\alpha$  and  $\beta$  rational. Observe that  $\cup\{J : J \in \mathcal{J}_+\} = (0, \infty) \setminus 2\pi\mathbb{Z}$ . Let  $\mathcal{J}_- = \{-\beta, -\alpha\} : [\alpha, \beta] \in \mathcal{J}_+\}$ , and  $\mathcal{J} = \mathcal{J}_+ \cup \mathcal{J}_-$ . Then  $\bigcup_{J \in \mathcal{J}} J = \mathbb{R} \setminus 2\pi\mathbb{Z}$ . Let  $J_1, J_2, \dots$  be an enumeration of  $\mathcal{J}$ , and let  $L_1 = J_1$ , and

$$L_{j+1} = J_{j+1} \setminus (J_1 \cup \cdots \cup J_j) \quad \text{for } j \geq 1.$$

Then  $\{L_j : j \in \mathbb{N}\}$  is a measurable partition of  $\mathbb{R} \setminus 2\pi\mathbb{Z}$ .

Let  $G \subseteq \mathbb{R}$  be a measurable set. Clearly we may assume  $G \cap 2\pi\mathbb{Z} = \emptyset$ . Let  $G_j = G \cap L_j$ . Then  $\{G_j\}$  is a measurable partition of  $G$  satisfying our requirements. If  $G$  is Borel, then each  $G_j$  is Borel.

We adapt the previous proof to the general case. Replace  $F$  with  $F_{A^l}$ ,  $E$  with the  $n$ -cube  $[-\pi, \pi) \times \cdots \times [-\pi, \pi)$ , and  $d(\cdot)$  and  $t(\cdot)$  with the corresponding projections from  $\mathbb{R}^n \setminus \{0\}$  to  $F_{A^l}$  and from  $\mathbb{R}^n$  to  $E$ , respectively. If  $K \subset \mathbb{R}^n$  has the property in paragraph one relative to these, the same argument shows that  $K$  is the intersection of two dilation- $A$  wavelet sets. The boundary  $\partial C$  of the  $n$ -cube  $C = [0, 2\pi) \times \cdots \times [0, 2\pi)$  is an  $m$ -null set. Let  $Q = \cup\{(\partial C) + 2\pi\ell : \ell \in \mathbb{Z}^{(n)}\}$ . By construction  $\partial F_{A^l}$  is also an  $m$ -null set. If  $J = B_r(y)$  is a ball in  $\mathbb{R}^n$  contained in one of the annuli  $(A^l)^l F_{A^l}$  and which is also bounded away from  $Q$ , then  $J$  satisfies the property of  $K$ . Let  $\mathcal{J}$  be the set of all such balls that have rational center and radius. Enumerate  $\mathcal{J}$ , define  $L_j$  as above, and observe that  $\{L_j : j \in \mathbb{N}\}$  is a partition of  $\mathbb{R}^n$  modulo a null set. As above, if  $G \subseteq \mathbb{R}^n$  is a measurable, the partition  $\{G \cap L_j : j \in \mathbb{N}\}$  satisfies our requirements.  $\square$

**Remark.** Theorem 1 can be improved in several further ways.

- 1. It is not necessary that  $m$  be nonatomic in Theorem 1. All that is needed is that  $\{\theta\}$  is not an atom for  $m$ .
- 2. The hypothesis that  $E$  contains a nhood of  $\theta$  in Theorem 1 can be replaced with the hypothesis that for each  $\epsilon > 0$  there exists  $\delta \in \mathcal{D}$  such that  $\delta(F) \subseteq E \cap B_\epsilon(\theta)$ . If we let  $\tilde{F} = \cup\{\delta(F) : \delta \in \mathcal{D}\} \cup \{\theta\}$ , then this is equivalent to the requirement that  $E$  contain a subset of  $\tilde{F}$  which is a nhood of  $\theta$  in the relative topology of  $\tilde{F}$  in  $X$ . Remark 5 generalizes as well.
- 3. Theorem 1 remains true, in the general form of Remark 5 and 1, 2 above, if we drop the hypotheses that  $E$  and  $F$  are bounded and  $F$  is bounded away from  $\theta$ . To adapt the proof, write  $E = \bigcup_{i=0}^{\infty} E_i$ ,  $F = \bigcup_{i=0}^{\infty} F_i$ ,  $\{E_i\}$ ,  $\{F_i\}$  disjoint, bounded,  $F_i$  bounded away from  $\theta$ , and such that  $E_0$  and  $F_0$  play the role of  $E$ ,  $F$  in the proof of Theorem 1; so  $E_0$  contains a nhood of  $\theta$  and  $F_0$  has a nonempty interior. Then, for  $k \geq 1$ , in the  $k^{\text{th}}$  induction step (in which  $G_{k1}$  and  $G_{k2}$  are constructed), replace  $E$  with  $E_0 \cup E_1 \cup \dots \cup E_k$  and  $F$  with  $F_0 \cup F_1 \cup \dots \cup F_k$ . The proof, thus modified, is easily seen to be valid.

□

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