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HYPERREFLEXIVITY AND A DUAL PRODUCT CONSTRUCTION¹

BY

DAVID R. LARSON

ABSTRACT. We show that an example of a nonhyperreflexive CSL algebra recently constructed by Davidson and Power is a special case of a general and natural reflexive subspace construction. Completely different techniques of proof are needed because of absence of symmetry. It is proven that if \mathcal{S} and \mathcal{T} are reflexive proper linear subspaces of operators acting on a separable Hilbert space, then the hyperreflexivity constant of $(\mathcal{S}_\perp \otimes \mathcal{T}_\perp)^\perp$ is at least as great as the product of the constants of \mathcal{S} and \mathcal{T} .

This paper was inspired by the interesting “key example” in the recent paper [2] by Davidson and Power in which a nonhyperreflexive CSL algebra was constructed. In an attempt to completely understand this result we obtained a distance constant inequality of a more general nature, which we present here.

Let H, K be separable Hilbert spaces—finite or infinite dimensional—and let \mathcal{S}, \mathcal{T} be linear subspaces of $L(H), L(K)$, which are reflexive in the Loginov-Shulman sense. (\mathcal{S} is reflexive iff whenever $T \in L(H)$ is such that $Tx \in [Sx]$, $x \in H$, then $T \in \mathcal{S}$, where $[\cdot]$ means closure.) Let $\mathcal{K}(\mathcal{S}), \mathcal{K}(\mathcal{T})$ be the constants of hyperreflexivity of \mathcal{S} and \mathcal{T} as defined in [4]. We recall that a subspace \mathcal{S} of $L(H)$ is hyperreflexive if there is a constant C such that for operators T in $L(H)$,

$$d(T, \mathcal{S}) \leq C \sup \{ \|P^\perp TQ\| : P, Q \text{ are projections with } P^\perp \mathcal{S}Q = 0 \},$$

and the optimal constant is denoted $\mathcal{K}(\mathcal{S})$. If \mathcal{S} is reflexive but not hyperreflexive, then we define $\mathcal{K}(\mathcal{S}) = +\infty$. We make use of preannihilator techniques, and refer the reader to [1, 4, 5, 7] for details. As shown in [4], the reflexive subspaces of $L(H)$ are precisely those for which the preannihilator in $\mathcal{L}_* \equiv C_1$ is the $\|\cdot\|_1$ -closed linear span of rank ≤ 1 operators, where $\|\cdot\|_1$ denotes trace-class norm. Since $\mathcal{S}_\perp, \mathcal{T}_\perp$ are generated by rank ≤ 1 operators, so is the tensor product of preannihilators $\mathcal{S}_\perp \otimes \mathcal{T}_\perp$. By this we mean the $\|\cdot\|_1$ -closed linear subspace of the ideal of trace-class operators on $L(H \otimes K)$ generated by the elementary tensors $\{f \otimes g : f \in \mathcal{S}_\perp, g \in \mathcal{T}_\perp\}$, where $H \otimes K$ denotes the usual tensor product Hilbert space. (When we write $\mathcal{S} \otimes \mathcal{T}$, we will mean the σ -weakly closed linear subspace of $L(H \otimes K)$ generated by $\{S \otimes T : S \in \mathcal{S}, T \in \mathcal{T}\}$.) Thus the annihilator

$$(\mathcal{S}_\perp \otimes \mathcal{T}_\perp)^\perp = \{A \in L(H \otimes K) : \text{Tr}(Ah) = 0, h \in \mathcal{S}_\perp \otimes \mathcal{T}_\perp\}$$

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is a reflexive subspace of $L(H \otimes K)$. We will call this the *dual* product of \mathcal{S} and \mathcal{T} since it is in a sense dual to the usual tensor product, and will adopt the notation $\mathcal{S} * \mathcal{T} = (\mathcal{S}_\perp \otimes \mathcal{T}_\perp)^\perp$. We extend this term, and notation, to arbitrary σ -weakly closed subspaces.

For reflexive \mathcal{S} and \mathcal{T} , Theorem 8 states that $\mathcal{S} * \mathcal{T}$ is the smallest reflexive subspace containing $\mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$. In special cases (and perhaps in general) this coincides with the σ -weak closure of $\mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$. This is the case in finite dimensions (Proposition 1).

It is clear that for subspaces \mathcal{S}_i we have $(\mathcal{S}_1 * \mathcal{S}_2) * \mathcal{S}_3 = \mathcal{S}_1 * (\mathcal{S}_2 * \mathcal{S}_3)$ (or rather, equivalence), so we may drop parentheses with no ambiguity. An n -fold dual product $\mathcal{S}_1 * \cdots * \mathcal{S}_n$ will have the n -fold tensor product $(\mathcal{S}_1)_\perp \otimes \cdots \otimes (\mathcal{S}_n)_\perp$ as preannihilator.

The main result of this paper, Theorem 9, states that the inequality $\mathcal{K}(\mathcal{S} * \mathcal{T}) \geq \mathcal{K}(\mathcal{S}) \cdot \mathcal{K}(\mathcal{T})$ always holds for reflexive *proper* subspaces \mathcal{S} , \mathcal{T} . (If either $\mathcal{S} = L(H)$ or $\mathcal{T} = L(K)$, then $\mathcal{S} * \mathcal{T} = L(H \otimes K)$, so the inequality need not hold. These are the only exceptions.)

A special case arises when \mathcal{D} is the algebra of 3×3 diagonal operators acting on a 3-dimensional Hilbert space. Then, since it is known (M. D. Choi, unpublished) that $\mathcal{K}(\mathcal{D}) \geq \sqrt{9/8}$, the n -fold dual product $\mathcal{D} * \cdots * \mathcal{D}$ has constant $\geq (9/8)^{n/2}$. This is seen to be the Davidson-Power example. The subspace $\mathcal{D} * \cdots * \mathcal{D}$ is a bimodule over the n -fold tensor product $\mathcal{D} \otimes \cdots \otimes \mathcal{D}$, a m.a.s.a.; hence

$$\begin{pmatrix} \mathcal{D} \otimes \cdots \otimes \mathcal{D} & \mathcal{D} * \cdots * \mathcal{D} \\ 0 & \mathcal{D} \otimes \cdots \otimes \mathcal{D} \end{pmatrix}$$

is a CSL algebra. It can be shown directly, as in [2], or via duality, as in Theorem 12, that the hyperreflexivity constant for this algebra is at least as great as that of $\mathcal{D} * \cdots * \mathcal{D}$. Theorem 9 can be viewed as a generalization of the induction step in the Davidson-Power construction. Since averaging techniques utilizing symmetry do not apply, proofs are necessarily different. Prior to their example, inequalities of this nature were not suspected.

We note that while \mathcal{D} is an algebra, $\mathcal{D} * \mathcal{D}$ is not. Hence, analysis of multi-dual-products such as $\mathcal{D} * \cdots * \mathcal{D}$ requires reflexive subspace theory. Also, we note that Propositions 1 and 3 are not used in the proofs of our main results, but are given for perspective on these.

The next lemma will be used repeatedly.

LEMMA 0. *Let $\mathcal{S} \subseteq L(H)$, $\mathcal{T} \subseteq L(K)$ be linear subspaces. Then $\mathcal{S} * \mathcal{T} \supseteq \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$.*

PROOF. If $f \in \mathcal{S}_\perp$, $g \in \mathcal{T}_\perp$, then for each $S \in \mathcal{S}$, $T \in \mathcal{T}$, $A \in L(H)$, $B \in L(K)$ we have

$$\begin{aligned} \text{Tr}[(S \otimes B + A \otimes T)(f \otimes g)] &= \text{Tr}[(sf) \otimes (Bg)] + \text{Tr}[(Af) \otimes (Tg)] \\ &= \text{Tr}(Sf) \cdot \text{Tr}(Bg) + \text{Tr}(Af) \cdot \text{Tr}(Tg) = 0 \end{aligned}$$

since $\text{Tr}(Sf) = 0$ and $\text{Tr}(Tg) = 0$. Since the operators $S \otimes B + A \otimes T$ generate $\mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$, and the operators $f \otimes g$ generate $\mathcal{S}_\perp \otimes \mathcal{T}_\perp = (\mathcal{S} * \mathcal{T})_\perp$, we conclude that $(\mathcal{S} * \mathcal{T})_\perp \subseteq (\mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T})_\perp$, and hence that $\mathcal{S} \otimes \mathcal{T} \supseteq \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$. \square

PROPOSITION 1. *Let H, K be finite dimensional Hilbert spaces, and let $\mathcal{S} \subseteq L(H)$, $\mathcal{T} \subseteq L(K)$ be linear subspaces. Then $\mathcal{S} * \mathcal{T} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$.*

PROOF. Let n_1, n_2 be the dimensions of H, K , respectively, and let m_1, m_2 be the vector space dimensions of \mathcal{S}, \mathcal{T} , respectively. Then $\dim(L(H)) = n_1^2$, $\dim(L(K)) = n_2^2$, $\dim(\mathcal{S}_\perp) = n_1^2 - m_1$ and $\dim(\mathcal{T}_\perp) = n_2^2 - m_2$. So $\dim(\mathcal{S}_\perp \otimes \mathcal{T}_\perp) = (n_1^2 - m_1)(n_2^2 - m_2)$, and thus

$$\dim(\mathcal{S} \otimes \mathcal{T}) = n_1^2 n_2^2 - (n_1^2 - m_1)(n_2^2 - m_2) = n_1^2 m_2 + m_1 n_2^2 - m_1 m_2.$$

If X, Y are finite dimensional vector spaces over \mathbf{C} , and if $X_1 \subseteq X, Y_1 \subseteq Y$ are linear subspaces, then $(X_1 \otimes Y) \cap (X \otimes Y_1) = X_1 \otimes Y_1$. Thus $(\mathcal{S} \otimes L(K)) \cap (L(H) \otimes \mathcal{T}) = \mathcal{S} \otimes \mathcal{T}$. So

$$\begin{aligned} \dim(\mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}) &= \dim(\mathcal{S} \otimes L(K)) + \dim(L(H) \otimes \mathcal{T}) - \dim(\mathcal{S} \otimes \mathcal{T}) \\ &= m_1 n_2^2 + n_1^2 m_2 - m_1 m_2 = \dim(\mathcal{S} * \mathcal{T}). \end{aligned}$$

So by Lemma 0 we must have $\mathcal{S} * \mathcal{T} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$. \square

If $\mathcal{A} \subseteq L(H)$ is a reflexive algebra and $T \in L(H)$, the Arveson estimate for the distance from T to \mathcal{A} is $\alpha(T, \mathcal{A}) = \sup\{\|P^\perp T P\| : P \in \text{lat } \mathcal{A}\}$. For reflexive subspaces $\mathcal{S} \subseteq L(H)$ the estimate is defined [4] by $\alpha(T, \mathcal{S}) = \sup\{\|P^\perp T Q\| : P, Q \text{ are projections with } P^\perp \mathcal{S} Q = 0\}$. This agrees with the “ PTP^\perp ” formula when $I \in \mathcal{S}$. There is a “projection free” characterization of this estimate which proves useful. Let $d(\cdot, \cdot)$ denote distance.

LEMMA 2. *Let $\mathcal{S} \subseteq L(H)$ be a reflexive subspace. Then $\alpha(T, \mathcal{S}) = \sup\{d(Tx, \mathcal{S}x) : x \in H, \|x\| = 1\}$, $T \in L(H)$.*

PROOF. We have $d(Tx, \mathcal{S}x) = \|P^\perp Tx\|$, where P is the orthogonal projection onto $[\mathcal{S}x]$. Let $\|x\| = 1$ and let Q be the projection onto $\mathbf{C}x$. Then $\|P^\perp Tx\| = \|P^\perp TQ\|$. Since $P^\perp \mathcal{S} Q = 0$, the inequality “ \geq ” follows.

Conversely, suppose P, Q are projections with $P^\perp \mathcal{S} Q = 0$. Let $\varepsilon > 0$ be given, and choose $x \in QH, \|x\| = 1$, such that $\|P^\perp Tx\| \geq \|P^\perp TQ\| - \varepsilon$. Then

$$d(Tx, \mathcal{S}x) \geq d(Tx, PH) = \|P^\perp Tx\| \geq \|P^\perp TQ\| - \varepsilon.$$

Since ε was arbitrary, we have $\|P^\perp TQ\| \leq \sup\{d(T, \mathcal{S}x) : x \in H, \|x\| = 1\}$. Taking the supremum over all pairs $\{P, Q\}$ with $P^\perp \mathcal{S} Q = 0$ completes the proof. \square

Lemma 2 points out that only cyclic projections P need be considered in distance estimate computations. Also, taking this as the definition permits natural extension of the concept to general normed linear spaces.

If $\mathcal{S} \neq L(H)$ is reflexive and $T \notin \mathcal{S}$, we write $\mathcal{K}(T, \mathcal{S}) = d(T, \mathcal{S})/\alpha(T, \mathcal{S})$. So $\mathcal{K}(\mathcal{S}) = \sup\{\mathcal{K}(T, \mathcal{S}) : T \in L(H), T \notin \mathcal{S}\}$. By convention $\mathcal{K}(L(H)) = 1$. \mathcal{S} is hyperreflexive if $\mathcal{K}(\mathcal{S}) < \infty$, and is nonhyperreflexive otherwise.

We first give an initial generalization of the Davidson-Power induction step in which use is made of symmetry. The proof is more direct than that of our general result, so is included for perspective.

PROPOSITION 3. Let \mathcal{S} be a reflexive subspace of $L(H)$, with $\mathcal{S} \neq L(H)$. Let

$$\tilde{\mathcal{F}} = \begin{pmatrix} * & \mathcal{S} & \mathcal{S} \\ \mathcal{S} & * & \mathcal{S} \\ \mathcal{S} & \mathcal{S} & * \end{pmatrix}$$

be the subspace of all 3×3 operator matrices with diagonal elements arbitrary and off-diagonal elements in \mathcal{S} . Then $\mathcal{K}(\tilde{\mathcal{F}}) \geq \sqrt{9/8} \cdot \mathcal{K}(\mathcal{S})$.

PROOF. First, suppose $\mathcal{K}(\mathcal{S})$ is finite. Fix $\varepsilon > 0$. Choose $T \in L(H)$ for which $\mathcal{K}(T, \mathcal{S}) \geq \mathcal{K}(\mathcal{S}) - \varepsilon$. Let

$$\tilde{T} = \begin{pmatrix} T & T & T \\ T & T & T \\ T & T & T \end{pmatrix}.$$

The averaging technique used in the proof of Theorem 1.1 in [2] yields without modification that $d(\tilde{T}, \tilde{\mathcal{F}}) = \frac{3}{2} \cdot d(T, \mathcal{S})$. We need only show that $\alpha(\tilde{T}, \tilde{\mathcal{F}}) \leq \sqrt{2} \alpha(T, \mathcal{S})$, for then

$$\mathcal{K}(\tilde{\mathcal{F}}) \geq \mathcal{K}(\tilde{T}, \tilde{\mathcal{F}}) \geq \sqrt{9/8} \mathcal{K}(T, \mathcal{S}) > \sqrt{9/8} (\mathcal{K}(\mathcal{S}) - \varepsilon),$$

and since ε was arbitrary the desired inequality would follow.

To show that $\alpha(\tilde{T}, \tilde{\mathcal{F}}) \leq \sqrt{2} \alpha(T, \mathcal{S})$, it is useful to use Lemma 2. Let

$$\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

be an arbitrary unit vector in $H \otimes H_3$. Then

$$\tilde{T}\tilde{x} = \begin{pmatrix} Tz \\ Tz \\ Tz \end{pmatrix},$$

where $z = x_1 + x_2 + x_3$. Descriptively, we have

$$\tilde{\mathcal{F}}\tilde{x} = \begin{pmatrix} * & \mathcal{S} & \mathcal{S} \\ \mathcal{S} & * & \mathcal{S} \\ \mathcal{S} & \mathcal{S} & * \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} L(H)x_1 + \mathcal{S}x_2 + \mathcal{S}x_3 \\ \mathcal{S}x_1 + L(H)x_2 + \mathcal{S}x_3 \\ \mathcal{S}x_1 + \mathcal{S}x_2 + L(H)x_3 \end{pmatrix}.$$

We consider three cases:

(1) If neither x_1 , x_2 nor $x_3 = 0$, then $\tilde{\mathcal{F}}\tilde{x} = H \otimes H_3$, so $d(\tilde{T}\tilde{x}, \tilde{\mathcal{F}}\tilde{x}) = 0$.

(2) If precisely one of x_1, x_2, x_3 is 0, then without loss of generality we may assume $x_1 = 0$ by noting that $\tilde{\mathcal{F}}$ is invariant under the group of unitary transformations corresponding to permutation of basis vectors in H_3 . We have $z = x_2 + x_3$, and

$$\tilde{\mathcal{F}}\tilde{x} = \begin{pmatrix} \mathcal{S}x_2 + \mathcal{S}x_3 \\ L(H) \\ L(H) \end{pmatrix},$$

so

$$d(\tilde{T}\tilde{x}, \tilde{\mathcal{F}}\tilde{x}) = d(Tz, \mathcal{S}x_2 + \mathcal{S}x_3) \leq d(Tz, \mathcal{S}z).$$

We have $\|z\| \leq \sqrt{2}$. If $z = 0$, then $d(\tilde{T}\tilde{x}, \tilde{\mathcal{S}}\tilde{x}) = 0$. If $z \neq 0$, let $w = z/\|z\|$. Then

$$d(\tilde{T}\tilde{x}, \tilde{\mathcal{S}}\tilde{x}) \leq \sqrt{2} d(Tw, \mathcal{S}w) \leq \sqrt{2} \alpha(T, \mathcal{S}),$$

as desired.

(3) If precisely two of x_1, x_2, x_3 are 0, via permutation as above, we may assume $x_1 = x_2 = 0$. Then $z = x_3$, so $\|z\| = 1$. We have

$$\tilde{\mathcal{S}}\tilde{x} = \begin{pmatrix} \mathcal{S}z \\ \mathcal{S}z \\ L(H) \end{pmatrix},$$

so

$$d(\tilde{T}\tilde{x}, \tilde{\mathcal{S}}\tilde{x}) = \sqrt{2} d(Tz, \mathcal{S}z) \leq \sqrt{2} \alpha(T, \mathcal{S}).$$

Now from cases (1)–(3) we have

$$\begin{aligned} \alpha(\tilde{T}, \tilde{\mathcal{S}}) &= \sup \{ d(\tilde{T}\tilde{x}, \tilde{\mathcal{S}}\tilde{x}) : \tilde{x} \in H \otimes H_3, \|\tilde{x}\| = 1 \} \\ &\leq \sqrt{2} \alpha(T, \mathcal{S}), \end{aligned}$$

as required. For the case $\mathcal{K}(\mathcal{S}) = \infty$, let $n \geq 1$ be arbitrary and choose T with $\mathcal{K}(T, \mathcal{S}) \geq n$. The same argument as above shows that $\mathcal{K}(\tilde{T}, \tilde{\mathcal{S}}) \geq \sqrt{9/8}n$. Hence $\mathcal{K}(\tilde{\mathcal{S}}) = +\infty$. \square

A simple duality computation shows that the preannihilator of $\tilde{\mathcal{S}}$ in Proposition 3 has the form

$$\tilde{\mathcal{S}}_{\perp} = \begin{pmatrix} 0 & \mathcal{S}_{\perp} & \mathcal{S}_{\perp} \\ \mathcal{S}_{\perp} & 0 & \mathcal{S}_{\perp} \\ \mathcal{S}_{\perp} & \mathcal{S}_{\perp} & 0 \end{pmatrix},$$

where \mathcal{S}_{\perp} is the preannihilator of \mathcal{S} . The preannihilator of \mathcal{D}_3 has the form

$$\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix},$$

so $\tilde{\mathcal{S}}_{\perp} = \mathcal{S}_{\perp} \otimes (\mathcal{D}_3)_{\perp}$, and hence $\tilde{\mathcal{S}} = (\mathcal{S}_{\perp} \otimes (\mathcal{D}_3)_{\perp})^{\perp} = \mathcal{S} * \mathcal{D}_3$. This suggests that a generalization is possible. The next proposition is used in place of an averaging technique.

PROPOSITION 4. *Let $\mathcal{S} \subseteq L(H)$, $\mathcal{T} \subseteq L(K)$ be σ -weakly closed subspaces. If $A \in L(H)$ and $B \in L(K)$ are arbitrary, then $d(A \otimes B, \mathcal{S} * \mathcal{T}) = d(A, \mathcal{S}) \cdot d(B, \mathcal{T})$.*

PROOF. Let $\mathcal{R} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$. By Lemma 0 we have $\mathcal{R} \subseteq \mathcal{S} * \mathcal{T}$, so for each $S \in \mathcal{S}$ and $T \in \mathcal{T}$ we have $(A - S) \otimes (B - T) - A \otimes B \in \mathcal{R} \subseteq \mathcal{S} * \mathcal{T}$. Thus

$$d(A \otimes B, \mathcal{S} * \mathcal{T}) = d((A - S) \otimes (B - T), \mathcal{S} * \mathcal{T}) \leq \|A - S\| \cdot \|B - T\|.$$

It follows that $d(A \otimes B, \mathcal{S} * \mathcal{T}) \leq d(A, \mathcal{S}) \cdot d(B, \mathcal{T})$.

For the reverse inequality, let $\varepsilon > 0$ be given and choose $f \in \mathcal{S}_\perp$, $g \in \mathcal{T}_\perp$ with $\|f\|_1 = \|g\|_1 = 1$, such that $\text{Tr}(Af) > d(A, \mathcal{S}) - \varepsilon$, and $\text{Tr}(Bg) > d(B, \mathcal{T}) - \varepsilon$. Let $h = f \otimes g$. We have

$$\begin{aligned} |\text{Tr}[(A \otimes B)h]| &= |\text{Tr}(Af \otimes Bg)| = |\text{Tr}(Af)| \cdot |\text{Tr}(Bg)| \\ &> (d(A, \mathcal{S}) - \varepsilon) \cdot (d(B, \mathcal{T}) - \varepsilon). \end{aligned}$$

Since h is a norm -1 operator in $\mathcal{S}_\perp \otimes \mathcal{T}_\perp$, and this is the preannihilator of $\mathcal{S} * \mathcal{T}$ by definition, this implies that $d(A \otimes B, \mathcal{S} * \mathcal{T}) > (d(A, \mathcal{S}) - \varepsilon) \cdot (d(B, \mathcal{T}) - \varepsilon)$. Since ε was arbitrary, the proof is complete. \square

LEMMA 5. *Let $\mathcal{S} \subseteq L(H)$ be a linear subspace, and let x be a vector in $H \otimes K$. Let F be the smallest projection in $L(H)$ such that $(F \otimes I)x = x$. Let P be the orthogonal projection onto $[\mathcal{S}FH]$. Then $P \otimes I$ is the orthogonal projection onto $[(\mathcal{S} \otimes L(K))x]$.*

PROOF. Let $\{e_1, e_2, \dots\}$ be any orthonormal basis for K . Then there is a sequence $\{x_i\}$ of vectors in H with $\sum \|x_i\|^2 = \|x\|^2$ such that $x = \sum x_i \otimes e_i$. Let E_i be the projection onto $\mathbb{C}e_i$. If $S \in \mathcal{S}$, $T \in L(K)$, then $(S \otimes TE_i)x = Sx_i \otimes Te_i$. Hence $[(\mathcal{S} \otimes L(K))x]$ contains all vectors of the form $Sx_i \otimes y$ for arbitrary $S \in \mathcal{S}$, $y \in K$, for each i . Let F' be the projection onto the closed span of vectors $\{x_i; i = 1, 2, \dots\}$. Then $F' \geq F$, and we have

$$\begin{aligned} [(\mathcal{S} \otimes L(K))x] &\supseteq [\mathcal{S}F'H] \otimes K \supseteq [\mathcal{S}FH] \otimes K \\ &= [(\mathcal{S} \otimes L(K))(FH \otimes K)] \supseteq [(\mathcal{S} \otimes L(K))x], \end{aligned}$$

so

$$[(\mathcal{S} \otimes L(K))x] = [\mathcal{S}FH] \otimes K = (P \otimes I)(H \otimes K). \quad \square$$

LEMMA 6. *Let H be a Hilbert space, let $\mathcal{S} \subseteq L(H)$ be a linear subspace, and let $h \in \mathcal{S}_\perp$ be a rank-1 operator. Then $hP = 0$, where P is the orthogonal projection onto $[\mathcal{S}hH]$.*

PROOF. Write $h = v \otimes u$, where u, v are vectors such that $hw = (w, v)u$, $w \in H$. Then $[\mathcal{S}hH] = [\mathcal{S}u]$. If $S \in \mathcal{S}$ we have $0 = \text{Tr}(Sh) = (Su, v)$, so $[\mathcal{S}u] \perp v$; hence $Pv = 0$. Then $hP = (Pv) \otimes u = 0$. \square

LEMMA 7. *Let $\mathcal{S} \subseteq L(H)$, $\mathcal{T} \subseteq L(K)$ be linear subspaces, and let $\mathcal{R} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$. Let $x \in H \otimes K$. Let $F \in L(H)$, $E \in L(K)$ be the smallest projections such that $F \otimes I$ and $I \otimes E$ contain x in their range, and let P be the projection onto $[\mathcal{S}FH]$ and Q the projection onto $[\mathcal{T}EK]$. The projection onto $[\mathcal{R}x]$ is then $(P^\perp \otimes Q^\perp)^\perp$.*

PROOF. We have $[\mathcal{R}x] = [(\mathcal{S} \otimes L(K))x] \vee [(L(H) \otimes \mathcal{T})x]$. By Lemma 5, the projections onto $[(\mathcal{S} \otimes L(K))x]$ and $[(L(H) \otimes \mathcal{T})x]$ are $P \otimes I$ and $I \otimes Q$, respectively. The projection onto $[\mathcal{R}x]$ is then $(P \otimes I) \vee (I \otimes Q)$, and since $P \otimes I$ and $I \otimes Q$ commute this reduces to $P \otimes I + I \otimes Q - P \otimes Q$. The orthogonal complement is then

$$I \otimes I - P \otimes I - I \otimes Q + P \otimes Q = P^\perp \otimes I - P^\perp \otimes Q = P^\perp \otimes Q^\perp,$$

so $\text{proj}[\mathcal{R}x] = (P^\perp \otimes Q^\perp)^\perp$. \square

If \mathcal{S} is a linear subspace of $L(H)$, we adopt the notation $\text{ref}(\mathcal{S})$ to mean the smallest reflexive subspace of $L(H)$ containing \mathcal{S} . Thus $\text{ref}(\mathcal{S}) = \{T \in L(H): Tx \in [\mathcal{S}x], x \in H\}$.

THEOREM 8. *Let $\mathcal{S} \subseteq L(H)$, $\mathcal{T} \subseteq L(K)$ be reflexive subspaces. Then $\mathcal{S} * \mathcal{T}$ is the smallest reflexive subspace containing $\mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$.*

PROOF. Let $\mathcal{R} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$, and let $\hat{\mathcal{R}} = \text{ref}(\mathcal{R})$. By definition, \mathcal{R} and $\hat{\mathcal{R}}$ have the same closed cyclic subspaces. Since $\mathcal{S} * \mathcal{T}$ is reflexive and contains \mathcal{R} it contains $\hat{\mathcal{R}}$. To show equality, it will suffice to show that every rank-1 operator in $\hat{\mathcal{R}}_{\perp}$ is in $(\mathcal{S} * \mathcal{T})_{\perp} = \mathcal{S}_{\perp} \otimes \mathcal{T}_{\perp}$. Let h be a rank-1 operator in $\hat{\mathcal{R}}_{\perp}$, and let x be a nonzero vector in the range of h . By Lemma 6, $h = hG^{\perp}$, where G is the projection onto $[\hat{\mathcal{R}}x] = [\mathcal{R}x]$. Let $F \in L(H)$, $E \in L(K)$ be the smallest projections such that $(F \otimes I)x = x = (I \otimes E)x$, and let $P = \text{proj}[\mathcal{S}FH]$, $Q = \text{proj}[\mathcal{T}EK]$. Then by Lemma 7, $G^{\perp} = P^{\perp} \otimes Q^{\perp}$. We have $(F \otimes E)x = x$; hence

$$h = (F \otimes E)h = (F \otimes E)h(P^{\perp} \otimes Q^{\perp}) \in (F \otimes E)(\mathcal{L}_{*}(H \otimes K))(P^{\perp} \otimes Q^{\perp}),$$

where $\mathcal{L}_{*}(H \otimes K)$ denotes the ideal of trace-class operators on $H \otimes K$. Since $\mathcal{L}_{*}(H \otimes K)$ is the trace-class norm closed span of elementary tensors $\{f \otimes g: f \in \mathcal{L}_{*}(H), g \in \mathcal{L}_{*}(K)\}$, the space $(F \otimes E)(\mathcal{L}_{*}(H \otimes K))(P^{\perp} \otimes Q^{\perp})$ is the closed span of elementary tensors $\{(FfP^{\perp}) \otimes (EgQ^{\perp}): f \in \mathcal{L}_{*}(H), g \in \mathcal{L}_{*}(K)\}$.

But for f arbitrary and $S \in \mathcal{S}$ we have $\text{Tr}(SFfP^{\perp}) = \text{Tr}(P^{\perp}SFf) = 0$ since $P^{\perp}\mathcal{S}F = 0$. So $FfP^{\perp} \in S_{\perp}$. Similarly, $EgQ^{\perp} \in \mathcal{T}_{\perp}$ for all $g \in \mathcal{L}_{*}(K)$. So each $(FfP^{\perp}) \otimes (EgQ^{\perp}) \in \mathcal{S}_{\perp} \otimes \mathcal{T}_{\perp}$, and hence $h \in \mathcal{S}_{\perp} \otimes \mathcal{T}_{\perp}$. \square

THEOREM 9. *Let $\mathcal{S} \subseteq L(H)$, $\mathcal{T} \subseteq L(K)$ be reflexive proper subspaces. Then $\mathcal{K}(\mathcal{S} * \mathcal{T}) \geq \mathcal{K}(\mathcal{S}) \cdot \mathcal{K}(\mathcal{T})$.*

PROOF. Let $A \in L(H)$, $B \in L(K)$ be arbitrary. By Proposition 4 we have $d(A \otimes B, \mathcal{S} * \mathcal{T}) = d(A, \mathcal{S}) \cdot d(B, \mathcal{T})$. We will show that in general $\alpha(A \otimes B, \mathcal{S} * \mathcal{T}) \leq \alpha(A, \mathcal{S}) \cdot \alpha(B, \mathcal{T})$, and hence if $A \notin \mathcal{S}$ and $B \notin \mathcal{T}$ then $\mathcal{K}(A \otimes B, \mathcal{S} * \mathcal{T}) \geq \mathcal{K}(A, \mathcal{S}) \cdot \mathcal{K}(B, \mathcal{T})$. Taking of suprema over all such A, B then yields $\mathcal{K}(\mathcal{S} * \mathcal{T}) \geq \mathcal{K}(\mathcal{S}) \cdot \mathcal{K}(\mathcal{T})$, since by hypothesis $\mathcal{S} \neq L(H)$ and $\mathcal{T} \neq L(K)$.

We utilize Lemma 2. Let x be a unit vector in $H \otimes K$. Let $\mathcal{R} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$. By Theorem 8, $\mathcal{S} * \mathcal{T} = \text{ref}(\mathcal{R})$, so $\mathcal{S} * \mathcal{T}$ and \mathcal{R} have the same cyclic subspaces. As in the proof of Theorem 8, let G be the projection onto $[\mathcal{R}x] = [(\mathcal{S} * \mathcal{T})x]$, and let $F \in L(H)$, $E \in L(K)$ be the smallest projections such that $(F \otimes I)x = (I \otimes E)x$. Let $P = \text{proj}[\mathcal{S}FH]$, $Q = \text{proj}[\mathcal{T}EK]$. By Lemma 7, $G = (P^{\perp} \otimes Q^{\perp})^{\perp}$. Since $(F \otimes E)x = x$ we have

$$\begin{aligned} d[(A \otimes B)x, (\mathcal{S} * \mathcal{T})x] &= \|G^{\perp}(A \otimes B)x\| = \|(P^{\perp} \otimes Q^{\perp})(A \otimes B)(F \otimes E)x\| \\ &= \|((P^{\perp}AF) \otimes (Q^{\perp}BE))x\| \leq \|P^{\perp}AF\| \cdot \|Q^{\perp}BE\| \leq \alpha(A, \mathcal{S}) \cdot \alpha(B, \mathcal{T}) \end{aligned}$$

since $P^{\perp}\mathcal{S}F = 0$ and $Q^{\perp}\mathcal{T}E = 0$.

Since x was an arbitrary unit vector, we have

$$\alpha(A \otimes B, \mathcal{S} * \mathcal{T}) \leq \alpha(A, \mathcal{S}) \cdot \alpha(B, \mathcal{T}), \text{ as required. } \square$$

The proof of Theorem 9 can be improved slightly to show that for arbitrary $A \in L(H)$, $B \in L(K)$ the inequality $\alpha(A \otimes B, \mathcal{S} * \mathcal{T}) \leq \alpha(A, \mathcal{S}) \cdot \alpha(B, \mathcal{T})$ is in fact an equality. We capture this fact.

COROLLARY 10. *Let $\mathcal{S} \subseteq L(H)$ and $\mathcal{T} \subseteq L(K)$ be reflexive subspaces, and let $A \in L(H)$, $B \in L(K)$ be arbitrary. Then $\alpha(A \otimes B, \mathcal{S} * \mathcal{T}) = \alpha(A, \mathcal{S}) \cdot \alpha(B, \mathcal{T})$.*

PROOF. The inequality “ \leq ” is contained in the proof of Theorem 9. For the converse, let $F, P \in L(H)$ and $E, Q \in L(K)$ be arbitrary projections satisfying $P^\perp \mathcal{S} F = 0$ and $Q^\perp \mathcal{T} E = 0$. Then if $\mathcal{R} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$, we have $(P^\perp \otimes Q^\perp) \mathcal{R} (F \otimes E) = 0$, so since $\mathcal{S} * \mathcal{T} = \text{ref}(\mathcal{R})$ by Theorem 8, we have $(P^\perp \otimes Q^\perp)(\mathcal{S} * \mathcal{T})(F \otimes E) = 0$. Since $\|(P^\perp \otimes Q^\perp)(A \otimes B)(F \otimes E)\| = \|P^\perp A F\| \cdot \|Q^\perp B E\|$, we have

$$\begin{aligned} \alpha(A \otimes B, \mathcal{S} * \mathcal{T}) &= \sup\{\|L(A \otimes B)M\| : L, M \text{ are projections} \\ &\quad \text{in } L(H \otimes K) \text{ with } L(\mathcal{S} \otimes \mathcal{T})M = 0\} \\ &\geq \|P^\perp A F\| \cdot \|Q^\perp B E\|. \end{aligned}$$

So since P, F, Q, E were arbitrary, we conclude that $\alpha(A \otimes B, \mathcal{S} * \mathcal{T}) \geq \alpha(A, \mathcal{S}) \cdot \alpha(B, \mathcal{T})$. Finally, we note that equality is trivially true if either $\mathcal{S} = L(H)$ or $\mathcal{T} = L(K)$. \square

From Corollary 10 and Proposition 4 we conclude that $\mathcal{K}(A \otimes B, \mathcal{S} * \mathcal{T}) = \mathcal{K}(A, \mathcal{S}) \cdot \mathcal{K}(B, \mathcal{T})$ whenever \mathcal{S}, \mathcal{T} are reflexive proper subspaces with $A \notin \mathcal{S}$, $B \notin \mathcal{T}$. That is, the basic inequality is an equality when restricted to the class of elementary tensors. It can happen, however, that for some operator $T \in L(H \otimes K)$, which is not an elementary tensor, we have $\mathcal{K}(T, \mathcal{S} * \mathcal{T}) > \mathcal{K}(\mathcal{S}) \cdot \mathcal{K}(\mathcal{T})$, and thus the inequality in Theorem 9 may be strict. The following simple example shows this.

EXAMPLE 11. Let

$$S = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} : \lambda \in \mathbf{C} \right\}$$

be regarded as a subspace of operators acting on 2-dimensional Hilbert space. An elementary computation shows that S is reflexive. An application of [6, Lemma 3.3] after interchanging rows, and either [4, Proposition 3 or 5, Theorems 1.1 or 1.2] to the preannihilator

$$S_\perp = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix},$$

shows that $\mathcal{K}(S) = 1$. Since

$$\mathcal{S}_\perp \otimes \mathcal{S}_\perp = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ 0 & * & 0 & * \\ * & * & * & * \end{pmatrix}$$

we have

$$S * S = \begin{pmatrix} * & * & * & 0 \\ * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

acting on 4-dimensional Hilbert space. Let

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $P(S * S) \subseteq S * S$, and thus by [5, Lemma 1.3] the compression $P(S * S)|_{PH}$ is reflexive with hyperreflexivity constant no greater than that of $S * S$. But this compression has diagram

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and by [5, Example 4.7] this has constant $\geq \sqrt{9/8}$. Thus $\mathcal{K}(S * S) \geq \sqrt{9/8} > 1 = \mathcal{K}(S) \cdot \mathcal{K}(S)$. So in this case the inequality of Theorem 9 is strict. \square

THEOREM 12. *Let n be a positive integer, and for $1 \leq i \leq n$ let \mathcal{A}_i be a reflexive proper subalgebra of $L(H_i)$ for H_i a separable Hilbert space. Suppose the tensor product $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$, acting on $H = H_1 \otimes \cdots \otimes H_n$, is reflexive. Let*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n & \mathcal{A}_1 * \cdots * \mathcal{A}_n \\ 0 & \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \end{pmatrix}.$$

Then \mathcal{A} is a reflexive subalgebra of $L(H \otimes H)$, and $\mathcal{K}(\mathcal{A}) \geq \mathcal{K}(\mathcal{A}_1) \cdot \mathcal{K}(\mathcal{A}_2) \cdots \mathcal{K}(\mathcal{A}_n)$.

PROOF. \mathcal{A} is an algebra since $\mathcal{A}_1 * \cdots * \mathcal{A}_n$ is a bimodule over $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. A simple calculation shows that

$$\mathcal{A}_\perp = \begin{pmatrix} (\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n)_\perp & L_*(H) \\ (\mathcal{A}_1 * \cdots * \mathcal{A}_n)_\perp & (\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n)_\perp \end{pmatrix}.$$

Since $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ is reflexive, $(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n)_\perp$ is generated by rank-1 operators. Since $\mathcal{A}_1 \cdots \mathcal{A}_n$ are reflexive, $(\mathcal{A}_1 * \cdots * \mathcal{A}_n)_\perp = (\mathcal{A}_1)_\perp \otimes \cdots \otimes (\mathcal{A}_n)_\perp$ is also generated by rank-1 operators. Hence \mathcal{A}_\perp is generated by rank-1 operators, so \mathcal{A} is reflexive.

To show $\mathcal{K}(\mathcal{A}) \geq \mathcal{K}(\mathcal{A}_1) \cdots \mathcal{K}(\mathcal{A}_n)$ we utilize [4, Proposition 3]. Let P be the orthogonal projection from $H \oplus H$ onto H . Let $S = \mathcal{A}_1 * \cdots * \mathcal{A}_n$. By Theorem 9, $\mathcal{K}(S) \geq \mathcal{K}(\mathcal{A}_1) \cdots \mathcal{K}(\mathcal{A}_n)$. Let $\mathcal{C}_1(\mathcal{A})$, $\mathcal{C}_1(S)$ denote the closed convex hulls of the rank ≤ 1 operators in the unit balls of \mathcal{A}_\perp , S , respectively. Then clearly

$$P^\perp \mathcal{C}_1(\mathcal{A}) P = \begin{pmatrix} 0 & 0 \\ \mathcal{C}_1(S) & 0 \end{pmatrix}.$$

Let $R(\mathcal{A})$, $R(S)$ be the largest radii such that $\{f \in \mathcal{A}_\perp : \|f\|_1 \leq R(\mathcal{A})\} \subseteq \mathcal{C}_1(\mathcal{A})$ and $\{g \in S_\perp : \|g\|_1 \leq R(S)\} \subseteq \mathcal{C}_1(S)$. It follows that $R(\mathcal{A}) \leq R(S)$. By [4, Proposition 3] we have $\mathcal{K}(\mathcal{A}) = 1/R(\mathcal{A})$ and $\mathcal{K}(S) = 1/R(S)$, so $\mathcal{K}(\mathcal{A}) \geq \mathcal{K}(S)$, as required. \square

REMARKS. The requirement that $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ be reflexive will be met if each \mathcal{A}_i is finite dimensional, and more generally, if each \mathcal{A}_i has property S_n (Kraus [3]). (It is, of course, an open question whether the tensor product of reflexive algebras is necessarily reflexive.) As in the special case of the “key example” in [2], Theorem 12 gives a means of constructing reflexive algebras of arbitrarily large distance constant. If each \mathcal{A}_i is a CSL algebra and so contains a m.a.s.a., then \mathcal{A} will also contain a m.a.s.a., so will be a CSL algebra. A direct sum of such algebras, with increasing constants, will be nonhyperreflexive.

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