COMPLETELY RANK-NONINCREASING LINEAR MAPS

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1. Introduction

Suppose H is a Hilbert space, $B(H)$ is the set of (bounded linear) operators on H, $\mathcal{F}(H)$ is the set of finite-rank operators on H, and $\mathcal{K}(H)$ is the set of compact operators on H. Suppose M is a Hilbert space, $\mathcal{A}$ is a unital C*-subalgebra of $B(H)$, the map $\pi : \mathcal{A} \to B(M)$ is a unital *-homomorphism, $\phi, \psi : \mathcal{A} \to B(M)$ are linear maps with ϕ unital and completely positive and ψ completely bounded. The theorems of W. Stinespring [**12**] and Wittstock [**14**] say that there is are unital representations ρ, σ of $\mathcal{A}$, an isometry V, and operators A, B with $\|A\| \; \|B\| = \|\psi\|_{cb}$ such that

$$\phi(x) = V^* \rho(x) V,$$

and

$$\psi(x) = A \sigma(x) B$$

for every x in $\mathcal{A}$. Two unital representations π_1, π_2 of $\mathcal{A}$ are *approximately (unitarily) equivalent*, denoted $\pi_1 \sim_a \pi_2$, if there is a net $\{U_\lambda\}$ of unitary operators such that

$$\lim_\lambda \|U_\lambda^* \pi_1(x) U_\lambda - \pi_2(x)\| = 0$$

for every $x \in \mathcal{A}$. It was proved by D. Voiculescu [**13**] that if $\mathcal{A}$ and π_1, π_2 are separable, then $\pi_1 \sim_a \pi_2$ implies that there is a sequence $\{U_n\}$ of unitary operators such that, for every $x \in \mathcal{A}$

$$\lim_{n \to \infty} \|U_n^* \pi_1(x) U_n - \pi_2(x)\| = 0,$$

and such that for every $n \in \mathbb{N}$, and every $x \in \mathcal{A}$,

$$U_n^* \pi_1(x) U_n - \pi_2(x) \text{ is compact.}$$

The following results were proved by the first author [**5**],[**6**], [**7**] ; the first result is a reformulation of D. Voiculescu's beautiful characterization of approximate equivalence [**13**] when H is separable. Note that $id_{\mathcal{A}}$ denotes the identity representation on $\mathcal{A}$. For an operator $T \in B(H)$ we let rank$T$ denote the Hilbert-space dimension of the closure of the range of T.

Theorem 1. *Suppose $\mathcal{A}, H, M$ are separable and π, ϕ, ψ are as above. Then*

1. $\pi \sim_a id_{\mathcal{A}}$ *if and only if* $\pi | \mathcal{A} \cap \mathcal{F}(H) \sim_a id_{\mathcal{A}} | \mathcal{A} \cap \mathcal{F}(H)$ *if and only*

$$rank \pi(A) = rank A$$

   for every $A \in \mathcal{A}$.
2. $\pi \oplus \pi_1 \sim_a id_{\mathcal{A}}$ *for some representation* π_1 *of* $\mathcal{A}$ *if and only if* $\pi | \mathcal{F}(H) \oplus \pi_2 \sim_a id_{\mathcal{A} \cap \mathcal{F}(H)}$ *for some representation* π_2 *of* $\mathcal{A} \cap \mathcal{F}(H)$ *if and only if*

$$rank \pi(A) \le rank A$$

   for every $A \in \mathcal{A}$.
3. *The following are equivalent:*

a *There is a unital representataion* ρ *of* $\mathcal{A}$ *with* $\rho \sim_a id_{\mathcal{A}}$ *and an isometry* V *such that* $\phi(x) = V^* \rho(x) V$ *for every* $x \in \mathcal{A}$.

b ϕ *is rank-nonincreasing and there is a representation* ρ_1 *of* $\mathcal{A} \cap \mathcal{F}(H)$ *with* $\rho_1 \sim_a id_{\mathcal{A}\cap\mathcal{F}(H)}$ *and an isometry* W *such that* $\phi(x) = W^* \rho_1(x) W$ *for every* $x \in \mathcal{A} \cap \mathcal{F}(H)$.

c *There is a sequence* $\{V_n\}$ *o f isometries such that* $V_n^* A V_n \to \phi(A)$ *in the weak operator topology for every* $A \in \mathcal{A}$.

4. *The following are equivalent:*

a *There is a unital representataion* σ *of* $\mathcal{A}$ *with* $\sigma \sim_a id_{\mathcal{A}}$ *and operators* A, B *with* $\|A\| \|B\| = \|\psi\|_{cb}$ *such that* $\psi(x) = A\sigma(x)B$ *for every* $x \in \mathcal{A}$

b ψ *is rank-nonincreasing and there is a representation* ρ_1 *of* $\mathcal{A} \cap \mathcal{F}(H)$ *with* $\rho_1 \sim_a id_{\mathcal{A}\cap\mathcal{F}(H)}$ *and operators* A_1, B_1 *such that* $\psi(x) = A_1 \sigma_1(x) B_1$ *for every* $x \in \mathcal{A} \cap \mathcal{F}(H)$.

c *there are norm-bounded sequences* $\{C_n\}, \{D_n\}$ *such that* $C_n A B_n \to \psi(A)$ *in the weak operator topology for every* $A \in \mathcal{A}$.

The beauty of the first two statements in Theorem 1 is that they give purely algebraic characterizations (in terms of rank) of very geometric relationships. It is tempting to hope that the statements in (3) and (4) in Theorem 1 are equivalent to the respective statements that ϕ and ψ are rank-nonincreasing. However, this is not case (see section 2).

It is the purpose of this paper to provide a different characterization, solely in terms of rank, of the statements (3) and (4) in Theorem 1. More precisely, if $\mathcal{S}$ is a linear subspace of $B(H)$ and $\phi : \mathcal{S} \to B(M)$ is a linear mapping, then, for each $n \in \mathbb{N}$, we define the maps $\phi_n : \mathcal{M}_n(\mathcal{S}) \to \mathcal{M}_n(B(M))$ by

$$\phi_n((s_{ij})) = (\phi(s_{ij})).$$

We say that ϕ is *completely rank-nonincreasing* if, for each $n \in \mathbb{N}$ and each $(s_{ij}) \in \mathcal{M}_n(\mathcal{S})$

$$\text{rank}\phi_n((s_{ij})) \le \text{rank}(s_{ij}).$$

The following is our main theorem.

Theorem 2. *Suppose* H *and* M *are separable Hilbert spaces,* $\mathcal{A}$ *is a separable unital C*-subalgebra of* $B(H)$, $\phi, \psi : \mathcal{A} \to B(M)$ *are linear maps with* ϕ *unital and completely positive and* ψ *completely bounded. Then*

1. *There is a unital representataion* ρ *of* $\mathcal{A}$ *with* $\rho \sim_a id_{\mathcal{A}}$ *and an isometry* V *such that* $\phi(x) = V^* \rho(x) V$ *for every* $x \in \mathcal{A}$ *if and only if* ϕ *is completely rank-nonincreasing.*
2. *There is a unital representataion* σ *of* $\mathcal{A}$ *with* $\sigma \sim_a id_{\mathcal{A}}$ *and operators* A, B *with* $\|A\| \|B\| = \|\psi\|_{cb}$ *such that* $\psi(x) = A\sigma(x)B$ *for every* $x \in \mathcal{A}$ *if and only if* ψ *is completely rank-nonincreasing.*

Remark 1. *We can see the significance of the preceding theorem with a simple application. Suppose* $T \in B(H)$ *and* $A \in B(M)$. *It follows from Stinespring's* *theorem* [**12**] *that there is a Hilbert space* $H' \supset M$ *and operator* $S \in B(H')$ *with an operator matrix*

$$S = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

and a unital representation $\pi : C^*(T) \to B(H')$ *with* $\pi(T) = S$ *if and only if there is a unital completely positive map* $\phi : C^*(T) \to B(M)$ *with* $\phi(T) = A$ *and* $\phi(T^*T) = A^*A$.

Adding the condition that ϕ *is completely rank-nonincreasing is equivalent to being able to choose* S *so that, for every* $\varepsilon > 0$, *there is a unitary operator* U_ε *and a compact operator* K_ε *with* $\|K_\varepsilon\| < \varepsilon$ *such that*

$$U_\varepsilon^* T U_\varepsilon = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} + K_\varepsilon.$$

The notion of completely rank-nonincreasing maps arose from an attempt to characterize the linear maps on a linear subspace of $B(H)$ that are point-strong limits of similarities or point-strong limits of skew-compressions introduced in [**9**]. Suppose $\mathcal{S}$ is a linear subspace of $B(H)$ and $\phi : \mathcal{S} \to B(M)$ is linear. We are not assuming that $\mathcal{S}$ is norm-closed or that ϕ is bounded. We say that ϕ is a *similarity* if there is an invertible operator W such that, for every $S \in \mathcal{S}$, $\phi(S) = W^{-1}SW$. We say that ϕ is a *compression* if there is an operator V such that, for every $S \in \mathcal{S}$, $\phi(S) = V^*SV$. Finally, we say that ϕ is a *skew-compression* if there are operators A, B such that, for every $S \in \mathcal{S}$, $\phi(S) = ASB$. If $\{\phi_\lambda\}$ is a net of maps on $\mathcal{S}$, we say that $\phi_\lambda \to \phi$ *point-strongly* (resp., point-weakly, point-norm) if, for every $S \in \mathcal{S}$, $\phi_\lambda(S) \to \phi(S)$ in the strong (resp., weak, norm) operator topology.

In [**9**] the authors proved the following results that, in a sense, parallel those of Theorem 1.

Theorem 3. *: The following statements are true.*

1. *The map* ϕ *is a point-strong limit of skew-compressions if and only if* $\phi|\mathcal{S} \cap \mathcal{F}(H)$ *is a point-weak limit of skew-compressions.*
2. *Suppose* $1 \in \mathcal{S}$. *Then* ϕ *is a point-strong limit of similarities if and only if* $\phi(1) = 1$ *and* $\phi|\mathcal{S} \cap \mathcal{F}(H)$ *is a point-weak limit of skew-compressions.*

It was shown in [**9**] that the problem of characterizing the maps that are point-strong limits of similarities or skew-compressions reduces to the case when H is finite-dimensional. In finite-dimensions the problems of characterizing limits of similarities is equivalent to determining closures of joint similarity orbits, a problem studied in [**4**]. This is because if $\dim H < \infty$, then every linear subspace $\mathcal{S}$ of $B(H)$ is spanned by finitely many elements $S_1, S_2, \ldots, S_k$, and a linear map $\phi : \mathcal{S} \to B(H)$ is a limit of similarities if and only if $(\phi(S_1), \ldots, \phi(S_k))$ is in the closure of the joint similarity orbit of $(S_1, S_2, \ldots, S_k)$.

This paper contains counterexamples to conjectures in [**4**], and we replace them with what we believe are the correct conjectures.

2. The basic conjecture

We begin this section with an example that shows that the condition on $\phi\,|(\mathcal{A} \cap \mathcal{F}(H))$ in part (3) of Theorem 1 cannot be dropped. In other words, the characterizations for representations solely in terms of rank cannot be directly carried over to completely positive and completely bounded cases. In fact, the rank-decreasing condition does not even imply that a linear map is a point-strong limit of skew-compressions. A simple counterexample is based on the following elementary fact:

There do not exist nets $\{e_\lambda\}$ and $\{f_\lambda\}$ in $\mathbb{C}^2$ such that, for every 2×2 matrix A,

$$(Ae_\lambda, f_\lambda) \to tr(A),$$

where tr denotes the normalized trace on $\mathcal{M}_2$.

This follows from the fact that the above assertion is equivalent to the rank-two matrix $\frac{1}{2}I_2$ being the weak*-limit in the dual space of $\mathcal{M}_2$ (which is $\mathcal{M}_2$) of the net rank-one elements $e_\lambda \otimes f_\lambda$.

To construct the counterexample, let $\phi = tr : \mathcal{M}_2 \to \mathbb{C}$. Clearly ϕ is linear, unital, completely positive, and completely bounded, and, for every $A \in \mathcal{M}_2$,

$$\text{rank}\phi(A) \le \text{rank}A.$$

However, it follows from the preceding observation that ϕ is not a point-weak limit of skew-compressions. To get an infinite-dimensional example, we can assume H is infinite-dimensional, let $V : \mathbb{C}^2 \to H$ be any isometry, let $\mathcal{A} = \mathbb{C}I + \mathcal{K}(H)$, let P be a rank-one projection, and define $\phi : \mathcal{A} \to B(H)$ by

$$\phi(zI + K) = zI + [tr(V^*KV)]\,P$$

for every $z \in \mathcal{C}$ and every $K \in \mathcal{K}(H)$. Since ϕ is the sum of a unital * homomorphism and a completely positive map, we see that ϕ is unital, completely positive, completely bounded, and, for every $A \in \mathcal{A}$, we have rank$\phi(A) \le$rank$A$. However, if Q is the projection onto the range of V, the restriction of ϕ to $Q\mathcal{K}(H)Q$ looks exactly like tr on $\mathcal{M}_2$, and hence ϕ is not even a point-weak limit of skew-compressions.

Although being rank-nonincreasing is not sufficient for a map to be a point-strong limit of skew-compressions, it may still be possible to find a characterization solely in terms of rank. Suppose $\mathcal{S}$ is a linear subspace of $B(H)$ and $\phi : \mathcal{S} \to B(M)$ is linear. As mentioned above, for each positive integer n, the map $\phi_n : \mathcal{M}_n(S) \to \mathcal{M}_n(B(M))$ is defined by

$$\phi_n\left(s_{ij}\right) = \left(\phi(s_{ij})\right).$$

It is easy to see that if $\phi(S) = \lim_\lambda A_\lambda S B_\lambda$ is a point-strong limit of skew-compressions, then, for every $n \in \mathbb{N}$ and every $(S_{ij}) \in \mathcal{M}_n\left(\mathcal{S}\right)$, we have

$$\phi_n\left((S_{ij})\right) = \lim_\lambda \begin{pmatrix} A_\lambda & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_\lambda \end{pmatrix} (S_{ij}) \begin{pmatrix} B_\lambda & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B_\lambda \end{pmatrix}.$$

Hence a necessary condition for ϕ to be a limit of skew-compressions is that ϕ be completely rank-nonincreasing. We now make our fundamental conjecture. We will prove our main theorem (Theorem 2) by verifying this conjecture when the domain is a C*-algebra. We can also give a characterization in terms of rank of the *elementary operators* on $B(H)$, i.e., linear combinations of skew-compressions (See Theorem 7)

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CONJECTURE 1. *Suppose $\mathcal{S}$ is a linear subspace of $B(H)$ and $\phi : \mathcal{S} \to B(M)$ is linear. Then ϕ is a point-strong limit of skew-compressions if and only if ϕ is completely rank-nonincreasing.*

We are not yet able to completely settle the above conjecture, but we will reduce it to the case where ϕ is a linear functional.

Let's return to our example where ϕ is the normalized trace on $\mathcal{M}_2$. If we define the matrix $T \in \mathcal{M}_2(\mathcal{M}_2)$ by

$$T = \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \end{pmatrix},$$

we see that $\mathrm{rank}(T) = 1$, $\phi_2(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has rank 2. Thus, although ϕ is rank-nonincreasing, ϕ_2 is not. This gives an easy way to see that ϕ is not a limit of skew-compressions.

EXAMPLE 1. *Let $\mathcal{S}$ be the set of upper triangular 2×2 matrices and let $\phi : \mathcal{S} \to \mathbb{C}$ be the trace. There is no rank-one 2×2 matrix K such that $\phi(S) = tr(SK)$ for every $S \in \mathcal{S}$. However, for every $S \in \mathcal{S}$, $\phi(S) = \lim_m \left(S \begin{pmatrix} 1 \\ \frac{1}{n} \end{pmatrix}, \begin{pmatrix} 1 \\ n \end{pmatrix} \right)$, so ϕ is a point-strong limit of skew compressions and is therefore completely rank-nonincreasing. It is not difficult to show that every completely rank-nonincreasing linear functional on $\mathcal{S}$ is a limit of skew-compressions.*

3. The MainResults

Our results require a more general notion of completely rank-nonincreasing. If $k \in \mathbb{N}$, we say that a map $\phi : \mathcal{S} \to B(M)$ is *k-rank nonincreasing* if

$$\mathrm{rank}\phi(S) \le k\mathrm{rank}S$$

for every $S \in \mathcal{S}$. We say ϕ is *completely k-rank nonincreasing* if ϕ_n is k-rank nonincreasing for every $n \in \mathbb{N}$. This is the first step in proving our main result (Theorem 2).

LEMMA 1. *Suppose T is a trace class operator and $\phi : \mathcal{K}(H) \to \mathbb{C}$ is defined by $\phi(A) = trace(TA)$, then the smallest k for which ϕ is completely k-rank nonincreasing is $rank(T)$.*

PROOF. Suppose $\{e_1, \ldots, e_n\}$ is a linearly independent set such that $\{Te_1, \ldots, Te_n\}$ is orthonormal. Then if, for $e, f \in H$, $e \otimes f$ denotes the operator defined by $(e \otimes f)(h) = (h, f)e$, then

$$\phi(e \otimes f) = trace(T(e \otimes f)) = trace(Te \otimes f) = (Te, f).$$

Let W be the $n \times n$ matrix over $\mathcal{K}(H)$ defined by $W = (e_j \otimes Te_i)$. Then $rankW = 1$, but $\phi_n(W) = ((Te_j, Te_i))$ is the identity matrix, which has rank n. Hence, if ϕ is completely k-rank-nonincreasing, then $k \ge rankT$. On the other hand, if $rankT = 1$, say $T = u \otimes v$, then $\phi(A) = (Au, v)$ is a skew compression, which is completely rank-nonincreasing. If $rankT = k < \infty$, then T is the sum of k rank-one transformations, so ϕ is the sum of k completely rank-nonincreasing maps, which means ϕ is completely k-rank-nonincreasing. □

We reduce our conjecture to the case of linear functionals. The key idea is a classical identification of the set of all linear maps from a vector space X into $\mathcal{M}_N$ and the set of linear functionals on $\mathcal{M}_N(X)$. This correspondence has been used in the study of completely positive and completely bounded maps [**1**], [**11**]. Suppose $\phi : X \to \mathcal{M}_N$ is linear. We can write

$$\phi(x) = (\phi_{ij}(x)),$$

where each ϕ_{ij} is a linear functional on X. We define a linear functional $\widehat{\phi} : \mathcal{M}_N(X) \to \mathbb{C}$ by

$$\widehat{\phi}(x_{ij}) = \frac{1}{N}\sum_{i,j=1}^{N} \phi_{ij}(x_{ij}).$$

Let $E_{ij}, 1 \leq i,j \leq k$, the $k \times k$ complex matrix with a 1 in the (i,j)-entry and 0's in the other entries. If $x \in X$, let $xE_{ij} \in \mathcal{M}_k(X)$ denote the $k \times k$ matrix with x in the (i,j)-entry and 0's in the other entries. We can write each matrix $(x_{ij}) \in \mathcal{M}_k(X)$ as

$$(x_{ij}) = \sum_{i,j=1}^{N} x_{ij}E_{ij},$$

and we can recover ϕ from $\widehat{\phi}$ by noting that

$$\phi_{ij}(x) = N\widehat{\phi}(xE_{ij}).$$

Theorem 4. *Suppose N, k are positive integers, $\mathcal{S}$ is a linear subspace of $B(H)$ and $\phi : \mathcal{S} \to \mathcal{M}_N$ is linear. Then*

1. *ϕ is completely k-rank-nonincreasing if and only if $\widehat{\phi}$ is completely k-rank-nonincreasing.*
2. *ϕ is a point-strong limit of skew-compressions if and only if $\widehat{\phi}$ is a point-strong limit of skew-compressions.*

Proof. Suppose ϕ is completely k-rank-nonincreasing. We will show that $\widehat{\phi}$ is k-rank-nonincreasing. The proof that $\widehat{\phi}$ is completely k-rank-nonincreasing follows similarly. We have that

$$\widehat{\phi}(s_{ij}) = \frac{1}{N}(1,1,\ldots,1)\phi_N(s_{ij}E_{ij})\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$$

and that the matrices (s_{ij}) and $\left(s_{ijE_{ij}}\right)$ have the same rank. Since ϕ_N is k -rank-nonincreasing, we conclude that $\widehat{\phi}$ is k -rank-nonincreasing. Conversely, suppose $\widehat{\phi}$ is completely k-rank-nonincreasing. It follows, for each $x \in \mathcal{S}$, that

$$\phi(x) = N\left(\widehat{\phi}\right)_N (xE_{ij}).$$

Since $\left(\widehat{\phi}\right)_N$ is k-rank-nonincreasing and x has the same rank as (xE_{ij}), we conclude that ϕ is k-rank-nonincreasing. A similar argument shows that ϕ is actually completely k-rank-nonincreasing. Next suppose ϕ is a point-strong limit of skew-compressions, i.e., there are nets $\{A_\lambda\}$ and $\{B_\lambda\}$ of operators such that, for every $S \in \mathcal{S}$,

$$\widehat{\phi}(S) = \lim_\lambda A_\lambda S B_\lambda$$

with convergence in the strong operator topology. It follows from the above argument, that if $W^{(N)}$ denotes a direct sum of N copies of W, then, for every $(s_{ij}) \in \mathcal{M}_N(\mathcal{S})$, we have

$$\widehat{\phi}\,(s_{ij}) = \lim_\lambda \frac{1}{N}(1,1,\ldots,1)A_\lambda^{(N)}\,(s_{ij}E_{ij})\,B_\lambda^{(N)}.$$

However, there is a unitary matrix U such that, for every $(s_{ij}) \in \mathcal{M}_N(\mathcal{S})$,

$$(s_{ij}E_{ij}) = U^* \begin{pmatrix} (s_{ij}) & 0 \\ 0 & 0 \end{pmatrix} U.$$

Also there is a partial isometry V such that

$$\begin{pmatrix} (s_{ij}) & 0 \\ 0 & 0 \end{pmatrix} = V^*\,(s_{ij})\,V.$$

Hence

$$\widehat{\phi}\,(s_{ij}) = \lim_\lambda \frac{1}{N}(1,1,\ldots,1)A_\lambda^{(N)}U^*V^*\,(s_{ij})\,VUB_\lambda^{(N)},$$

which shows that $\widehat{\phi}$ is a point-strong limit of skew-compressions. A similar argument shows that if $\widehat{\phi}$ is a point-strong limit of skew-compressions, then so is ϕ. $\square$

COROLLARY 1. *Suppose $\phi : \mathcal{K}(H) \to \mathcal{M}_n$ is linear and continuous. Then ϕ is completely k-rank-nonincreasing if and only if there are operators C and D such that*

$$\phi(T) = C^*T^{(k)}D$$

for every $T \in \mathcal{K}(H)$.

PROOF. The "if" part is easy; we only show the "only if" part. Since $\mathcal{M}_n(\mathcal{K}(H))$ is isomorphic to $\mathcal{K}(H \otimes \mathbb{C}^n)$, it follows from the preceding theorem that we can assume $n = 1$. Thus there is a trace-class operator K such that, for every $A \in \mathcal{K}(H)$,

$$\phi(A) = tr(AK).$$

We want to show that $\text{rank}(K) \le k$. Write $K = \sum_j \lambda_j e_j \otimes f_j$, with $\{e_1,\ldots\}$ and $\{f_1 \ldots\}$ orthonormal sets and $\lambda_j > 0$ and $\sum_j \lambda_j < \infty$. Suppose m is a positive integer and $m \le \text{rankK}K$, and let T be any invertible operator such that $T^*f_j = \frac{1}{\lambda_j}e_j$ for $1 \le j \le m$ and such that $T^*\left(\{f_1,\ldots,f_m\}^\perp\right) = \{e_1,\ldots,e_m\}^\perp$. Then $KT = \sum_j \lambda_j\,(e_j \otimes f_j)\,T = \sum_j \lambda_j\,(e_j \otimes T^*f_j)$ is reduced by $sp\{e_1,\ldots,e_m\}$ and its restriction to $sp\{e_1,\ldots,e_m\}$ is the identity. Consider the matrix $W \in \mathcal{M}_m(\mathcal{K}(H))$ defined as

$$W = \left(T\,(e_j \otimes e_i)\right)_{1 \le i,j \le m}.$$

Then $\text{rank}(W) = 1$, but $\phi_m(W) = (tr\,[T\,(e_j \otimes e_i)\,K]) = (tr\,[(e_j \otimes e_i)\,KT]) = I_m$. But ϕ is completely k-rank-nonincreasing, so $m = \text{rank}(\phi_m(W)) \le k\text{rank}(W) = k$. Hence $\text{rank}K \le k$, and $K = \sum_{j=1}^{m} \lambda_j e_j \otimes f_j$, so $\phi(A) = \sum_{j=1}^{m} (A\lambda_j e_j, f_j)$. Define $C, D : \mathbb{C} \to \mathbb{C}^k$ by $C(\lambda) = \lambda \begin{pmatrix} f_1 \\ \vdots \\ f_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and $D(\lambda) = \begin{pmatrix} \lambda_1 e_1 \\ \vdots \\ \lambda_m e_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then, for every $A \in \mathcal{K}(H)$, we have

$$\phi(A) = C^* AD.$$

□

The following lemma is a key result. It allows us to keep track of the cb-norms (in terms of $\|C\|\,\|D\|$ in Corollary 1) of representations of completely bounded completely k-rank-nonincreasing maps on $\mathcal{K}(H)$.

LEMMA 2. *Suppose $\phi : \mathcal{K}(H) \to \mathcal{M}_n$ is linear, continuous and completely k-rank-nonincreasing, m is a cardinal, and A, B are matrices such that, for every $T \in \mathcal{K}(H)$,*

$$\phi(T) = AT^{(m)}B.$$

Then there is a projection P such that

1. *P is in the commutant of $\mathcal{K}(H)^{(m)} = \{T^{(m)} : T \in \mathcal{K}(H)\}$*
2. *$\mathcal{K}(H)^{(m)}\,|ran(P) = \{T^{(m)}\,|ran(P) : T \in \mathcal{K}(H)\}$ is unitarily equivalent to $\mathcal{K}(H)^{(k)}$ (i.e., there is a unitary U such that $T^{(m)}|ran(P) = U^* T^{(k)} U$ for every $T \in \mathcal{K}(H)$),*
3. *$\mathcal{K}(H)^{(m)}\,|ran(P)$ has a cyclic vector, and*
4. *for every $T \in \mathcal{K}(H)$,*

$$\phi(T) = APT^{(m)}PB = APU^* T^{(k)} UPB.$$

PROOF. We first consider the case when $n = 1$. In this case we have, by the preceding corollary, a trace-class operator K with $\text{rank}(K) = k$ such that, for every $T \in \mathcal{M}_s$, $\phi(T) = tr(TK)$. The equation $\phi(T) = AT^{(m)}B$ translates to $\phi(T) = (Tu, v)$ with $u, v \in H^{(m)}$. Write $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}$. Thus we have $\phi(T) = \sum_{j=1}^{m} (Tu_j, v_j) = tr(T \sum_{j=1}^{m} u_j \otimes v_j)$, which implies $K = \sum_{j=1}^{m} u_j \otimes v_j$. Since $\text{rank}(K) = k$, it follows that $m \ge k$. If the desired conclusion fails, we can assume m is the smallest cardinal for which failure exists. Since the restriction of $\mathcal{K}(H)^{(m)}$ to a nontrivial reducing subspace is unitarily equivalent to $\mathcal{K}(H)^{(t)}$ for some $t \le m$, and since the range of A is finite-dimensional, we have $m < \infty$. It follows from the minimality of m that $\mathcal{K}(H)^{(m)}u = \mathcal{K}(H)^{(m)}v = H^{(m)}$. Hence, $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_m\}$ must be linearly independent. Choose operators

$V, W \in \mathcal{K}(\mathcal{H})$ so that $Vu_j = W^*v_j = e_j$ for $1 \leq j \leq m$, where $\{e_1, \ldots, e_s\}$ is some orthonormal set in H. It then follows that

$$VKW = V\left(\sum_{j=1}^{m} u_j \otimes v_j\right)W$$

$$= \sum_{j=1}^{m} Vu_j \otimes W^*v_j = \sum_{j=1}^{m} e_j \otimes e_j.$$

Comparing ranks, we see that $m \leq k$, which implies $m = k$, contradicting the assumption that the desired conclusion fails. We now turn to the general case. Define $\widehat{\phi} : \mathcal{M}_n(\mathcal{K}(\mathcal{H})) \to \mathbb{C}$ as above. For $1 \leq j \leq n$, let $V_j : \mathbb{C} \to \mathbb{C}^n$ be the map into the $j^{th}-$ coordinate. The equation $\phi(T) = AT^{(m)}B$ therefore yields $\phi_{ij}(T) = V_i^*\phi(T)V_j = V_i^*AT^{(m)}BV_j$, and

$$\widehat{\phi}((T_{ij})) = \sum_{i,j=1}^{n} V_i^*AT_{ij}^{(m)}BV_j = C\left(T_{ij}^{(m)}\right)D$$

where $C = (V_1^*A, \ldots, V_n^*A)$ and $D = \begin{pmatrix} BV_1 \\ \vdots \\ BV_n \end{pmatrix}$. It follows from the case in which $n = 1$ that there is a projection Q in the commutant of $\mathcal{M}_n(\mathcal{K}(\mathcal{H})^{(m)}) = \left\{\left(T_{ij}^{(m)}\right) : T_{ij} \in \mathcal{K}(\mathcal{H})\right\}$ such that $\mathcal{M}_n(\mathcal{K}(\mathcal{H})^{(m)})\,|ran(Q)$ is unitarily equivalent to $\mathcal{M}_n(\mathcal{K}(\mathcal{H})^{(k)})$ and such that

$$\widehat{\phi}((T_{ij})) = CQ\left(T_{ij}^{(m)}\right)QD$$

always holds. However, being in the commutant of $\mathcal{M}_n(\ \mathcal{K}(\mathcal{H})^{(m)})$, Q must have the form $Q = diag(P, P, \ldots, P)$ for some projection P in the commutant of $\mathcal{K}(\mathcal{H})^{(m)}$. It follows that $\mathcal{K}(\mathcal{H})^{(m)}\,|ran(P)$ is unitarily equivalent to $\mathcal{K}(\mathcal{H})^{(k)}$ and we always have

$$\widehat{\phi}((T_{ij})) = (V_1^*AP, \ldots, V_n^*AP)\left(T_{ij}^{(m)}\right)\begin{pmatrix} PBV_1 \\ \vdots \\ PBV_n \end{pmatrix}.$$

If we define ψ by

$$\psi(T) = APT^{(m)}PB,$$

it is clear that $\widehat{\psi} = \widehat{\phi}$; hence, $\phi = \psi$. □

We now want to extend the preceding lemma with $\mathcal{K}(H)$ replaced with an arbitrary C*-subalgebra of $\mathcal{K}(H)$. Suppose $\mathcal{S}$ is a C*-subalgebra of $\mathcal{K}(H)$. It follows from [**2**] that we can write H as a direct sum $H = H_0 \oplus \sum_{i\in I}^{\oplus} H_i^{m_i}$ (each m_i a positive integer) and we can, up to unitary equivalence, write $\mathcal{S}$ as the C*-direct sum

$$\mathcal{S} = \{0 \oplus \sum_{i\in I}^{\oplus} K_i^{(m_i)} : K_i \in \mathcal{K}(H_i), \{\|K_i\|\} \in c_0(I)\}.$$

This gives, for each $i \in I$, a representation $\pi_i : \mathcal{S} \to \mathcal{K}(H_i)$ so that, for every $S \in \mathcal{S}$,

$$S = 0 \oplus \sum_{i \in I}^{\oplus} \pi_i^{(m_i)}(S),$$

that is, the identity representation $id_{\mathcal{S}}$ on $\mathcal{S}$ is unitarily equivalent to

$$id_{\mathcal{S}} = 0 \oplus \sum_{i \in I}^{\oplus} \pi_i^{(m_i)}.$$

Moreover, in [**2**] it is shown that if $\rho : \mathcal{S} \to B(H_\rho)$ is any * homomorphism of $\mathcal{S}$, then there is a Hilbert space M_0 and a family $\{\kappa_i : i \in I\}$ of cardinals (possibly 0) such that ρ is unitarily equivalent to

$$0 \oplus \sum_{i \in I}^{\oplus} \pi_i^{(\kappa_i)}(S)$$

on the Hilbert space $M_0 \oplus \sum_{i \in I}^{\oplus} H_i^{\kappa_i}$.

THEOREM 5. *Suppose $\mathcal{S}$ is a C^*-subalgebra of $\mathcal{K}(H)$, $\rho : \mathcal{S} \to B(H_\rho)$ is a * homomorphism, A, B are operators and ϕ is defined on $\mathcal{S}$ by*

$$\phi(S) = A\rho(S)B.$$

If ϕ is completely k-rank-nonincreasing, then there is a projection P in the commutant of $\rho(\mathcal{S})$ such that the restriction of ρ to $ranP$ is unitarily equivalent to a subrepresentation (direct summand) of $id_{\mathcal{S}}^{(k)}$ and such that, for every $S \in \mathcal{S}$,

$$\phi(S) = AP\rho(S)PB.$$

PROOF. Write $id_{\mathcal{S}} = 0 \oplus \sum_{i \in I}^{\oplus} \pi_i^{(m_i)}$ and $\rho = 0 \oplus \sum_{i \in I}^{\oplus} \pi_i^{(\kappa_i)}(S)$ on $H_\rho = M_0 \oplus \sum_{i \in I}^{\oplus} H_i^{\kappa_i}$. For each $j \in I$, define $\tau_j : \mathcal{K}(H_j) \to \mathcal{S}$ so that for every $K \in \mathcal{K}(H_j)$, $\tau_j(K) = 0 \oplus \sum_{i \in I}^{\oplus} K_i^{(m_i)}$, where $K_j = K$, and $K_i = 0$ when $i \neq j$. Clearly, $\phi \circ \tau_j$ is completely $m_j k$-rank-nonincreasing. Since $(\phi \circ \tau_j)(K) = A(\rho \circ \tau_j)(K)B$, it follows from the preceding lemma that there is a projection P_j in the commutant of $(\rho \circ \tau_j)(\mathcal{K}(H_j))$ such that $ran(P_j) \subset H_j^{\kappa_j}$ and $(\rho \circ \tau_j)(\mathcal{K}(H_j))|ran(P_j)$ is unitarily equivalent to $\mathcal{K}(H_j)^{(n_j)}$ for some $n_j \leq m_j k$ (where n_j is the minimal integer such that $\phi \circ \tau_j$ is completely n_j-rank-nonincreasing) and such that $(\phi \circ \tau_j)(K) = A(\rho \circ \tau_j)(K)B$ for every $K \in \mathcal{K}(H_j)$. Let $P = \sum_{j \in I} P_j$. Since the ranges of the P_j's are orthogonal, P is a projection, and P is in the commutant of $\rho(\mathcal{S})$ and $\phi(S) = AP\rho(S)PB$ for every $S \in \mathcal{S}$. Also $\rho|ranP$ is unitarily equivalent to $\sum_{j \in I}^{\oplus} \pi_j^{(n_j)}$, which is clearly a direct summand of $id_{\mathcal{S}}^{(k)}$. □

COROLLARY 2. *If $\mathcal{S}$ is a C^*-subalgebra of $\mathcal{K}(H)$ and $\phi : \mathcal{S} \to B(M)$ is a completely bounded, completely rank-nonincreasing linear map on $\mathcal{S}$, then there are operators A, B such that $\|A\| \, \|B\| = \|\phi\|_{cb}$ such that, for every $S \in \mathcal{S}$,*

$$\phi(S) = ASB.$$

Moreover, if ϕ is completely positive, we can choose $A = B^*$.

The preceding corollary, combined with Theorem 1 clearly implies Theorem 2. The following is an equivalent formulation of Theorem 2.

THEOREM 6. *Suppose $\mathcal{A}$ us a unital separable C*-algebra, H is a separable Hilbert space, $\pi : \mathcal{A} \to B(H)$ is a unital * homomorphism, and $\phi : \mathcal{A} \to B(M)$ is a unital completely bounded linear map. The following are equivalent:*

1. *There is a representation ρ of $\mathcal{A}$ that is approximately equivalent to π and operators A, B with $\|A\| \, \|B\| = \|\phi\|_{cb}$ such that, for every $S \in \mathcal{A}$,*
$$\phi(S) = A\rho\,(S)\,B.$$
2. *For every $n \in \mathbb{N}$, and for every $T \in \mathcal{M}_n(\mathcal{A})$,rank$(\phi_n(T)) \leq$rank$(\pi_n(T))$.*

Moreover, if ϕ is completely positive, we can choose A, B in (1) so that $A = B^*$.

PROOF. We can assume that $\mathcal{S} \subset B(H)$ and π is the identity map on $\mathcal{S}$. Then the condition in (2) is that ϕ is completely rank-nonincreasing. It follows from the results in [**6**],[**7**] that statement (1) holds if and only if it holds for $\phi\,|(\mathcal{A} \cap \mathcal{K}(H))$. Hence the implication (2) $\Rightarrow$ (1) follows from the preceding theorem. The reverse implication is obvious. □

We can show that our main conjecture is true for C*-algebras of operators.

COROLLARY 3. *Suppose $\mathcal{S}$ is a separable unital C*-algebra, H, M are separable Hilbert spaces, $\pi : \mathcal{S} \to B(H)$ is a unital * homomorphism, and $\phi : \mathcal{S} \to B(M)$ is a (not necessarily bounded) linear map. Then ϕ is a point-strong limit of skew-compressions of π if and only if, for every $n \in \mathbb{N}$ and every $T \in \mathcal{M}_n(\mathcal{S})$, rank$(\phi_n(T)) \leq$ rank$(\pi_n(T))$.*

PROOF. We can assume $\mathcal{S} \subset B(H)$ and π is the identity map on $\mathcal{S}$. Suppose ϕ is completely rank-nonincreasing. To show that ϕ is a point-strong limit of skew compressions, it follows from [**9**] that it is enough to show that $\phi\,|(\mathcal{S} \cap \mathcal{F}(H))$ is a point-strong limit of skew-compressions. In turn we need only look at finite dimensional subspaces of $\mathcal{S} \cap \mathcal{F}(H)$. Since every finite subset of $\mathcal{S} \cap \mathcal{F}(H)$ generates a finite-dimensional C*-algebra contained in $\mathcal{S} \cap \mathcal{F}(H) \subset \mathcal{S} \cap \mathcal{K}(H)$, the desired conclusion follows from the preceding theorem. □

An operator $\phi : B(H) \to B(H)$ is called *elementary* if there are finitely many operators $A_1, B_1, \ldots, A_n, B_n$ so that, for every $T \in B(H)$,

$$\phi(T) = \sum_{j=1}^{n} A_j T B_j.$$

Call the smallest possible n in the above representation the *degree* of ϕ. It is clear that elementary operators on $B(H)$ are weak*-weak* continuous and completely bounded. The continuity implies that such maps are determined by their restrictions to $\mathcal{K}(H)$. Also the above representation is equivalent to a representation of the form $\phi(T) = XT^{(n)}Y$. We can therefore describe elementary operators in terms of rank.

THEOREM 7. *The following are true for a linear map $\phi : B(H) \to B(H)$:*

1. *ϕ is elementary with degree$(\phi) \leq n$ if and only if ϕ is completely bounded, weak*-weak* continuous, and completely n-rank-nonincreasing.*
2. *In the subset of completely bounded weak*-weak* continuous linear maps on $B(H)$, the set of all elementary operators of degree at most n ($n < \infty$) is closed under point-weak limits.*

3. *If ϕ is elementary with degree n, then there are operators $A_1, B_1, \dots, A_n, B_n$ so that $\left\|\sum_{j=1}^n A_j A_j^*\right\|^{\frac{1}{2}} \left\|\sum_{j=1}^n B_j^* B_j\right\|^{\frac{1}{2}} = \|\phi\|_{cb}$ and, for every $T \in B(H)$,*

$$\phi(T) = \sum_{j=1}^{n} A_j T B_j.$$

EXAMPLE 2. *Define $\phi : \mathcal{M}_n \to \mathcal{M}_n$ be the transpose map. It is clear that ϕ is linear, completely bounded, and trivially weak*-weak* continuous. Hence ϕ is an elementary map. If $T = (E_{ij}) \in \mathcal{M}_n(\mathcal{M}_n)$, then rank$(T) = 1$, and rank$(\phi_n(T))$= n^2. Hence the degree of ϕ must be at least n^2. On the other hand, every linear map on $\mathcal{M}_n$ has degree at most n^2, since, for $A = (a_{ij})$, $\phi(A) = \sum_{i,j} a_{ij}\phi(E_{ij})$ and each summand is a rank non-increasing linear map. Hence the degree of ϕ is exactly n^2.*

EXAMPLE 3. *Suppose $A = (a_{ij})$ is an $n \times n$ matrix. We let ϕ :$\mathcal{M}_n \to \mathcal{M}_n$ be Schur multiplication by A, i.e., $\phi((t_{ij})) = (a_{ij}t_{ij})$. We then have the map $\widehat{\phi}$: $\mathcal{M}_n(\mathcal{M}_n) \to \mathbb{C}$ is given by the formula*

$$\widehat{\phi}(A) = trace(TA),$$

where $T = (a_{ij}E_{ij})$. However, T is unitarily equivalent to $A \oplus 0$, so rank$T$ =rank$A$. It follows from Lemma 1 that ϕ has degree equal to rank$A$.

4. Closures of joint similarity orbits

We now construct examples to provide negative solutions to two conjectures in [**4**]. If $T = (T_1, \dots, T_n) \in (\mathcal{M}_d)^n$ is an n-tuple of $d \times d$ matrices, the *similarity orbit* $\mathfrak{S}(T)$ of T is defined as

$$\mathfrak{S}(T) = \{(A^{-1}T_1A, \dots, A^{-1}T_nA) : A \in \mathcal{M}_d,\ A \text{ invertible}\}.$$

The following two conjectures appear in [**4**]

1. [**4**, Conjecture 8.14] Suppose $T \in (\mathcal{M}_d)^n$. Then $S \in \mathfrak{S}(T)^-$ if and only if rank$p(S) \leq$rank$p(T)$ for all noncommutative polynomials p.
2. [**4**, Conjecture 9.1] Suppose $T \in (\mathcal{M}_d)^n$. Then $A \in \mathfrak{S}(T)^-$ if and only if for every noncommutative polynomial p, $p(A) \in \mathfrak{S}(p(T))^-$.

The first conjecture was proved when $n = 1$ in [**3**]. Hence it follows immediately that these two conjectures are actually the same.

EXAMPLE 4. *Define* $T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $T_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $T_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $T_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, *and let* $T =$

$(T_1, T_2, T_3, T_4, T_5)$. Let $A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. *Since the product of any two elements of $\{T_2, T_3, T_4, T_5\}$ is 0, the algebra $\mathcal{A}$ generated by T_1, T_2, T_3, T_4, T_5 equals the linear span of T_1, T_2, T_3, T_4, T_5. Furthermore, the linear mapping $\phi : \mathcal{A} \to \mathcal{M}_4$ defined by $f(T_j) = A_j$ $(1 \leq j \leq 5)$ is an algebra homomorphism. Moreover, the algebra $\mathcal{A}$ can be described as the set of all 2×2 block matrices of the form $\begin{pmatrix} \lambda & A \\ 0 & \lambda \end{pmatrix}$ with λ a scalar and $A \in \mathcal{M}_2$. Then the mapping ϕ can be expressed as*

$$\phi\left(\begin{pmatrix} \lambda & A \\ 0 & \lambda \end{pmatrix}\right) = \begin{pmatrix} \lambda & tr(A)E \\ 0 & \lambda \end{pmatrix},$$

where $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $S = \begin{pmatrix} \lambda & A \\ 0 & \lambda \end{pmatrix}$. If $\lambda \neq 0$, then rank$S$ =rank$\phi(S) = 4$. If $S = 0$, then rank$S$ =rank$\phi(S) = 0$. If $\lambda = 0$ and $S \neq 0$, then rank$\phi(S) \leq 1 \leq$rank$S$.

*It follows that if $A = (A_1, A_2, A_3, A_4, A_5)$ and $T = (T_1, T_2, T_3, T_4, T_5)$, then, for every polynomial p,rank$p(A) \leq$rank$p(T)$. It follows from a result of [**3**] that, for every polynomial p, $p(A) \in \mathcal{S}(p(T))^-$. However, the statement $A \in \mathcal{S}(T)^-$ is equivalent to the statement that ϕ is a point-norm limit of similarities. However, the restriction of ϕ to $sp\{T_2, T_3, T_4, T_5\}$ looks exactly like tr on $\mathcal{M}_2$, so ϕ is not a point-norm limit of similarities. This shows that Conjectures 8.14 and 9.1 in* [**4**] *are both false.*

Using the ideas in the preceding paragraph, we can construct a simpler counterexample to the conjectures in [**4**].

Example 5. *It follows immediately from the Jordan canonical form that every complex matrix is similar to its transpose. However, the transpose map is not a homomorphism. But it is a homomorphism on commutative algebras of matrices. Let $T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and let A_j be the transpose of T_j for $j = 1, 2$. It follows that, for any polynomial $p(x, y)$, $p(A_i, A_2)$ is the transpose of $p(T_1, T_2)$, so we have $p(T_1, T_2)$ and $p(A_i, A_2)$ are similar for every polynomial $p(x, y)$. However, the map ϕ that sends $p(T_1, T_2)$ to $p(A_1, A_2)$ is not completely rank-nonincreasing, since*

$$rank\phi_2 \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} > rank \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}.$$

Note that if dim$H < \infty$ and $\phi(1) = 1$, then ϕ is a limit of similarities if and only if ϕ is a limit of skew-compressions. To see this, assume that $\phi(S) = \lim_{n\to\infty} A_n S B_n$ for every $S \in \mathcal{S}$. Thus $A_n B_n \to \phi(1) = 1$. Hence, for sufficiently large n, $A_n B_n$ is invertible, which, in finite dimensions, implies that both A_n and

B_n are invertible. Thus

$$\phi(S) = \lim_{n\to\infty} (A_n B_n)^{-1} A_n S B_n = \lim_{n\to\infty} B_n^{-1} S B_n.$$

Hence our main conjecture, when restricted to unital maps in finite-dimensions gives what we feel is the correct conjecture for joint similarity orbits.

Conjecture 2. *Suppose* $S, T \in (\mathcal{M}_d)^n$. *Then* $S \in \mathfrak{S}(T)^- \Leftrightarrow$ *the map that sends* $p(T)$ *to* $p(S)$ *for each noncommutative polynomial is well-defined and completely rank-nonincreasing.*

Note that our main conjecture implies in finite-dimensions, that a unital completely rank-nonincreasing linear map must be a limit of similarities. It has been proved in [**8**] that, in finite-dimensions, a unital completely rank-nonincreasing linear map on a linear space $\mathcal{S}$ of matrices can be uniquely extended to a completely rank-nonincreasing algebra homomorphism on the algebra generated by $\mathcal{S}$.

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