

# PATCH AND CROSSOVER PLANAR DYADIC WAVELET SETS

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ABSTRACT. A single dyadic orthonormal wavelet on the plane  $\mathbb{R}^2$  is a measurable square integrable function  $\psi(x, y)$  whose translates along the coordinate axes followed by dilates by positive and negative integral powers of 2 generate an orthonormal basis for  $\mathcal{L}^2(\mathbb{R}^2)$ . A planar dyadic wavelet set  $E$  is a measurable subset of  $\mathbb{R}^2$  with the property that the inverse Fourier Transform of the normalized characteristic function  $\frac{1}{2\pi}\chi(E)$  of  $E$  is a single dyadic orthonormal wavelet. While constructive characterizations are known, no algorithm is known for constructing all of them. The purpose of this paper is to construct two new distinct uncountably infinite families of dyadic orthonormal wavelet sets in  $\mathbb{R}^2$ . We call these the crossover and patch families. Concrete algorithms are given for both constructions.

## INTRODUCTION

Wavelet theory is interesting to mathematicians both for its applications to signal analysis and image analysis, and also because of the rich mathematical structure underlying the theory of wavelets. Wavelet sets are measurable sets whose normalized characteristic functions are the Fourier Transforms of wavelets. Planar dyadic wavelet sets are interesting mathematically because they are fractal-like and there are hands-on methods of working with them and constructing new examples. They are also interesting because, while constructive characterizations are known, no algorithm is known for constructing all of them. There are open problems associated with their classification, and algorithms for constructing new examples or classes of examples can be useful for counterexamples to conjectures as well as for their intrinsic mathematical beauty.

A single dyadic *orthonormal wavelet* on the plane  $\mathbb{R}^2$  is a (Lebesgue) measurable square-integrable function  $\psi(x, y)$  whose translates along the coordinate axes followed by dilates by positive and negative integral powers of 2 generate an orthonormal basis for  $L^2(\mathbb{R}^2)$ . A planar dyadic *wavelet set*  $E$  is a measurable subset of  $\mathbb{R}^2$  with the property that the inverse Fourier Transform of the normalized characteristic function  $\frac{1}{2\pi}\chi(E)$  of  $E$  is a single dyadic orthonormal wavelet. As usual, the Fourier transform on  $\mathcal{L}^2(\mathbb{R}^2)$  is the tensor product of two copies of the Fourier transform on  $\mathcal{L}^2(\mathbb{R})$ . In this paper we discuss two algorithms which generate two distinct uncountably infinite classes of dyadic orthonormal wavelet sets in  $\mathbb{R}^2$ . We

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denote these classes the *crossover* and *patch* classes and denote the algorithms for these constructions the crossover and patch algorithms. A free parameter in both of the algorithms is a partition of the so-called inner square,  $[-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$ , into four measurable subsets  $X_{\ominus}, X_{\oplus}, Y_{\ominus}, Y_{\oplus}$ , with the property that  $X_{\ominus}$  is contained in the left half of the inner square,  $X_{\oplus}$  is contained in the right half,  $Y_{\ominus}$  is contained in the bottom half, and  $Y_{\oplus}$  is contained in the top half. Notice that if the boundary of any two of these sets is the same, if this boundary has Lebesgue measure 0, then these two sets are still essentially disjoint although their boundaries are the same.

Wavelets for dilations other than 2 (the dyadic case) on the line  $\mathbb{R}$  and in  $\mathbb{R}^n$  have been investigated by many authors. In higher dimensions both scalar dilations and matrix dilations have been studied. However, much of the interesting work in the literature has been for the dyadic case, which is the case we focus on.

For completeness we give the form used for abstract matrix dilations: A dilation  $A$  wavelet is a function on  $\mathbb{R}^n$  whose dilations by integral powers of  $A$  and translations along the coordinate axes (or more generally, translations along some other so-called *full-rank lattice* form an orthonormal basis for the space of all square-integrable measurable functions over  $\mathbb{R}^n$  (with respect to Lebesgue measure). In precise terms, a function  $f$  on  $\mathbb{R}^n$  is a dilation  $A$  wavelet if and only if it is measurable with respect to product Lebesgue measure and  $\{|det A|^{\frac{m}{2}} f(A^m t - l) : m \in \mathbb{Z}, l \in \mathbb{Z}^n\}$  is an orthonormal basis of  $L^2(\mathbb{R}^n)$ . A dilation  $A$  wavelet set is a measurable set  $W$  for which the inverse Fourier transform of the normalized characteristic function is a dilation- $A$  orthonormal wavelet. The dyadic case is where  $A := 2I_2$ , where  $I_2$  is the identity matrix in two dimensions.

Existence of wavelet sets in  $\mathbb{R}^n$  was first demonstrated in 1994 [5]. The proof given used an algorithmic approach which generated wavelet sets that were unbounded and had 0 as a limit point, rendering them difficult to visualize [5, 15]. It showed that there are uncountably many such sets in  $\mathbb{R}^2$  for many matrix dilations, including the dyadic case. Subsequently, several authors [13, 6, 2, 3, 1, 8] constructed wavelet sets in  $\mathbb{R}^2$  which were more easily pictured due to their qualities of being bounded and bounded away from 0 and had other interesting structural properties. Two such sets were included in the final remarks section of [4]. Recent papers that construct new planar wavelet sets with reasonable graphics and interesting properties can be found in [15] and [12]. A brief history of wavelet sets can be found in section 5 of [15].

In the summer of 2007, the first three authors were undergraduate student participants in the Texas A&M Mathematics REU on *matrix analysis and wavelets* (funded by the NSF), which was mentored by D. Larson. They set out to classify multiple categories of wavelet sets in  $\mathbb{R}^2$  using an algorithmic approach. The present paper is the upshot of that project. Two algorithms were obtained. The wavelet sets in Figures 1 and 2 are called *crossover* wavelet sets because in their generation, regions are added to or subtracted from alternating sides of the inner square. Alternatively, *patch* wavelet sets are created by adding regions to the same side of the square for each translation, such as the set in Figure 3. This category of sets is so named because in computer networking a patch cable is the opposite of a crossover cable.

## 1. PRELIMINARIES

We begin with some basic formal definitions.

A *single dyadic orthonormal wavelet* is a function  $\psi \in \mathcal{L}^2(\mathbb{R})$  (Lebesgue measure) with the property that the set  $\{2^{\frac{n}{2}}\psi(2^n \cdot -l) \mid n, l \in \mathbb{Z}\}$  forms an orthonormal basis for  $\mathcal{L}^2(\mathbb{R})$  (c.f. [11, 14]). More generally, if  $A$  is any real invertible  $n \times n$  matrix, then a single function  $\psi \in L^2(\mathbb{R}^n)$  is a multivariate orthonormal wavelet for  $A$  if

$$\{|\det A|^{\frac{n}{2}}\psi(A^n \cdot -l) \mid n \in \mathbb{Z}, l \in \mathbb{Z}^{(n)}\}$$

is an orthonormal basis of  $L^2(\mathbb{R}^n)$ . It was shown in [5] that if  $A$  is *expansive* (equivalently, all eigenvalues of  $A$  are required to have absolute value strictly greater than 1) then orthonormal wavelets for  $A$  always exist. The dyadic case is the case where  $A := 2I_2$  (two times the identity matrix on  $\mathbb{R}^n$ ). This is the simplest (and most investigated) case.

We let  $\mathcal{F}$  denote the  $n$ -dimensional Fourier transform on  $\mathcal{L}^2(\mathbb{R}^n)$  defined by

$$(\mathcal{F}f)(s) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-s \circ t} f(t) dm$$

for all  $f \in L^2(\mathbb{R}^n)$ . Here,  $s \circ t$  denotes the real inner product. A measurable set  $E \subseteq \mathbb{R}^n$  is defined to be a *wavelet set* for a dilation matrix  $A$  if

$$\mathcal{F}^{-1}\left(\frac{1}{\sqrt{\mu(E)}}\chi_E\right)$$

is an orthonormal wavelet for  $A$ , where  $\mathcal{F}^{-1}$  denotes the inverse Fourier Transform.

In this paper we will not explicitly use properties of  $\mathcal{F}$  and  $\mathcal{F}^{-1}$ ; however we state the formal definition of Fourier Transform because it is needed to give the proper definition of a wavelet set. The Fourier Transform is a unitary transform from  $\mathcal{L}(\mathbb{R}^n)$ , where  $\mathbb{R}^n$  is usually considered as a multivariate time domain, to another copy of  $\mathcal{L}(\mathbb{R}^n)$ , where  $\mathbb{R}^n$  is considered a multivariate *frequency* domain. We will do our work with wavelet sets entirely in the frequency domain. We can do this because there is a set-theoretic characterization of wavelet sets, the *Working Principle Criterion* (see Prop. 1.1 below), which allows one to construct and otherwise work with wavelet sets without using the Fourier Transform.

A sequence of measurable subsets  $\{E_n\}$  of a measurable set  $E$  is called a *measurable partition* of  $E$  if the relative complement of  $\bigcup_n E_n$  in  $E$  is a null set (i.e. has measure zero) and  $E_n \cap E_m$  is a null set whenever  $n \neq m$ .

Measurable subsets  $E$  and  $F$  of  $\mathbb{R}$  are called  *$2\pi$ -translation congruent* to each other, denoted by  $E \sim_{2\pi} F$ , if there exists a measurable partition  $\{E_n\}$  of  $E$ , such that  $\{E_n + 2n\pi\}$  is a measurable partition of  $F$ . Similarly,  $E$  and  $F$  are called *2-dilation congruent* to each other, denoted by  $E_2 \sim F$ , if there is a measurable partition  $\{E_n\}$  of  $E$ , such that  $\{2^n E_n\}$  is a measurable partition of  $F$ . A measurable set  $E$  is called a  *$2\pi$ -translation generator of a measurable partition of  $\mathbb{R}$*  if  $\{E + 2n\pi\}_{n \in \mathbb{Z}}$  forms a measurable partition of  $\mathbb{R}$ . Similarly, a measurable set  $F$  is called a *2-dilation generator of a measurable partition of  $\mathbb{R}$*  if  $\{2^n F\}_{n \in \mathbb{Z}}$  forms a measurable partition of  $\mathbb{R}$ .

Lemma 4.3 in [4] gives the following characterization of dyadic wavelet sets in  $\mathbb{R}$ , which was also obtained independently in [7] using different techniques. *Let  $E \subseteq \mathbb{R}$  be a measurable set. Then  $E$  is a dyadic wavelet set if and only if  $E$  is both a  $2\pi$ -translation generator of a measurable partition of  $\mathbb{R}$  and a 2-dilation generator of a measurable partition of  $\mathbb{R}$ .*

In [5] this criterion was generalized to the multivariate case. We will consider only the dyadic planar case in this paper because we will only use the criterion

for that case, although the criterion actually goes through for the arbitrary expansive case [5, 6]. So from [5] we have that  $E$  is a dyadic wavelet set in  $\mathbb{R}^n$  if and only if  $E$  is both a  $2\pi$ -translation generator of a measurable partition of  $\mathbb{R}^n$  and a 2-dilation generator of a measurable partition of  $\mathbb{R}^n$ . Here, to achieve a translation partition one translates by all integral multiples of  $2\pi$  separately in each coordinate direction. To achieve a dilation partition one dilates by all integral powers of 2 *simultaneously* in all coordinates. For example, it is clear that the set  $[-\pi, \pi) \times [-\pi, \pi)$  is a  $2\pi$ -translation generator of a measurable partition of  $\mathbb{R}^2$  and  $G_{TO} \setminus ([-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2}))$  is a 2-dilation generator of a measurable partition of  $\mathbb{R}^2$ .

Properly generalizing the one dimensional definition to the planar case, we say that two Lebesgue measurable sets  $A, B \subset \mathbb{R}^2$  are  $2\pi$ -translation congruent if there is a measurable partition  $\{A_{k,l} : k, l \in \mathbb{Z}\}$  of  $A$  such that  $\{A_{k,l} + \begin{bmatrix} 2k\pi \\ 2l\pi \end{bmatrix} : k, l \in \mathbb{Z}\}$  is a measurable partition of  $B$ , and they are 2-dilation congruent if there is a measurable partition  $\{A_n : n \in \mathbb{Z}\}$  of  $A$  such that  $\{2^n A_n : n \in \mathbb{Z}\}$  is a measurable partition of  $B$ .

Translation congruence and dilation congruence are both equivalence relations on the class of all measurable subsets. If a set  $A$  is  $2\pi$ -translation congruent to a  $2\pi$ -translation generator of a measurable partition of  $\mathbb{R}^2$ , it is clear that  $A$  itself is a  $2\pi$ -translation generator of a measurable partition of  $\mathbb{R}^2$ . Moreover, sets  $A$  and  $B$  which are both  $2\pi$ -translation generators of measurable partitions of  $\mathbb{R}^2$  are necessarily translation congruent to each other. All this is in [5], and other expositions can be found in [4, 10, 15]. This yields a useful criterion.

In the following proposition (and in the rest of the paper), let  $G_{TO} := [-\pi, \pi) \times [-\pi, \pi)$  and  $G_{SO} := G_{TO} \setminus ([-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2}))$ .

**Proposition 1.1.** (*Working Principle Criterion*) *A measurable set  $W \subseteq \mathbb{R}^2$  is a dyadic wavelet set if and only if  $W$  is  $2\pi$ -translation congruent to  $G_{TO}$  and 2-dilation congruent to  $G_{SO}$ .*

## 2. THE CROSSOVER ALGORITHM

We first consider a special case of a wavelet set to illustrate the *crossover algorithm*. We will then generalize this to obtain Theorem 2.1.

**Example 2.1.** *Let*

$$\begin{aligned} X_{\ominus} &= \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ X_{\oplus} &= \left[\frac{\pi}{4}, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ Y_{\ominus} &= \left[-\frac{\pi}{4}, \frac{\pi}{4}\right) \times \left[-\frac{\pi}{2}, 0\right), \\ Y_{\oplus} &= \left[-\frac{\pi}{4}, \frac{\pi}{4}\right) \times \left[0, \frac{\pi}{2}\right). \end{aligned}$$

We can generate a wavelet set in the plane using the above partition of the inner square using an algorithm (the crossover algorithm) which we will illustrate using this example.

Let  $X_{\ominus 1} := X_{\ominus}$ . Start by adding the vector  $\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$  to the set  $X_{\ominus}$ , translating it (i.e. *crossing it over*) to the right half-plane. The set formed is  $[\frac{3\pi}{2}, \frac{7\pi}{4}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ . Call this  $X_{\ominus 2}$ . Secondly, scale  $X_{\ominus 2}$  by  $\frac{1}{2}$  to obtain  $[\frac{3\pi}{4}, \frac{7\pi}{8}] \times [-\frac{\pi}{4}, \frac{\pi}{4}]$ . Call this  $X_{\ominus 3}$ . Thirdly, translate  $X_{\ominus 3}$  to the opposite half-plane by adding  $\begin{bmatrix} -2\pi \\ 0 \end{bmatrix}$  to obtain  $[-\frac{5\pi}{4}, -\frac{9\pi}{8}] \times [-\frac{\pi}{4}, \frac{\pi}{4}]$ , and call this set  $X_{\ominus 4}$ . Notice that  $X_{\ominus 4}$  is on the same side half-plane as  $X_{\ominus}$ . Finally, scale  $X_{\ominus 4}$  by  $\frac{1}{2}$  to form the set  $[-\frac{5\pi}{8}, -\frac{9\pi}{16}] \times [-\frac{\pi}{8}, \frac{\pi}{8}]$  and call it  $X_{\ominus 5}$ . Continue these four steps inductively for  $X_{\ominus}$ .

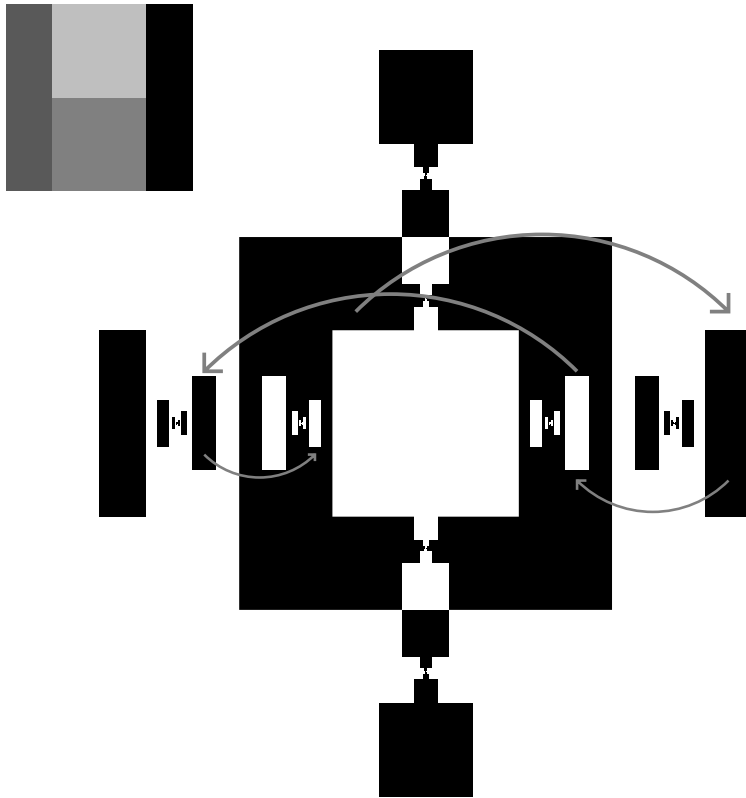


FIGURE 1. The 2-dimensional wavelet set formed in example 1.

We perform four similar steps on the set  $X_{\oplus}$ ; however, we translate by  $\begin{bmatrix} -2\pi \\ 0 \end{bmatrix}$  for the first step (instead of  $\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$ ) and translate by  $\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$  for the third step (instead of  $\begin{bmatrix} -2\pi \\ 0 \end{bmatrix}$ ). We obtain the following from the first four steps:

$$\begin{aligned} X_{\oplus 2} &= \left[-\frac{7\pi}{4}, -\frac{3\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ X_{\oplus 3} &= \left[-\frac{7\pi}{8}, -\frac{3\pi}{4}\right) \times \left[-\frac{\pi}{4}, \frac{\pi}{4}\right), \\ X_{\oplus 4} &= \left[\frac{9\pi}{8}, \frac{5\pi}{4}\right) \times \left[-\frac{\pi}{4}, \frac{\pi}{4}\right), \\ X_{\oplus 5} &= \left[\frac{9\pi}{16}, \frac{5\pi}{8}\right) \times \left[-\frac{\pi}{8}, \frac{\pi}{8}\right). \end{aligned}$$

Continue this process inductively for  $X_{\oplus}$  as well. Perform similar steps for  $Y_{\oplus}$  and  $Y_{\ominus}$ , translating by  $\begin{bmatrix} 0 \\ \pm 2\pi \end{bmatrix}$  instead of  $\begin{bmatrix} \pm 2\pi \\ 0 \end{bmatrix}$ , beginning with a translation of  $\begin{bmatrix} 0 \\ 2\pi \end{bmatrix}$  for  $Y_{\ominus}$  and a translation of  $\begin{bmatrix} 0 \\ -2\pi \end{bmatrix}$  for  $Y_{\oplus}$ .

Let  $W$  be the set

$$\begin{aligned} &\left(\bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}]\right) \cup \left(G_{TO} \setminus \left[\bigcup_{i=1}^{\infty} [X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}]\right]\right) \\ &= \left(\bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}]\right) \cup \left(G_{SO} \setminus \left[\bigcup_{i=2}^{\infty} [X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}]\right]\right). \end{aligned}$$

We can think of  $W$  as being the union of  $G_{TO}$  and the exterior “black pieces” of the form  $X_{\oplus 2n}$ ,  $X_{\ominus 2n}$ ,  $Y_{\oplus 2n}$ ,  $Y_{\ominus 2n}$ , with “white spaces” of the form  $X_{\oplus 2n+1}$ ,  $X_{\ominus 2n+1}$ ,  $Y_{\oplus 2n+1}$ ,  $Y_{\ominus 2n+1}$  removed from  $G_{TO}$ .

This set  $W$  (see Figure 1) is a wavelet set. To see this, let  $G(X_{\ominus odd}) := \bigcup_{i=1}^{\infty} X_{\ominus 2i-1}$  and  $G(X_{\ominus even}) := \bigcup_{i=1}^{\infty} X_{\ominus 2i}$ . Observe that  $G(X_{\ominus odd}) \setminus X_{\ominus} \subset G_{SO}$  and  $G(X_{\ominus even}) \not\subset G_{SO}$ .

Similarly, define sets for  $X_{\oplus}$ ,  $Y_{\ominus}$ ,  $Y_{\oplus}$  with analogous characteristics. Observe that  $W$  is translation congruent to  $G_{TO}$  modulo  $2\pi$  because  $[(\bigcup_{i=0}^{\infty} X_{\ominus 4i+2}) \cup (\bigcup_{i=1}^{\infty} X_{\oplus 4i})] - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = [(\bigcup_{i=0}^{\infty} X_{\ominus 4i+1}) \cup (\bigcup_{i=0}^{\infty} X_{\oplus 4i+3})]$ ,  $[(\bigcup_{i=0}^{\infty} X_{\oplus 4i+2}) \cup (\bigcup_{i=1}^{\infty} X_{\ominus 4i})] + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = [(\bigcup_{i=0}^{\infty} X_{\oplus 4i+1}) \cup (\bigcup_{i=0}^{\infty} X_{\ominus 4i+3})]$ , and  $[(\bigcup_{i=0}^{\infty} X_{\oplus 4i+1}) \cup (\bigcup_{i=0}^{\infty} X_{\ominus 4i+3})] \cup [(\bigcup_{i=0}^{\infty} X_{\ominus 4i+1}) \cup (\bigcup_{i=0}^{\infty} X_{\oplus 4i+3})] = G(X_{\ominus odd}) \cup G(X_{\oplus odd})$ , so that all gaps in the set  $G_{TO}$  due to the crossover algorithm applied to the sets  $X_{\ominus 1}$  and  $X_{\oplus 1}$  are filled when we translate the black sets formed by the

crossover algorithm applied to the sets  $X_{\ominus 1}$  and  $X_{\oplus 1}$  by multiples of  $\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$ . Similar results will apply for  $Y_{\ominus}$  and  $Y_{\oplus}$ .

Moreover,  $W$  is dilation congruent to  $G_{SO}$  because  $\frac{1}{2}G(X_{\ominus even}) = G(X_{\ominus odd}) \in G_{SO}$  (i.e., the even pieces of the form  $X_{\ominus n}$  scale into the odd pieces of the form  $X_{\ominus n}$ ), with similar results for  $G(X_{\oplus even}), G(Y_{\ominus even}),$  and  $G(Y_{\oplus even})$ .

The set of steps indicated above applied to all four pieces of the partition of the inner square is a special case of the *crossover algorithm*. An uncountably infinite family of wavelet sets in  $\mathbb{R}^2$  can be similarly constructed using a natural generalization of this. This *general* crossover algorithm will be presented rigorously in the later sections in the context of the the proof of Theorem 1 and the constructions involved in the proof.

A brief description of the general crossover algorithm is the following:

- (1) Partition the inner square into a maximum of four subsets satisfying the conditions given in the statement of Theorem 1 below. (These conditions are necessary because not all partitions of the inner square will lead to a wavelet set in this way.)
- (2) Translate one piece of the partition by  $\begin{bmatrix} \pm 2\pi \\ 0 \end{bmatrix}$  or  $\begin{bmatrix} 0 \\ \pm 2\pi \end{bmatrix}$ , moving it out of the inner square to the opposite side of the  $x$ - or  $y$ -axis.
- (3) Dilate the set formed in step 2 into  $G_{SO}$  by  $\frac{1}{2}$ .
- (4) Translate the set formed in step 3 in the opposite direction (compared to the first translation), i.e., by  $\begin{bmatrix} \mp 2\pi \\ 0 \end{bmatrix}$  or  $\begin{bmatrix} 0 \\ \mp 2\pi \end{bmatrix}$ .
- (5) Dilate the set formed in step 4 into  $G_{SO}$  by  $\frac{1}{2}$ .
- (6) Repeat these steps inductively for this piece, and perform steps 2–5 on the other pieces of the partition inductively as well.  $\blacklozenge$

**Theorem 2.1.** (*Crossover Algorithm*) *Let  $\{X_{\ominus}, X_{\oplus}, Y_{\ominus}, Y_{\oplus}\}$  be a partition of the set  $[-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$  such that  $X_{\ominus}$  is contained in the left half of the inner square (i.e., the maximum possible  $x$ -coordinate of any point in the set  $X_{\ominus}$  is 0),  $X_{\oplus}$  is contained in the right half of the inner square (i.e., the minimum possible  $x$ -coordinate of any point in the set  $X_{\oplus}$  is 0),  $Y_{\ominus}$  is contained in the bottom half of the inner square (i.e., the maximum possible  $y$ -coordinate of the set  $Y_{\ominus}$  is 0), and  $Y_{\oplus}$  is contained in the top half of the inner square (i.e., the minimum possible  $y$ -coordinate of the set  $Y_{\oplus}$  is 0). Then the set  $W$  generated by this partition, under translation by  $\begin{bmatrix} \pm 2\pi \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ \pm 2\pi \end{bmatrix}$  and dilation by powers of 2 using steps (1)-(6) above, defined as*

$$\left[ \left( \bigcup_{i=1}^{\infty} [X_{\ominus 2^i} \cup X_{\oplus 2^i} \cup Y_{\ominus 2^i} \cup Y_{\oplus 2^i}] \right) \cup G_{TO} \right] \setminus \left[ \bigcup_{i=1}^{\infty} [X_{\ominus 2^{i-1}} \cup X_{\oplus 2^{i-1}} \cup Y_{\ominus 2^{i-1}} \cup Y_{\oplus 2^{i-1}}] \right],$$

is a dyadic wavelet set in  $\mathbb{R}^2$ .

**Remark 2.1.** *If either both  $X_{\oplus}$  and  $X_{\ominus}$  are defined or both  $Y_{\oplus}$  and  $Y_{\ominus}$  are defined, then the other two sets are automatically determined due to our constraints.*

3. EXPRESSIONS FOR  $X_{\ominus n}, X_{\oplus n}, Y_{\ominus n}$  AND  $Y_{\oplus n}$ 

Before proving our main result, Theorem 2.1, we first give rigorous expressions for the sets  $X_{\ominus n}, X_{\oplus n}, Y_{\ominus n}$  and  $Y_{\oplus n}$  that we will need. We begin with the sets of the form  $X_{\ominus n}$ . Suppose first that  $n$  is odd and  $n \geq 3$ . We can derive the formula for  $X_{\ominus n}$  in terms of  $n$  by looking at the first few terms. Let  $X_{\ominus 1} := X_{\ominus}$ , which has the above constraints listed according to the theorem. We can then find the next few odd terms using the crossover algorithm described in the example.

$$\begin{aligned} X_{\ominus 3} &= \frac{1}{2} \left( X_{\ominus} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \right), \\ X_{\ominus 5} &= \frac{1}{2} \left( X_{\ominus 3} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \right) = \left( \frac{X_{\ominus}}{4} + \frac{1}{4} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \right), \\ X_{\ominus 7} &= \frac{1}{2} \left( X_{\ominus 5} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \right) = \left( \frac{X_{\ominus}}{8} + \frac{1}{8} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \right), \end{aligned}$$

and in general we find that  $X_{\ominus n+2} = \frac{1}{2} \left( X_{\ominus n} + (-1)^{\frac{n-1}{2}} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \right)$ . Solving this recurrence relation we find that

$$\begin{aligned} X_{\ominus n} &= \frac{X_{\ominus}}{2^{\frac{n-1}{2}}} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \left( \frac{1}{2^{\frac{n-1}{2}}} - \dots + \frac{(-1)^{\frac{n-3}{2}}}{2} \right) \\ &= \frac{X_{\ominus}}{2^{\frac{n-1}{2}}} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n-3}{2}} \left( \frac{(-1)^{\frac{n-3}{2}}}{2^{\frac{n-1}{2}}} + \frac{(-1)^{\frac{n-3}{2}-1}}{2^{\frac{n-1}{2}-1}} + \dots - \frac{1}{4} + \frac{1}{2} \right) \\ &= \frac{X_{\ominus}}{2^{\frac{n-1}{2}}} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n-3}{2}} \sum_{i=1}^{\frac{n-1}{2}} \frac{(-1)^{i-1}}{2^i} \\ &= \frac{X_{\ominus}}{2^{\frac{n-1}{2}}} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \frac{1}{3} (-1)^{\frac{n-3}{2}} \left( 1 - \left( -\frac{1}{2} \right)^{\frac{n-1}{2}} \right) \end{aligned}$$

using the formula for a geometric series summation. In order to formally verify that our formula for  $X_{\ominus n}$  solves the recurrence relation, we merely plug in the expressions for  $X_{\ominus n+2}$  and  $X_{\ominus n}$  and carry out basic computations.

Now suppose  $n'$  is even and  $n' > 2$ . We can derive the formula for  $X_{\ominus n'}$  in terms of  $n'$  similarly.

$$\begin{aligned}
 X_{\ominus 2} &= X_{\ominus} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}, \\
 X_{\ominus 4} &= \frac{X_{\ominus 2}}{2} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \frac{X_{\ominus}}{2} + \frac{1}{2} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}, \\
 X_{\ominus 6} &= \frac{X_{\ominus 4}}{2} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \frac{X_{\ominus}}{4} + \frac{1}{4} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}, \\
 X_{\ominus 8} &= \frac{X_{\ominus 6}}{2} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \frac{X_{\ominus}}{8} + \frac{1}{8} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix},
 \end{aligned}$$

and in general we find that  $X_{\ominus n'+2} = \frac{1}{2}X_{\ominus n'} + (-1)^{\frac{n'}{2}} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$ . Observe that in general  $\frac{X_{\ominus n'}}{2} = X_{\ominus n'+1}$  is true in general based on our construction using the crossover algorithm since  $n'$  is even. Since  $n' + 1$  is odd, we plug into our formula for  $X_{\ominus n}$  where  $n$  is odd to find  $X_{\ominus n'}$ .

$$\begin{aligned}
 X_{\ominus n'} &= 2X_{\ominus n'+1} = 2 \left[ \frac{X_{\ominus}}{2^{\frac{n'}{2}}} + \frac{1}{3} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n'-2}{2}} \left( 1 - \left( \frac{-1}{2} \right)^{\frac{n'}{2}} \right) \right] \\
 &= \left[ \frac{X_{\ominus}}{2^{\frac{n'-2}{2}}} + \frac{2}{3} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n'-2}{2}} \left( 1 - \left( \frac{-1}{2} \right)^{\frac{n'}{2}} \right) \right].
 \end{aligned}$$

Once again, the proof that the recurrence relation is satisfied involves plugging in the expressions for  $X_{\ominus n'+2}$  and  $X_{\ominus n'}$  and performing basic computations.

Notice that  $X_{\ominus n'} = X_{\ominus n'-1} + (-1)^{\frac{n'-2}{2}} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$  when  $n'$  is even, consistent with the crossover algorithm because of the following:

$$\begin{aligned}
 X_{\ominus n'} &= X_{\ominus n'-1} + (-1)^{\frac{n'-2}{2}} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \frac{X_{\ominus}}{2^{\frac{n'-2}{2}}} + \frac{1}{3} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n'-4}{2}} \left( 1 - \left( \frac{-1}{2} \right)^{\frac{n'-2}{2}} \right) + (-1)^{\frac{n'-2}{2}} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \\
 &= \frac{X_{\ominus}}{2^{\frac{n'-2}{2}}} + \frac{2}{3} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n'-2}{2}} \left( \frac{-1}{2} - \left( \frac{-1}{2} \right)^{\frac{n'}{2}} \right) + (-1)^{\frac{n'-2}{2}} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \\
 &= \frac{X_{\ominus}}{2^{\frac{n'-2}{2}}} + \frac{2}{3} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n'-2}{2}} \left( 1 - \left( \frac{-1}{2} \right)^{\frac{n'}{2}} \right).
 \end{aligned}$$

Thus, we can say in general that

$$X_{\ominus n} = \begin{cases} \frac{X_{\ominus}}{2^{\frac{n-1}{2}}} + \frac{1}{3} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n-3}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right), & \text{for } n \text{ odd} \\ \frac{X_{\ominus}}{2^{\frac{n'-2}{2}}} + \frac{2}{3} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n'-2}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n'}{2}}\right), & \text{for } n \text{ even} \end{cases},$$

but then clearly

$$X_{\oplus n} = \begin{cases} \frac{X_{\oplus}}{2^{\frac{n-1}{2}}} - \frac{1}{3} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n-3}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right), & \text{for } n \text{ odd} \\ \frac{X_{\oplus}}{2^{\frac{n'-2}{2}}} - \frac{2}{3} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n'-2}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n'}{2}}\right), & \text{for } n \text{ even} \end{cases}.$$

Analogously, we find

$$Y_{\ominus n} = \begin{cases} \frac{Y_{\ominus}}{2^{\frac{n-1}{2}}} + \frac{1}{3} \begin{bmatrix} 0 \\ 2\pi \end{bmatrix} (-1)^{\frac{n-3}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right), & \text{for } n \text{ odd} \\ \frac{Y_{\ominus}}{2^{\frac{n'-2}{2}}} + \frac{2}{3} \begin{bmatrix} 0 \\ 2\pi \end{bmatrix} (-1)^{\frac{n'-2}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n'}{2}}\right), & \text{for } n \text{ even} \end{cases},$$

and

$$Y_{\oplus n} = \begin{cases} \frac{Y_{\oplus}}{2^{\frac{n-1}{2}}} - \frac{1}{3} \begin{bmatrix} 0 \\ 2\pi \end{bmatrix} (-1)^{\frac{n-3}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right), & \text{for } n \text{ odd} \\ \frac{Y_{\oplus}}{2^{\frac{n'-2}{2}}} - \frac{2}{3} \begin{bmatrix} 0 \\ 2\pi \end{bmatrix} (-1)^{\frac{n'-2}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n'}{2}}\right), & \text{for } n \text{ even} \end{cases}$$

Comment: By our construction we have (analogously to the properties for  $X_{\ominus n}$ ) that for  $n'$  even,  $\frac{X_{\oplus n'}}{2} = X_{\oplus n'+1}$ ,  $\frac{Y_{\oplus n'}}{2} = Y_{\oplus n'+1}$ , and  $\frac{Y_{\ominus n'}}{2} = Y_{\ominus n'+1}$ . Moreover,

$$X_{\oplus n'} = X_{\oplus n'-1} - (-1)^{\frac{n'-2}{2}} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}, Y_{\ominus n'} = Y_{\ominus n'-1} + (-1)^{\frac{n'-2}{2}} \begin{bmatrix} 0 \\ 2\pi \end{bmatrix}, \text{ and } Y_{\oplus n'} = Y_{\oplus n'-1} - (-1)^{\frac{n'-2}{2}} \begin{bmatrix} 0 \\ 2\pi \end{bmatrix}.$$

#### 4. PROOF OF THEOREM 2.1

For the proof of Theorem 1 we will require three technical lemmas.

**Lemma 4.1.** *For all odd  $n \geq 3$ ,  $X_{\ominus n} \subseteq G_{SO}$ .*

*Proof.* Since all such  $X_{\ominus} \subseteq [-\frac{\pi}{2}, 0) \times [-\frac{\pi}{2}, \frac{\pi}{2})$  by definition, for all functions  $f$ ,  $f(X_{\ominus}) \subseteq f\left([-\frac{\pi}{2}, 0) \times \left[\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$ . Therefore, we only need to prove that for all odd  $n \geq 3$ ,  $\left[[-\frac{\pi}{2}, 0) \times \left[\frac{\pi}{2}, \frac{\pi}{2}\right)\right]_{\ominus n} \subseteq G_{SO}$ . Let  $S_{\ominus n} := \left[[-\frac{\pi}{2}, 0) \times \left[\frac{\pi}{2}, \frac{\pi}{2}\right)\right]_{\ominus n}$  represent the result of the  $n^{\text{th}}$  step of the crossover algorithm applied to  $[-\frac{\pi}{2}, 0) \times \left[\frac{\pi}{2}, \frac{\pi}{2}\right)$ .

Notice that  $S_{\ominus n}$  is of the form of the more general set  $X_{\ominus n}$ , and therefore we can use our derived bounds (see above on page 10) for  $X_{\ominus n}$  in terms of  $n$  to determine the bounds for  $S_{\ominus n}$ .

We begin by showing that  $S_{\ominus n} \subseteq [-\pi, \pi) \times [-\pi, \pi)$ . That this is satisfied for the vertical bounds of  $S_{\ominus n}$  is clear, so we will only consider the horizontal bounds. Note that when we use the phrase ‘‘vertical bound,’’ we refer to both upper and lower bounds. By a vertical upper bound, we mean to say that such a number is greater than or equal to all  $y$ -coordinates of the points in that set. When we use the word ‘‘horizontal bound of a set,’’ we refer to both left and right hand bounds of a set. By left hand bound, we mean to signify a number that is less than or equal to all of the horizontal coordinates of the points in that set.

Case 1:  $n = 4k + 1$  for some  $k \in \mathbb{Z}$ . Then  $S_{\ominus n}$  is on the left hand side of the origin. Thus the horizontal left hand bound (LHB) for  $S_{\ominus n}$  is  $\frac{-1}{2^{\frac{n-1}{2}}} \begin{bmatrix} \pi \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n-3}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right)$ , and we must show it is bounded below by  $\begin{bmatrix} -\pi \\ 0 \end{bmatrix}$ . In the  $x$ -coordinate,

$$\begin{aligned} -\pi &\leq \frac{-\pi}{2^{\frac{n-1}{2}}} + \frac{2\pi}{3} (-1)^{\frac{n-3}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right) \\ \iff 1 &\geq \frac{1}{2^{2k}} - \frac{2}{3} (-1)^{2k-1} \left(1 - \left(-\frac{1}{2}\right)^{2k}\right) \\ \iff 1 &\geq \frac{1}{2^{2k}} + \frac{2}{3} \left(1 - \frac{1}{2^{2k}}\right) \\ \iff \frac{1}{3} &\geq \frac{1}{2^{2k}} \left(1 - \frac{2}{3}\right) \\ \iff 1 &\geq \frac{1}{2^{2k}}, \end{aligned}$$

which is clearly true because  $3 \leq n = 4k + 1 \Rightarrow \frac{1}{2} \leq k$  and  $k \in \mathbb{Z}$ , so  $1 \leq k$ . Moreover, since the LHB on the  $x$ -coordinates of  $S_{\ominus n}$  is less than the horizontal right hand bound (RHB), we know that  $-\pi < \text{RHB}$ .

Case 2:  $n = 4k + 3$  for some  $k \in \mathbb{Z}$ . Then  $S_{\ominus n}$  is on the right hand side of the origin. Therefore, we want to show that the horizontal RHB on  $S_{\ominus n}$  is less than or equal to  $\pi$ , i.e. that

$$\begin{aligned}
& \frac{2\pi}{3}(-1)^{\frac{n-3}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right) \leq \pi \\
& \iff \frac{2}{3}(-1)^{2k} \left(1 - \left(-\frac{1}{2}\right)^{2k+1}\right) \leq 1 \\
& \iff \frac{2}{3} \left(1 + \left(\frac{1}{2}\right)^{2k+1}\right) \leq 1 \\
& \iff \left(1 + \left(\frac{1}{2}\right)^{2k+1}\right) \leq \frac{3}{2} \\
& \iff \left(\frac{1}{2}\right)^{2k} \leq 1,
\end{aligned}$$

which is trivially true since  $n$  is odd and  $n \geq 3 \Rightarrow 3 \leq 4k + 3 \Rightarrow 0 \leq k$ . We know that in this case,  $\text{LHB} \leq \text{RHB} \leq \pi$ , as needed. But then in all possible cases, it is true that  $S_{\ominus n} \subseteq [-\pi, \pi) \times [-\pi, \pi)$ , and therefore that  $X_{\ominus n} \subseteq [-\pi, \pi) \times [-\pi, \pi)$ .

Now we want to show that  $X_{\ominus n} \not\subseteq \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$  for all  $n \geq 3$ . Recall from our earlier discussion that this will follow from the fact that  $S_{\ominus n} \not\subseteq \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$ . We can prove this fact by merely showing that the horizontal bounds on the set  $S_{\ominus n}$  are not contained in the set  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ . Thus, the vertical bounds on the set  $S_{\ominus n}$  are irrelevant.

Case 1:  $n = 4k + 1$  for some  $k \in \mathbb{Z}$ . Recall  $S_{\ominus n}$  is on the left hand side of the origin, so we must show that the horizontal RHB on the  $x$ -coordinates of the set  $S_{\ominus n}$  is less than or equal to  $-\frac{\pi}{2}$ .

$$\begin{aligned}
-\frac{\pi}{2} & \geq \frac{2\pi}{3}(-1)^{\frac{n-3}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right) \\
& \iff -\frac{1}{2} \geq \frac{2}{3}(-1)^{2k-1} \left(1 - \left(-\frac{1}{2}\right)^{2k}\right) \\
& \iff -\frac{1}{2} \geq \frac{2}{3} \left(\frac{1}{2^{2k}} - 1\right) \\
& \iff \frac{1}{4} \geq \frac{1}{2^{2k}},
\end{aligned}$$

which is true since  $3 \leq n = 4k + 1 \Rightarrow \frac{1}{2} \leq k$  and  $k \in \mathbb{Z}$  (so  $\Rightarrow 1 \leq k$ ).

Case 2:  $n = 4k + 3$  for some  $k \in \mathbb{Z}$ . Then  $S_{\ominus n}$  is on the right hand side of the origin, and therefore we want to show that the horizontal LHB on  $S_{\ominus n}$  is greater than or equal to  $\frac{\pi}{2}$ .

$$\begin{aligned}
 \frac{\pi}{2} &\leq \frac{-\pi}{2^{\frac{n+1}{2}}} + \frac{2\pi}{3}(-1)^{\frac{n-3}{2}} \left(1 - \left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right) \\
 &\iff \frac{1}{2} \leq \frac{-1}{2^{2k+2}} + \frac{2}{3}(-1)^{2k} \left(1 - \left(-\frac{1}{2}\right)^{2k+1}\right) \\
 &\iff \frac{-1}{6} \leq \frac{1}{2^{k+1}} \left(\frac{2}{3} - \frac{1}{2}\right) \\
 &\iff -1 \leq \frac{1}{2^{k+1}},
 \end{aligned}$$

which is trivially true since  $0 < \frac{1}{2^{k+1}} \forall k$ .

Thus,  $\forall n$ , the horizontal bounds of  $S_{\ominus n}$  are not contained in the set  $[\frac{-\pi}{2}, \frac{\pi}{2}]$  but are contained in the set  $[-\pi, \pi]$ , and the vertical bounds are contained in the set  $[-\pi, \pi]$ . Thus  $S_{\ominus n} \subseteq G_{SO}$  for all odd  $n \geq 3$ . Recall from our earlier discussion that it therefore follows that  $X_{\ominus n} \subseteq G_{SO}$  for all odd  $n \geq 3$ , as needed.  $\square$

Analogously, for all odd  $n \geq 3$ ,  $X_{\oplus n} \subseteq G_{SO}$ ,  $Y_{\ominus n} \subseteq G_{SO}$ , and  $Y_{\oplus n} \subseteq G_{SO}$ .

**Lemma 4.2.**  $X_{\ominus n+4}$  and  $X_{\ominus n}$  are disjoint for all  $n > 1 \in \mathbb{Z}$ .

*Proof.* Let  $S_{\ominus n} := [[\frac{-\pi}{2}, 0) \times [\frac{-\pi}{2}, \frac{\pi}{2}]]_{\ominus n}$  represent the result of the  $n^{\text{th}}$  step of the crossover algorithm applied to  $[\frac{-\pi}{2}, 0) \times [\frac{-\pi}{2}, \frac{\pi}{2}]$ . Since all  $X_{\ominus} \subseteq [\frac{-\pi}{2}, 0) \times [\frac{-\pi}{2}, \frac{\pi}{2}]$ , for all functions  $f$ ,  $f(X_{\ominus n}) \subseteq f([\frac{-\pi}{2}, 0) \times [\frac{-\pi}{2}, \frac{\pi}{2}])$ , and therefore  $X_{\ominus n} \subseteq S_{\ominus n}$ . Therefore we will prove that all  $S_{\ominus n}$  and  $S_{\ominus(n+4)}$  are disjoint, from which our lemma follows immediately.

Consider odd  $n$  ( $n = 2k + 1$  for some  $k \in \mathbb{Z}$ ). The horizontal LHB of  $S_{\ominus(2k+1)}$  is  $\frac{-\pi}{2^{k+1}} + \frac{2\pi}{3}(-1)^{k-1} \left(1 - \left(\frac{-1}{2}\right)^k\right)$  (as follows from the rigorous definition of the set  $X_{\ominus n}$ ), and the RHB of  $S_{\ominus(2k+5)}$  is  $\frac{2\pi}{3}(-1)^{k+1} \left(1 - \left(\frac{-1}{2}\right)^{k+2}\right)$ . Therefore, the RHB of  $S_{\ominus(2k+5)}$  equals the LHB of  $S_{\ominus(2k+1)}$  iff

$$\begin{aligned}
 \frac{-\pi}{2^{k+1}} + \frac{2\pi}{3}(-1)^{k-1} \left(1 - \left(\frac{-1}{2}\right)^k\right) &= \frac{2\pi}{3}(-1)^{k+1} \left(1 - \left(\frac{-1}{2}\right)^{k+2}\right) \\
 \iff \frac{-1}{2^{k+1}} + \frac{2}{3}(-1)^{k-1} + \frac{2}{3} \frac{1}{(2)^k} &= \frac{2}{3}(-1)^{k+1} + \frac{2}{3} \frac{1}{(2)^{k+2}} \\
 \iff \frac{-1}{2} \frac{1}{2^k} = \frac{2}{3} \frac{1}{2^k} \left(\frac{1}{4} - 1\right), &
 \end{aligned}$$

which is clearly true. So for all odd  $n$ , the LHB of the set  $S_{\ominus n}$  is equivalent to the RHB of the set  $S_{\ominus(n+4)}$ . Recall that for  $n'$  even,  $\frac{1}{2}X_{\ominus n'} = X_{\ominus n'+1}$ , which implies  $\frac{1}{2}S_{\ominus n'} = S_{\ominus(n'+1)}$ , i.e.  $\frac{1}{2}S_{\ominus(n-1)} = S_{\ominus n}$  and  $\frac{1}{2}S_{\ominus(n+3)} = S_{\ominus(n+4)}$ . Thus, the LHB of  $S_{\ominus(n-1)}$  is equivalent to the RHB of the set  $S_{\ominus(n+3)}$ . But since  $n-1 \in \mathbb{Z}^+$  is even, both even and odd cases are satisfied. Thus we can say that for all  $n'' > 1 \in \mathbb{Z}$ , the LHB of the set  $S_{\ominus n''}$  is equivalent to the RHB of the set  $S_{\ominus(n''+4)}$ . Nonetheless, these two sets are still “essentially disjoint” because their intersection has measure 0 using Lebesgue Measure. Therefore, by our earlier argument, our lemma follows.  $\square$

**Lemma 4.3.** *All  $X_{\oplus n}, X_{\ominus n'}, Y_{\oplus n''}, Y_{\ominus n'''}$  are disjoint for all natural numbers  $n, n', n'',$  and  $n'''$ . Moreover,  $X_{\oplus i}$  and  $X_{\ominus j}$  are disjoint when  $i \neq j$ , with analogous properties following for sets of the form  $X_{\oplus n}, Y_{\ominus n}$ , and  $Y_{\oplus n}$ .*

*Proof.* First we show that all  $X_{\oplus n}, X_{\ominus n}$  are disjoint (a similar argument shows that all  $Y_{\oplus n}, Y_{\ominus n}$  are disjoint). Consider the maximal case for  $X_{\oplus 1}$  and  $X_{\ominus 1}$ , namely,  $X_{\oplus 1} = [0, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$  and  $X_{\ominus 1} = [-\frac{\pi}{2}, 0) \times [-\frac{\pi}{2}, \frac{\pi}{2})$ . Because all other  $X_{\oplus n}, X_{\ominus n}$  are copies of  $X_{\oplus 1}$  and  $X_{\ominus 1}$  that have been translated along the  $x$ -axis and scaled, we will consider only their  $x$ -coordinates. We note that because sets of the form  $X_{\oplus n}, X_{\ominus n}$  are never scaled by factors  $\alpha$  such that  $|\alpha| > 1$ , they are all contained in  $[-\infty, \infty) \times [-\frac{\pi}{2}, \frac{\pi}{2})$  (i.e., their vertical bounds are contained in the set  $[-\frac{\pi}{2}, \frac{\pi}{2})$ ). From the crossover algorithm, we have that the following hold for all  $m$  (except for  $4m + 1 = 1$ ) in the  $x$ -coordinate:

$$\begin{aligned} X_{\oplus 4m+1} &= \frac{1}{2} X_{\oplus 4m} & X_{\ominus 4m+1} &= \frac{1}{2} X_{\ominus 4m} \\ X_{\oplus 4m+2} &= X_{\oplus 4m+1} - 2\pi & X_{\ominus 4m+2} &= X_{\ominus 4m+1} + 2\pi \\ X_{\oplus 4m+3} &= \frac{1}{2} X_{\oplus 4m+2} & X_{\ominus 4m+3} &= \frac{1}{2} X_{\ominus 4m+2} \\ X_{\oplus 4m+4} &= X_{\oplus 4m+3} + 2\pi & X_{\ominus 4m+4} &= X_{\ominus 4m+3} - 2\pi \end{aligned}$$

$$\Rightarrow X_{\oplus 4(m+1)+1} = \frac{1}{4} X_{\oplus 4m+1} + \frac{\pi}{2} \qquad \Rightarrow X_{\ominus 4(m+1)+1} = \frac{1}{4} X_{\ominus 4m+1} - \frac{\pi}{2}.$$

Solving the recurrence relation for  $X_{\oplus 4m+1}$  and  $X_{\ominus 4m+1}$  and using the solution to obtain the other cases, we obtain the following:

$$\begin{aligned} X_{\oplus 4m+1} &= \frac{1}{4^m} X_{\oplus 1} + \frac{2}{3} \left(1 - \frac{1}{4^m}\right) \pi & X_{\ominus 4m+1} &= \frac{1}{4^m} X_{\ominus 1} + \frac{2}{3} \left(\frac{1}{4^m} - 1\right) \pi \\ X_{\oplus 4m+2} &= \frac{1}{4^m} X_{\oplus 1} - \frac{2}{3} \left(2 + \frac{1}{4^m}\right) \pi & X_{\ominus 4m+2} &= \frac{1}{4^m} X_{\ominus 1} + \frac{2}{3} \left(\frac{1}{4^m} + 2\right) \pi \\ X_{\oplus 4m+3} &= \frac{1}{2} \frac{1}{4^m} X_{\oplus 1} - \frac{1}{3} \left(2 + \frac{1}{4^m}\right) \pi & X_{\ominus 4m+3} &= \frac{1}{2} \frac{1}{4^m} X_{\ominus 1} + \frac{1}{3} \left(\frac{1}{4^m} + 2\right) \pi \\ X_{\oplus 4m+4} &= \frac{1}{2} \frac{1}{4^m} X_{\oplus 1} + \frac{1}{3} \left(4 - \frac{1}{4^m}\right) \pi & X_{\ominus 4m+4} &= \frac{1}{2} \frac{1}{4^m} X_{\ominus 1} + \frac{1}{3} \left(\frac{1}{4^m} - 4\right) \pi. \end{aligned}$$

Using our maximal  $X_{\oplus 1}$  and  $X_{\ominus 1}$ , we find that

$$\begin{aligned} X_{\oplus 4m+1} &= \left[\left(\frac{2}{3} - \frac{2}{3} \left(\frac{1}{4}\right)^m\right) \pi, \left(\frac{2}{3} - \frac{1}{6} \left(\frac{1}{4}\right)^m\right) \pi\right] \subset \left[0, \frac{2}{3} \pi\right) \\ X_{\oplus 4m+2} &= \left[\left(-\frac{4}{3} - \frac{2}{3} \left(\frac{1}{4}\right)^m\right) \pi, \left(-\frac{4}{3} - \frac{1}{6} \left(\frac{1}{4}\right)^m\right) \pi\right] \subset \left[-2\pi, -\frac{4}{3} \pi\right) \\ X_{\oplus 4m+3} &= \left[\left(-\frac{2}{3} - \frac{1}{3} \left(\frac{1}{4}\right)^m\right) \pi, \left(-\frac{2}{3} - \frac{1}{12} \left(\frac{1}{4}\right)^m\right) \pi\right] \subset \left[-\pi, -\frac{2}{3} \pi\right) \\ X_{\oplus 4m+4} &= \left[\left(\frac{4}{3} - \frac{1}{3} \left(\frac{1}{4}\right)^m\right) \pi, \left(\frac{4}{3} - \frac{1}{12} \left(\frac{1}{4}\right)^m\right) \pi\right] \subset \left[\pi, \frac{4}{3} \pi\right) \\ X_{\ominus 4m+1} &= \left[\left(\frac{1}{6} \left(\frac{1}{4}\right)^m - \frac{2}{3}\right) \pi, \left(\frac{2}{3} \left(\frac{1}{4}\right)^m - \frac{2}{3}\right) \pi\right] \subset \left[-\frac{2}{3} \pi, 0\right) \\ X_{\ominus 4m+2} &= \left[\left(\frac{1}{6} \left(\frac{1}{4}\right)^m + \frac{4}{3}\right) \pi, \left(\frac{2}{3} \left(\frac{1}{4}\right)^m + \frac{4}{3}\right) \pi\right] \subset \left[\frac{4}{3} \pi, 2\pi\right) \\ X_{\ominus 4m+3} &= \left[\left(\frac{1}{12} \left(\frac{1}{4}\right)^m + \frac{2}{3}\right) \pi, \left(\frac{1}{3} \left(\frac{1}{4}\right)^m + \frac{2}{3}\right) \pi\right] \subset \left[\frac{2}{3} \pi, \pi\right) \\ X_{\ominus 4m+4} &= \left[\left(\frac{1}{12} \left(\frac{1}{4}\right)^m - \frac{4}{3}\right) \pi, \left(\frac{1}{3} \left(\frac{1}{4}\right)^m - \frac{4}{3}\right) \pi\right] \subset \left[-\frac{4}{3} \pi, -\pi\right). \end{aligned}$$

Trivially we conclude that the eight different sets of intervals are disjoint. Within each set of intervals, note that both endpoints of the intervals either monotonically increase (for  $X_{\oplus n}$ ) or monotonically decrease (for  $X_{\ominus n}$ ) as  $n$  increases; we also find that the right endpoint of  $X_{\oplus 4m+k}$  is equal to the left endpoint of  $X_{\oplus 4(m+1)+k}$  for all possible values of  $k$ , and the left endpoint of  $X_{\ominus 4m+k}$  is equal to the right endpoint of  $X_{\ominus 4(m+1)+k}$  for all possible values of  $k$  (See the proof of Lemma 3.2.). Thus, all

the  $X_{\oplus 4m+k}$  and  $X_{\ominus 4m+k}$  are disjoint for all  $k$ , meaning we have proved that all  $X_{\oplus n}, X_{\ominus n}$  are disjoint. Analogously, all  $Y_{\oplus n}$  and  $Y_{\ominus n}$  are disjoint. Moreover, all  $X_{\ominus i}$  and  $X_{\oplus j}$  all  $X_{\oplus i}$  and  $X_{\oplus j}$  when  $i \neq j$ . Analogously, all  $Y_{\ominus i}$  and  $Y_{\oplus j}$  and all  $Y_{\oplus i}$  and  $Y_{\oplus j}$  when  $i \neq j$ .

To show that the sets of the form  $X_{\oplus n}, X_{\ominus n}, Y_{\oplus n}$  and  $Y_{\ominus n}$  are disjoint, consider the following: All the sets of the form  $X_{\oplus n}, X_{\ominus n}$  are contained in the region  $[-2\pi, 2\pi) \times [-\frac{\pi}{2}, \frac{\pi}{2})$ . Similarly, all the sets of the form  $Y_{\oplus n}, Y_{\ominus n}$  are contained in the region  $[-\frac{\pi}{2}, \frac{\pi}{2}) \times [-2\pi, 2\pi)$ . The intersection between these two regions is  $[-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$ —but the only sets in this region are  $X_{\oplus 1}, X_{\ominus 1}, Y_{\oplus 1}$ , and  $Y_{\ominus 1}$ , and by definition, these are disjoint, completing the proof.  $\square$

**Remark 4.1.** *The proof of Lemma 4.3 shows that all sets of the form  $X_{\oplus n}, X_{\ominus n}$  for odd  $n \geq 3$  are in the area  $[-\pi, \pi) \times [-\frac{\pi}{2}, \frac{\pi}{2}) \setminus [-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2}) \in G_{SO}$  (Exclusion from the inner square is due to the fact that  $X_{\oplus 1}, X_{\ominus 1}, Y_{\oplus 1}$  and  $Y_{\ominus 1}$  occupy that space and all  $X_{\ominus i}$  and  $X_{\oplus j}$  are disjoint if  $i \neq j$  (with analogous results for  $X_{\oplus}, Y_{\ominus}$  and  $Y_{\oplus}$ ), implying no other sets of the form  $X_{\ominus n}, X_{\oplus n}, Y_{\ominus n}$  and  $Y_{\oplus n}$  can occupy that space.), providing a short proof of Lemma 4.1.*

We can now prove our main result.

*Proof.* (of Theorem 2.1)

Let  $W$  be defined as in example 1 to be

$$\begin{aligned} & \left( \bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}] \right) \cup \left( G_{TO} \setminus \left[ \bigcup_{i=1}^{\infty} [X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}] \right] \right) \\ & = \left( \bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}] \right) \cup \left( G_{SO} \setminus \left[ \bigcup_{i=2}^{\infty} [X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}] \right] \right). \end{aligned}$$

**Part I:** We will first show that  $W$  is dilation congruent to  $G_{SO}$ . Let  $\Phi_{X_{\ominus}} : \{X_{\ominus i} : i \text{ is even } \geq 2\} \rightarrow \{X_{\ominus j} : j \text{ is odd } \geq 3\}$  be such that for all even  $n$ ,  $\Phi_{X_{\ominus}}(X_{\ominus n}) := \frac{1}{2}X_{\ominus n} = X_{\ominus n+1} \in G_{SO}$  from **lemma 4.1** and the discussion on the top of page 9 of our paper.

**Claim 1:**  $\Phi_{X_{\ominus}}$  is surjective. Take an arbitrary  $X_{\ominus j}$  such that  $j \geq 3$  is odd. Then  $\Phi_{X_{\ominus}}(X_{\ominus j-1}) = X_{\ominus j}$ .

**Claim 2:**  $\Phi_{X_{\ominus}}$  is injective. Suppose  $\Phi_{X_{\ominus}}(X_{\ominus j}) = \Phi_{X_{\ominus}}(X_{\ominus i})$  for some even  $i$  and  $j$ . Then  $X_{\ominus j+1} = X_{\ominus i+1}$ , and therefore  $X_{\ominus j} = 2X_{\ominus j+1} = 2X_{\ominus i+1} = X_{\ominus i}$ , as needed.

Therefore  $\Phi_{X_{\ominus}}$  is a bijection. Similarly, define  $\Phi_{X_{\oplus}}, \Phi_{Y_{\ominus}}$  and  $\Phi_{Y_{\oplus}}$ , which are all bijections by analogous arguments. But then  $\Phi_{X_{\oplus}}(\bigcup_{i=1}^{\infty} X_{\oplus 2i}) = \bigcup_{i=1}^{\infty} \Phi_{X_{\oplus}}(X_{\oplus 2i}) = \bigcup_{i=1}^{\infty} X_{\oplus 2i+1} \in G_{SO}$  (using Lemma 1), and all of the white spaces ( $X_{\oplus k}$  for  $k \geq 3$  and odd) in  $G_{SO}$  are filled because  $\Phi_{X_{\oplus}}$  is surjective, and all of the black pieces ( $X_{\ominus n}$  for  $n$  even) have been mapped into  $G_{SO}$  injectively so that no two distinct black pieces map to the same white space. Analogous properties follow for  $\Phi_{X_{\oplus}}, \Phi_{Y_{\ominus}}$ , and  $\Phi_{Y_{\oplus}}$ .

Let  $\Phi : \{X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i} : i \text{ is even } \geq 2\} \rightarrow \{X_{\ominus j} \cup X_{\oplus j} \cup Y_{\ominus j} \cup Y_{\oplus j} : j \text{ is odd } \geq 3\}$  be such that

$$\begin{aligned}
& \Phi(X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i}) \\
&= \Phi_{X_{\ominus}}(X_{\ominus i}) \cup \Phi_{X_{\oplus}}(X_{\oplus i}) \cup \Phi_{Y_{\ominus}}(Y_{\ominus i}) \cup \Phi_{Y_{\oplus}}(Y_{\oplus i}) \\
&= (X_{\ominus i+1} \cup X_{\oplus i+1} \cup Y_{\ominus i+1} \cup Y_{\oplus i+1}).
\end{aligned}$$

Then

$$\begin{aligned}
& \Phi\left(\bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}]\right) \\
&= \bigcup_{i=1}^{\infty} [\Phi_{X_{\ominus}}(X_{\ominus 2i}) \cup \Phi_{X_{\oplus}}(X_{\oplus 2i}) \cup \Phi_{Y_{\ominus}}(Y_{\ominus 2i}) \cup \Phi_{Y_{\oplus}}(Y_{\oplus 2i})] \\
&= \bigcup_{i=1}^{\infty} [X_{\ominus 2i+1} \cup X_{\oplus 2i+1} \cup Y_{\ominus 2i+1} \cup Y_{\oplus 2i+1}] \in G_{SO},
\end{aligned}$$

and  $\Phi$  is clearly a bijection, so  $\Phi$  maps all exterior black pieces into all interior white pieces such that no distinct black pieces map to the same white piece.

Thus,

$$\begin{aligned}
& \Phi\left(\bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}]\right) \cup \left[G_{SO} \setminus \left(\bigcup_{i=2}^{\infty} [X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}]\right)\right] \\
&= G_{SO},
\end{aligned}$$

as needed. Therefore,  $W$  is dilation congruent to  $G_{SO}$ .

**Part II:** We will now prove that  $W$  is translation congruent to  $G_{TO}$ . Let  $\Psi_{X_{\ominus}} : \{X_{\ominus i} : i \text{ is even } \geq 2\} \rightarrow \{X_{\ominus j} : j \text{ is odd } \geq 1\}$  be such that for all even  $n$ ,  $\Psi_{X_{\ominus}}(X_{\ominus n}) := X_{\ominus n} - (-1)^{\frac{n-2}{2}} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \frac{1}{2}X_{\ominus(n-2)} = X_{\ominus n-1} \in G_{TO}$  (using the discussion on the top of page 9, **lemma 4.1** and the fact that  $X_{\ominus 1} \in [-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$ ).

**Claim 1:**  $\Psi_{X_{\ominus}}$  is surjective. Take an arbitrary  $X_{\ominus j}$  such that  $j \geq 1$  is odd. Then  $\Phi_{X_{\ominus}}(X_{\ominus j+1}) = X_{\ominus j}$ .

**Claim 2:**  $\Psi_{X_{\ominus}}$  is injective. Suppose  $\Psi_{X_{\ominus}}(X_{\ominus j}) = \Psi_{X_{\ominus}}(X_{\ominus i})$  for some even  $i$  and  $j$ . Then  $X_{\ominus j-1} = X_{\ominus i-1}$ . Suppose that  $(j-1) \neq (i-1)$ . Then by **lemma 4.3**,  $X_{\ominus j-1} \cap X_{\ominus i-1} = \emptyset$ , which contradicts  $X_{\ominus j-1} = X_{\ominus i-1}$ . Thus it must be true that  $(j-1) = (i-1)$  and thus  $X_{\ominus i} = X_{\ominus j}$ , as needed.

Therefore  $\Psi_{X_{\ominus}}$  is a bijection. But then  $\Psi_{X_{\ominus}}(\bigcup_{i=1}^{\infty} X_{\ominus 2i}) = \bigcup_{i=1}^{\infty} \Psi_{X_{\ominus}}(X_{\ominus 2i}) = \bigcup_{i=1}^{\infty} X_{\ominus 2i-1}$ . Therefore all blank spaces in  $G_{TO}$  of the form  $X_{\ominus n}$  are filled when  $\Psi_{X_{\ominus}}$  acts on  $\bigcup_{i=1}^{\infty} X_{\ominus 2i}$  since  $\Psi_{X_{\ominus}}$  is onto. Moreover, all black pieces of the form  $X_{\ominus n}$  are contained in the set  $\bigcup_{i=1}^{\infty} X_{\ominus 2i}$  and therefore have been mapped into  $G_{TO}$ . Since  $\Psi_{X_{\ominus}}$  is injective, no two distinct black pieces will map to the same white piece.

Define similarly  $\Psi_{X_{\oplus}}$ ,  $\Psi_{Y_{\ominus}}$  and  $\Psi_{Y_{\oplus}}$ , which are all bijections by analogous arguments.

Define  $\Psi : \{X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i} : i \text{ is even } \geq 2\} \rightarrow \{X_{\ominus j} \cup X_{\oplus j} \cup Y_{\ominus j} \cup Y_{\oplus j} : j \text{ is odd } \geq 1\}$  to be such that

$$\begin{aligned}
 & \Psi (X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i}) \\
 &= \Psi_{X_{\ominus}} (X_{\ominus i}) \cup \Psi_{X_{\oplus}} (X_{\oplus i}) \cup \Psi_{Y_{\ominus}} (Y_{\ominus i}) \cup \Psi_{Y_{\oplus}} (Y_{\oplus i}) \\
 &= [X_{\ominus i-1} \cup X_{\oplus i-1} \cup Y_{\ominus i-1} \cup Y_{\oplus i-1}] \in G_{TO}.
 \end{aligned}$$

$\Psi$  is clearly a bijection, and therefore when  $\Psi$  acts on the entire domain (i.e.,

$$\begin{aligned}
 & \Psi \left( \bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}] \right) \\
 &= \bigcup_{i=1}^{\infty} \Psi (X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}) \\
 &= \bigcup_{i=1}^{\infty} [X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}] \in G_{TO},
 \end{aligned}$$

every white space (of the form  $X_{\ominus n}$  for odd  $n$ ) in  $G_{TO}$  is filled by some black piece (of the form  $X_{\oplus n'}$  for some even  $n'$ ) from the exterior of  $G_{TO}$  since  $\Psi$  is surjective). No two distinct black pieces map to the same white piece since  $\Psi$  is injective. Moreover, every black piece outside  $G_{TO}$  is contained in the domain of  $\Psi$ , and therefore every black piece outside  $G_{TO}$  is mapped into  $G_{TO}$ . Thus,

$$\begin{aligned}
 & \Psi \left( \bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}] \right) \cup \left[ G_{TO} \setminus \left( \bigcup_{i=1}^{\infty} [X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}] \right) \right] \\
 & \hspace{20em} = G_{TO},
 \end{aligned}$$

and thus by definition,  $W$  is dilation congruent modulo  $2\pi$  to  $G_{TO}$ . By definition  $W$  is a wavelet set. □

A different example of a partition of the inner square conforming to the requirements of Theorem 2.1 is shown in Figure 2 with the resulting wavelet set.

## 5. PATCH WAVELET SETS

All of the wavelet sets we have considered thus far are *crossover* wavelet sets. In this class, regions are added to or subtracted from alternating sides of the inner square. Alternatively, we could add or subtract regions to the same side of the square for each translation. Such wavelet sets are called *patch* wavelet sets. To illustrate the patch algorithm, we give an example. The reader will note that this example is actually a well-known wavelet set: The “wedding cake” wavelet set (see Figure 3). See [4], Example 6.6.1, and also [6].

**Patch Example 1:** Let

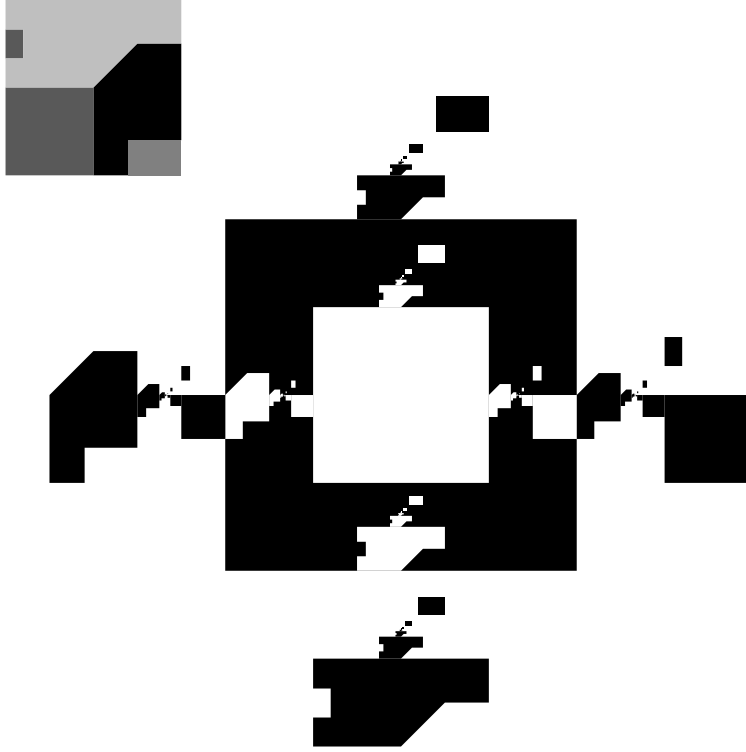


FIGURE 2. An arbitrary (conformant) partition, with wavelet set.

$$\begin{aligned}
 X_{\ominus} &= \left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \\
 X_{\oplus} &= \left[0, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \\
 Y_{\ominus} &= \emptyset, \\
 Y_{\oplus} &= \emptyset.
 \end{aligned}$$

Consider the piece  $X_{\ominus}$ . Start by translating  $X_{\ominus}$  by  $\begin{bmatrix} -2\pi \\ 0 \end{bmatrix}$  (keeping it on the same side of the origin) to obtain  $X_{\ominus 2}$ . We find that  $X_{\ominus 2} = \left[-\frac{5\pi}{2}, -2\pi\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$ . Secondly, scale  $X_{\ominus 2}$  by  $\frac{1}{4}$  to obtain  $X_{\ominus 3} = \left[-\frac{5\pi}{8}, -\frac{\pi}{2}\right) \times \left[-\frac{\pi}{8}, \frac{\pi}{8}\right)$ . Thirdly, translate  $X_{\ominus 3}$  in the same direction as that of the first translation (i.e., by  $\begin{bmatrix} -2\pi \\ 0 \end{bmatrix}$ ) to obtain  $X_{\ominus 4} = \left[-\frac{21\pi}{8}, -\frac{5\pi}{2}\right) \times \left[-\frac{\pi}{8}, \frac{\pi}{8}\right)$ . Finally, scale  $X_{\ominus 4}$  by  $\frac{1}{4}$  to form the set  $X_{\ominus 5} = \left[-\frac{21\pi}{32}, -\frac{5\pi}{32}\right) \times \left[-\frac{\pi}{32}, \frac{\pi}{32}\right)$ . Continue these two steps inductively for  $X_{\ominus}$ .

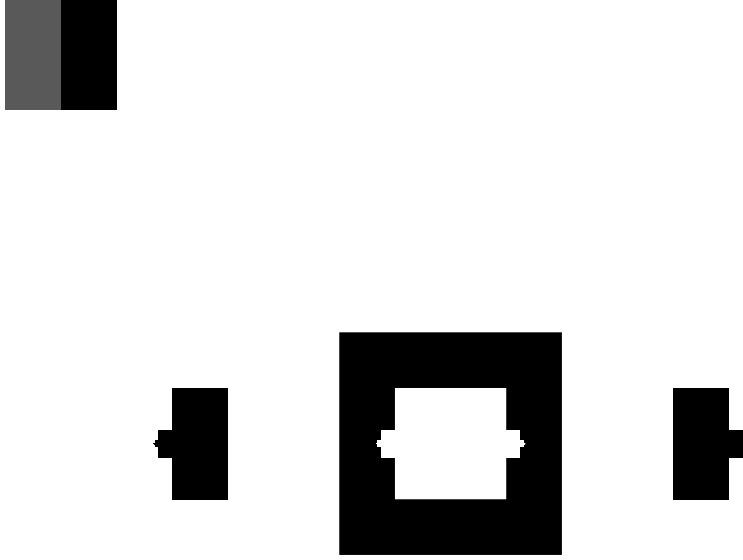


FIGURE 3. A patch wavelet set: The “wedding cake” set

We perform two similar steps on the set  $X_{\oplus}$  inductively as well; however, we translate by  $\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$  (instead of  $\begin{bmatrix} -2\pi \\ 0 \end{bmatrix}$ ). We obtain the following as a result from the first four steps of the patch algorithm:

$$\begin{aligned} X_{\oplus 2} &= \left[ 2\pi, \frac{5\pi}{2} \right) \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right), \\ X_{\oplus 3} &= \left[ \frac{\pi}{2}, \frac{5\pi}{8} \right) \times \left[ -\frac{\pi}{8}, \frac{\pi}{8} \right), \\ X_{\oplus 4} &= \left[ \frac{5\pi}{2}, \frac{21\pi}{8} \right) \times \left[ -\frac{\pi}{8}, \frac{\pi}{8} \right), \\ X_{\oplus 5} &= \left[ \frac{5\pi}{32}, \frac{21\pi}{32} \right) \times \left[ -\frac{\pi}{32}, \frac{\pi}{32} \right). \end{aligned}$$

Continue this process inductively for  $X_{\oplus}$  as well. In theory, we would perform similar steps for  $Y_{\oplus}$  and  $Y_{\ominus}$ , but in this example both are the null set, and thus we have no computations to carry out for the sets  $Y_{\ominus}$  and  $Y_{\oplus}$ .

Let  $W'$  be the set

$$\begin{aligned} &\left( \bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i}] \right) \cup \left( G_{TO} \setminus \left[ \bigcup_{i=1}^{\infty} [X_{\ominus 2i-1} \cup X_{\oplus 2i-1}] \right] \right) \\ &= \left( \bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i}] \right) \cup \left( G_{SO} \setminus \left[ \bigcup_{i=2}^{\infty} [X_{\ominus 2i-1} \cup X_{\oplus 2i-1}] \right] \right) \end{aligned}$$

(see Figure 3). Similarly to the crossover case, we can think of the set  $W'$  as being the union of  $G_{TO}$  combined with the exterior “black pieces” of the form  $X_{\oplus n}, X_{\ominus n}$  where  $n$  is even and with “white pieces” of the form  $X_{\oplus n}, X_{\ominus n}$  where  $n$  is odd erased from  $G_{TO}$ . The reader should check that this set  $W'$  is indeed a wavelet set.

This algorithm can be generalized as follows:

(1) Partition the inner square into a maximum of four pieces. The conditions on this partition are identical to those on the partition of the inner square using the crossover algorithm as given in Theorem 2.1, and the proof for the case of the patch algorithm is similar to the proof given for the crossover algorithm.

(2) Translate one piece of the partition by  $\begin{bmatrix} \pm 2\pi \\ 0 \end{bmatrix}$  or  $\begin{bmatrix} 0 \\ \pm 2\pi \end{bmatrix}$  so that the piece is translated out of the inner square to the same half of the plane that the original piece of the partition lied in.

(3) Dilate the set formed in step 2 into  $G_{SO}$  by  $\frac{1}{4}$ .

(4) Translate the set formed in step 3 out of  $G_{SO}$  in the same direction as the translation in step 2 (i.e., by  $\begin{bmatrix} \pm 2\pi \\ 0 \end{bmatrix}$  or  $\begin{bmatrix} 0 \\ \pm 2\pi \end{bmatrix}$ ).

(5) Dilate the set formed in step 4 into  $G_{SO}$  by  $\frac{1}{4}$ .

(6) Repeat steps 2 and 3 inductively for this piece of the partition, and perform the same steps inductively on the other three pieces of the partition of the inner square.  $\blacklozenge$

**Theorem 5.1.** (*Patch Algorithm*) Let  $\{X_{\ominus}, X_{\oplus}, Y_{\ominus}, Y_{\oplus}\}$  be a partition of the set  $[-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$  such that  $X_{\ominus}$  is contained in the left half of the inner square,  $X_{\oplus}$  is contained in the right half of the inner square,  $Y_{\ominus}$  is contained in the bottom half of the inner square, and  $Y_{\oplus}$  is contained in the top half of the inner square. Then the set,  $W$  defined as

$$\left[ \left( \bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}] \right) \cup G_{TO} \right] \setminus \left[ \bigcup_{i=1}^{\infty} [X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}] \right],$$

generated by this partition under translation by  $\begin{bmatrix} \pm 2\pi \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ \pm 2\pi \end{bmatrix}$  and dilation by powers of 2 using steps (1)-(6) above, is a dyadic wavelet set in  $\mathbb{R}^2$ .

*Proof.* Begin by showing the following for natural numbers  $n$  odd and  $n'$  even:

$$X_{\ominus n+2} = \frac{1}{4} \left( X_{\ominus n} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \right); \quad X_{\ominus n'+2} = \frac{1}{4} X_{\ominus n'} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}; \quad \frac{X_{\ominus n'}}{4} = X_{\ominus n'+1}; \quad \text{and}$$

$X_{\ominus n'} = X_{\ominus n'-1} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$ . First solve the recurrence relation for  $n$  odd and use this and the fact that  $\frac{X_{\ominus n'}}{4} = X_{\ominus n'+1}$  to obtain a form for  $n'$  odd. From this point

forward let  $n$  be an arbitrary odd or even natural number. We find that

$$X_{\ominus n} = \begin{cases} \frac{X_{\ominus} - \frac{1}{3} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}}{4^{\frac{n-1}{2}}} \left(1 - \left(\frac{1}{4}\right)^{\frac{n-1}{2}}\right), & \text{for } n \text{ odd} \\ \frac{X_{\ominus} - \frac{4}{3} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}}{4^{\frac{n'+1}{2}}} \left(1 - \left(\frac{1}{4}\right)^{\frac{n'-1}{2}}\right), & \text{for } n \text{ even} \end{cases}$$

Derive similar expressions for  $X_{\oplus n}$ ,  $Y_{\ominus n}$  and  $Y_{\oplus n}$ .

Show an analogous property to that of Lemma 4.1 for the patch algorithm. Once again use the maximal possible  $X_{\ominus n}$ , i.e. use  $S_{\ominus n} := \left[[-\frac{\pi}{2}, 0) \times [-\frac{\pi}{2}, \frac{\pi}{2})\right]_{\ominus n}$  (the result of the  $n^{\text{th}}$  step of the patch algorithm applied to  $[-\frac{\pi}{2}, 0) \times [-\frac{\pi}{2}, \frac{\pi}{2})$ ). Use our derived bounds for  $X_{\ominus n}$  in terms of  $n$  to determine the bounds for  $S_{\ominus n}$ . Begin by showing that  $S_{\ominus n} \subseteq [-\pi, \pi) \times [-\pi, \pi)$ . That this is satisfied for the vertical bounds of  $S_{\ominus n}$  is clear, so we will only consider the horizontal bounds. There is only one case to consider for the patch algorithm (because of the fact that the patch algorithm always translates the odd pieces out to the same side of the inner square rather than to alternating sides as in the crossover algorithm, leading to two cases for the crossover algorithm), i.e. the case that  $n = 2k + 1$  for some non-negative integer  $k$ . Second, show that  $X_{\ominus n} \not\subseteq [-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$  for all  $n \geq 3$  by showing that  $S_{\ominus n} \not\subseteq [-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$ . This follows from the fact that the horizontal bounds on the set  $S_{\ominus n}$  are not contained in the set  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ . Thus, the vertical bounds on the set  $S_{\ominus n}$  are irrelevant. Once again, here we find that there is only one case to consider (the case that  $n = 2k + 1$  for some non-negative integer  $k$ ). Conclude that  $S_{\ominus n} \subseteq G_{SO}$  for all odd  $n \geq 3$  and therefore that  $X_{\ominus n} \subseteq G_{SO}$  for all odd  $n \geq 3$ . Analogously, for all odd  $n \geq 3$ ,  $X_{\oplus n} \subseteq G_{SO}$ ,  $Y_{\ominus n} \subseteq G_{SO}$ , and  $Y_{\oplus n} \subseteq G_{SO}$ .

Show an analogous property for the patch case to Lemma 4.2 for the crossover algorithm, that  $X_{\ominus n+2}$  and  $X_{\ominus n}$  are disjoint for all  $n > 0 \in \mathbb{Z}$ . Modify the argument that was used for the crossover case by showing that the left hand bound of  $S_{\ominus 2k+1}$  equals the right hand bound of  $S_{\ominus 2k+3}$ .

Show an analogous property for the patch case to that of Lemma 4.3 for crossover sets. Namely show that all  $X_{\oplus n}, X_{\ominus n'}, Y_{\oplus n''}, Y_{\ominus n'''}$  are disjoint for all natural numbers  $n, n', n'',$  and  $n'''$ . Moreover,  $X_{\ominus i}$  and  $X_{\ominus j}$  are disjoint when  $i \neq j$ , with analogous properties following for sets of the form  $X_{\oplus n}, Y_{\ominus n}$ , and  $Y_{\oplus n}$ .

First show that all  $X_{\oplus n}, X_{\ominus n}$  are disjoint. Once again, consider the maximal case for  $X_{\oplus 1}$  and  $X_{\ominus 1}$ . Because all other  $X_{\oplus n}, X_{\ominus n}$  are copies of  $X_{\oplus 1}$  and  $X_{\ominus 1}$  that have been translated along the  $x$ -axis and scaled, consider only the  $x$ -coordinates. Because sets of the form  $X_{\oplus n}, X_{\ominus n}$  are never scaled by factors  $\alpha$  such that  $|\alpha| > 1$ , they are all contained in  $[-\infty, \infty) \times [-\frac{\pi}{2}, \frac{\pi}{2})$ . From the patch algorithm, observe that the following hold for all  $m$  (where  $2m + 1 > 1$ ) in the  $x$ -coordinate:

$$\begin{aligned} X_{\ominus 2m+1} &= \frac{1}{4} X_{\ominus 2m} \\ X_{\ominus 2m+2} &= X_{\ominus 2m+1} - 2\pi \\ \Rightarrow X_{\ominus 2(m+1)+1} &= \frac{1}{16} X_{\ominus 2m} - \frac{\pi}{2} \\ X_{\oplus 2m+1} &= \frac{1}{4} X_{\oplus 2m} \\ X_{\oplus 2m+2} &= X_{\oplus 2m+1} + 2\pi \\ \Rightarrow X_{\oplus 2(m+1)+1} &= \frac{1}{16} X_{\oplus 2m} + \frac{\pi}{2}. \end{aligned}$$

Solve these recurrence relations and find a collection of disjoint sets, each of which contains one of the following as a subset:  $X_{\ominus 2m+1}, X_{\ominus 2m+2}, X_{\oplus 2m+1}$  and  $X_{\oplus 2m+2}$ . Trivially conclude that the four different sets of intervals are disjoint. Within each set of intervals, note that both endpoints of the intervals either monotonically increase (for  $X_{\oplus n}$ ) or monotonically decrease (for  $X_{\ominus n}$ ) as  $n$  increases. Recall from our argument for the property similar to Lemma 4.2 (but for the patch case) that the left hand bound of  $S_{\ominus 2k+1}$  equals the right hand bound of  $S_{\ominus 2k+3}$ . We will also find that the right hand bound of  $S_{\ominus 2k+1}$  equals the left hand bound of  $S_{\ominus 2k+3}$ . Thereby conclude that all  $X_{\oplus n}, X_{\ominus n}$  are disjoint along with all  $X_{\ominus i}$  and  $X_{\ominus j}$  and all  $X_{\oplus i}$  and  $X_{\oplus j}$  when  $i \neq j$ . Analogously, all  $Y_{\oplus n}$  and  $Y_{\ominus n}$  are disjoint along with all  $Y_{\ominus i}$  and  $Y_{\ominus j}$  and all  $Y_{\oplus i}$  and  $Y_{\oplus j}$  when  $i \neq j$ .

To show that the sets of the form  $X_{\oplus n}, X_{\ominus n}, Y_{\oplus n}$  and  $Y_{\ominus n}$  are disjoint, consider the following: All the sets of the form  $X_{\oplus n}, X_{\ominus n}$  are contained in the region  $[-\infty, \infty) \times [-\frac{\pi}{2}, \frac{\pi}{2})$ . Similarly, all the sets of the form  $Y_{\oplus n}, Y_{\ominus n}$  are contained in the region  $[-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\infty, \infty)$ . The intersection between these two regions is  $[-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$ , but the only sets in this region are  $X_{\oplus 1}, X_{\ominus 1}, Y_{\oplus 1}$ , and  $Y_{\ominus 1}$ , and by definition, these are disjoint.

Define the set  $W$  in the same way it was defined in the proof of the theorem for the crossover case. To show that  $W$  is dilation congruent to  $G_{SO}$ , define the bijection  $\Phi_{X_{\ominus}} : \{X_{\ominus i} : i \text{ is even } \geq 2\} \rightarrow \{X_{\ominus j} : j \text{ is odd } \geq 3\}$  such that for all even  $n$ ,  $\Phi_{X_{\ominus}}(X_{\ominus n}) := \frac{1}{4}X_{\ominus n} = X_{\ominus n+1} \in G_{SO}$ . Observe that  $\Phi_{X_{\ominus}}(\bigcup_{i=1}^{\infty} X_{\ominus 2i}) = \bigcup_{i=1}^{\infty} \Phi_{X_{\ominus}}(X_{\ominus 2i}) = \bigcup_{i=1}^{\infty} X_{\ominus 2i+1} \in G_{SO}$  (using the property analogous to Lemma 1 but for the patch case), all of the white spaces ( $X_{\ominus k}$  for  $k \geq 3$  and odd) in  $G_{SO}$  are filled, and all of the black pieces ( $X_{\ominus n}$  for  $n$  even) have been mapped into  $G_{SO}$  injectively.

Similarly, define the bijections  $\Phi_{X_{\oplus}}, \Phi_{Y_{\ominus}}$  and  $\Phi_{Y_{\oplus}}$ . Analogous properties follow for  $\Phi_{X_{\oplus}}, \Phi_{Y_{\ominus}}$ , and  $\Phi_{Y_{\oplus}}$ . Let  $\Phi : \{X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i} : i \text{ is even } \geq 2\} \rightarrow \{X_{\ominus j} \cup X_{\oplus j} \cup Y_{\ominus j} \cup Y_{\oplus j} : j \text{ is odd } \geq 3\}$  be such that

$$\begin{aligned} & \Phi(X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i}) \\ &= \Phi_{X_{\ominus}}(X_{\ominus i}) \cup \Phi_{X_{\oplus}}(X_{\oplus i}) \cup \Phi_{Y_{\ominus}}(Y_{\ominus i}) \cup \Phi_{Y_{\oplus}}(Y_{\oplus i}) \\ &= (X_{\ominus i+1} \cup X_{\oplus i+1} \cup Y_{\ominus i+1} \cup Y_{\oplus i+1}). \end{aligned}$$

Using  $\Phi$ , show that  $W$  is dilation congruent to  $G_{SO}$ .

To show that  $W$  is translation congruent to  $G_{TO}$ , let  $\Psi_{X_{\ominus}} : \{X_{\ominus i} : i \text{ is even } \geq 2\} \rightarrow \{X_{\ominus j} : j \text{ is odd } \geq 1\}$  be such that for all even  $n$ ,  $\Psi_{X_{\ominus}}(X_{\ominus n}) := X_{\ominus n} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \frac{1}{2}X_{\ominus(n-2)} = X_{\ominus n-1} \in G_{TO}$ .  $\Psi_{X_{\ominus}}$  is a bijection. Observe that  $\Psi_{X_{\ominus}}(\bigcup_{i=1}^{\infty} X_{\ominus 2i}) = \bigcup_{i=1}^{\infty} \Psi_{X_{\ominus}}(X_{\ominus 2i}) = \bigcup_{i=1}^{\infty} X_{\ominus 2i-1}$ . Therefore all blank spaces in  $G_{TO}$  of the form  $X_{\ominus n}$  are filled when  $\Psi_{X_{\ominus}}$  acts on  $\bigcup_{i=1}^{\infty} X_{\ominus 2i}$  since  $\Psi_{X_{\ominus}}$  is onto. Moreover, all black pieces of the form  $X_{\ominus n}$  are contained in the set  $\bigcup_{i=1}^{\infty} X_{\ominus 2i}$  and therefore have been mapped into  $G_{TO}$ . Define similarly  $\Psi_{X_{\oplus}}, \Psi_{Y_{\ominus}}$  and  $\Psi_{Y_{\oplus}}$ , which are all bijections by analogous arguments.

Define the bijection  $\Psi : \{X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i} : i \text{ is even } \geq 2\} \rightarrow \{X_{\ominus j} \cup X_{\oplus j} \cup Y_{\ominus j} \cup Y_{\oplus j} : j \text{ is odd } \geq 1\}$  to be such that

$$\begin{aligned}
 & \Psi(X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i}) \\
 &= \Psi_{X_{\ominus}}(X_{\ominus i}) \cup \Psi_{X_{\oplus}}(X_{\oplus i}) \cup \Psi_{Y_{\ominus}}(Y_{\ominus i}) \cup \Psi_{Y_{\oplus}}(Y_{\oplus i}) \\
 &= [X_{\ominus i-1} \cup X_{\oplus i-1} \cup Y_{\ominus i-1} \cup Y_{\oplus i-1}] \in G_{TO}.
 \end{aligned}$$

Using  $\Phi$ , show  $W$  is dilation congruent modulo  $2\pi$  to  $G_{TO}$ . Conclude now that  $W$  is a wavelet set.  $\square$

## 6. CONCLUDING REMARKS

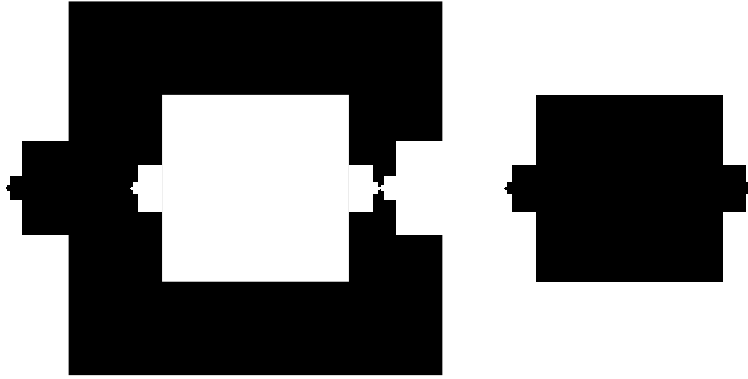


FIGURE 4. A wavelet set which has characteristics of both patch and crossover wavelet sets.

In figure 4, we partition the inner square in the following way:

$$\begin{aligned}
 X_{\ominus} &= \left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \\
 X_{\oplus} &= \left[0, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \\
 Y_{\ominus} &= \emptyset, \\
 Y_{\oplus} &= \emptyset.
 \end{aligned}$$

To the piece  $X_{\ominus}$  we apply the crossover algorithm. We obtain the following:

$$\begin{aligned}
 X_{\ominus 2} &= \left[\frac{3\pi}{2}, 2\pi\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \\
 X_{\ominus 3} &= \left[\frac{3\pi}{4}, \pi\right) \times \left[-\frac{\pi}{4}, \frac{\pi}{4}\right), \\
 X_{\ominus 4} &= \left[\frac{-5\pi}{4}, -2\pi\right) \times \left[-\frac{\pi}{4}, \frac{\pi}{4}\right), \\
 X_{\ominus 5} &= \left[\frac{-5\pi}{8}, -\pi\right) \times \left[-\frac{\pi}{8}, \frac{\pi}{8}\right).
 \end{aligned}$$

To the piece  $X_{\oplus}$ , we apply the patch algorithm and obtain the following as a result from the first four steps of the algorithm:

$$\begin{aligned}
X_{\oplus 2} &= \left[2\pi, \frac{5\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \\
X_{\oplus 3} &= \left[\frac{\pi}{2}, \frac{5\pi}{8}\right) \times \left[-\frac{\pi}{8}, \frac{\pi}{8}\right), \\
X_{\oplus 4} &= \left[\frac{5\pi}{2}, \frac{21\pi}{8}\right) \times \left[-\frac{\pi}{8}, \frac{\pi}{8}\right), \\
X_{\oplus 5} &= \left[\frac{5\pi}{32}, \frac{21\pi}{32}\right) \times \left[-\frac{\pi}{32}, \frac{\pi}{32}\right).
\end{aligned}$$

We continue application of the patch algorithm to the piece  $X_{\oplus}$  and application of the crossover algorithm to the piece  $X_{\ominus}$  inductively. Once again we let  $W$  be the set

$$\begin{aligned}
&\left(\bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i}]\right) \cup \left(G_{TO} \setminus \left[\bigcup_{i=1}^{\infty} [X_{\ominus 2i-1} \cup X_{\oplus 2i-1}]\right]\right) \\
&= \left(\bigcup_{i=1}^{\infty} [X_{\ominus 2i} \cup X_{\oplus 2i}]\right) \cup \left(G_{SO} \setminus \left[\bigcup_{i=2}^{\infty} [X_{\ominus 2i-1} \cup X_{\oplus 2i-1}]\right]\right),
\end{aligned}$$

where  $X_{\ominus n}$  is defined according to our definition for a set of this form operated on by the crossover algorithm (page 9-10) and  $X_{\oplus n}$  is defined according to our definition given for a set of this form operated on by the patch algorithm (page 20).

This set  $W$  (see Figure 4) is a wavelet set. To see this, let  $G(X_{\ominus odd}) := \bigcup_{i=1}^{\infty} X_{\ominus 2i-1}$  and  $G(X_{\oplus even}) := \bigcup_{i=1}^{\infty} X_{\oplus 2i}$ . Similarly, define sets for  $X_{\oplus}$ ,  $Y_{\ominus}$ ,  $Y_{\oplus}$  with analogous characteristics. Observe that  $W$  is translation congruent to  $G_{TO}$

modulo  $2\pi$  because  $\bigcup_{i=1}^{\infty} X_{\ominus 4i} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \bigcup_{i=0}^{\infty} X_{\ominus 4i+3}$ ,  $\bigcup_{i=0}^{\infty} X_{\ominus 4i+2} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} =$

$\bigcup_{i=0}^{\infty} X_{\ominus 4i+1}$  and  $\bigcup_{i=1}^{\infty} X_{\oplus 2i} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \bigcup_{i=0}^{\infty} X_{\oplus 2i+1}$ . Notice  $\bigcup_{i=0}^{\infty} X_{\oplus 2i+1} \cup \bigcup_{i=0}^{\infty} X_{\ominus 4i+1} \cup$

$\bigcup_{i=0}^{\infty} X_{\ominus 4i+3} = G(X_{\ominus odd}) \cup G(X_{\oplus odd})$ , and thus we observe that all of the white spaces in the set  $G_{TO}$  are filled when we translate the black sets on the exterior

of  $G_{TO}$  by multiples of  $\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$ . Moreover,  $W$  is dilation congruent to  $G_{SO}$  because

$\frac{1}{2}G(X_{\oplus even}) = G(X_{\ominus odd}) \in G_{SO}$  (i.e., the even pieces of the form  $X_{\ominus n}$  scale into the odd pieces of the form  $X_{\oplus n}$ ), and  $\frac{1}{4}G(X_{\oplus even}) = G(X_{\oplus odd})$ . Thus  $W$  is a wavelet set by definition.

Thanks to this example, we see that a wavelet set may demonstrate characteristics of both patch and crossover wavelet sets, and thereby not be classified as either type. The set contains both a ‘‘patch region’’ and a ‘‘crossover region.’’ Therefore, we have not made a complete classification of all two dimensional wavelet sets, but note that crossover wavelet sets seem to be maximally non-patch. Finding a broader algorithm which encompasses both the patch and crossover algorithms would be an interesting problem to consider.

As a final comment, we remark that crossover and patch wavelet sets make perfect sense in one-dimension (i.e. in  $\mathbb{R}^1$ ). The reader can easily prove that

all dyadic one-dimensional wavelet sets of two or three intervals are necessarily crossover wavelet sets. (Here crossover would mean through the origin.) On the other hand, the well known Journé wavelet set of 4 intervals (c.f. [4], Example 4.5(i)) is easily seen to be a *patch* wavelet set. A characterization is not known at this time of all finite interval patch wavelet sets.

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