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Operator Theory and Modulation Spaces

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ABSTRACT. This is a "problems" paper. We isolate some connections between operator theory and the theory of modulation spaces that were stimulated by a question of Feichtinger's regarding integral and pseudodifferential operators. We discuss several problems inspired by this question, and give a reformulation of the original question in operator-theoretic terms. A detailed discussion of the background and context for these problems is included, along with a solution of the problem for the case of finite-rank operators.

1. Introduction

The purpose of this article is to present some connections between two subareas of modern analysis: operator theory and the theory of modulation spaces. The Oberwolfach mini-workshop on *Wavelets, Frames, and Operator Theory*, which took place in February 2004, had as one of its central aims the forging of direct and indirect connections between these two areas. After Hans Feichtinger gave out a particular problem on modulation spaces in a workshop problem session, the two authors of this article set out to give a reformulation of the problem in operator-theoretic terms, in order to promote connections. The results of our discussions are given here. It is hoped that this article will promote the development of new inroads into both of these subjects. We think that we have isolated some interesting research problems, whose solution could conceivably impact mathematics beyond operator theory and beyond modulation space theory.

In Section 2 we give the statements of several problems in operator theory inspired by Feichtinger's original question, as well as an operator-theoretic reformulation of his question. Then in Section 3 we present some background and explain the precise relationship of the original question to the problems discussed in Section 2.

The form of the problems that we state in Section 2 appear to be natural ones accessible to specialists in operator theory, but of a type that perhaps would not have been considered by specialists without an external motivation. We think of

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them as *problems in operator theory that are motivated by a problem in modulation space theory.*

1.1. Notation. Throughout, H will denote an infinite-dimensional separable Hilbert space, with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. If x and y are vectors in H, then $x \otimes y$ will denote the rank one operator defined by $(x\otimes y)z = \langle z, y\rangle\, x$. The operator norm of $x \otimes y$ is then just the product of $\|x\|$ and $\|y\|$.

If $A \in \mathcal{B}(H)$ is a compact operator, then A has a countable set of nonnegative singular values $\{s_n(A)\}_{n=1}^\infty$, which we arrange in nonincreasing order. These can be defined by spectral theory to be the square-roots of the eigenvalues of the positive, self-adjoint operator A^*A, i.e., $s_n(A) = \lambda_n(A^*A)^{1/2}$. If A is self-adjoint then $s_n(A) = |\lambda_n(A)|$, and if A is positive (the case we are concerned with in this article), then $s_n(A) = \lambda_n(A)$. We say that A is *trace-class* if

$$\|A\|_{\mathcal{I}_1} \;=\; \sum_{n=1}^{\infty} s_n(A) \;<\; \infty.$$

The space $\mathcal{I}_1$ of trace-class operators is a Banach space under the norm $\|\cdot\|_{\mathcal{I}_1}$, see [**Sim79**], [**DS88**].

2. Some Problems in Operator Theory

Below we will introduce some terminology and provide a characterization of postive finite-rank operators that have a certain property, and then present several new problems.

2.1. Definitions and Observations. Fix an orthonormal basis $\mathcal{E} = \{e_k\}_{k\in\mathbf{N}}$ for H, and define

$$H^1 \;=\; \Bigl\{f \in H : \|f\|_1 = \sum_{k=1}^{\infty} |\langle f, e_k\rangle| < \infty\Bigr\}. \tag{2.1}$$

We have the following facts.

Lemma 2.1.

(a) *H^1 is a dense subspace of H.*

(b) *H^1 is a Banach space with respect to the norm $\|\cdot\|_1$.*

(c) *H^1 is isomorphic to ℓ^1.*

(d) *$\|e_k\|_1 = \|e_k\| = 1$ for every k.*

(e) *If $f \in H^1$, then $f = \sum_{k=1}^\infty \langle f, e_k\rangle\, e_k$, with convergence of this series in both norms $\|\cdot\|$ and $\|\cdot\|_1$.*

Let T be any positive trace-class operator in $\mathcal{B}(H)$. Let $\{\lambda_k\}$ denote the positive eigenvalues of T (either finitely many or converging to zero if infinitely many). Then the spectral representation of T is

$$T \;=\; \sum_k \lambda_k\,(g_k \otimes g_k) \;=\; \sum_k h_k \otimes h_k, \tag{2.2}$$

where $\{g_k\}$ are orthonormal eigenvectors of T and $h_k = \lambda_k^{1/2} g_k$. Since the eigenvalues of T coincide with its singular values, the trace of T coincides with its trace-class norm. In particular,

$$\sum_k \|h_k\|^2 \;=\; \sum_k \lambda_k \;=\; \operatorname{trace}(A) \;=\; \|A\|_{\mathcal{I}_1} \;<\; \infty. \tag{2.3}$$

Let us say that T is of *Type A* with respect to the orthonormal basis $\mathcal{E}$ if, for the eigenvectors $\{h_k\}$ as above, we have that

$$\sum_k \|h_k\|_1^2 \;<\; \infty. \tag{2.4}$$

Note that this is just the (somewhat unusual) formula displayed above for the trace of T with the H^1-norm used in place of the usual Hilbert space norm of the vectors $\{h_k\}$.

Let us also say that T is of *Type B* with respect to the orthonormal basis $\mathcal{E}$ if there is *some* sequence of vectors $\{v_k\}$ in H such that

$$\sum_k \|v_k\|_1^2 \;<\; \infty \qquad \text{and} \qquad T \;=\; \sum_k v_k \otimes v_k,$$

where the convergence of this series is in the strong operator topology. Note that the definition of Type B does depend on the choice of orthonormal basis, for reasons which will become apparent later.

Example 2.2. Not every positive trace-class operator is of Type A. For example, choose any vector $f \in H \setminus H^1$. Then $T = f \otimes f$ is positive and trace-class (in fact, has rank one), but is clearly not of Type A with respect to $\mathcal{E}$. In particular, positive finite-rank operators need not be of Type A.

In fact, we can characterize all finite-rank operators that are of Type A or Type B. To do this, we need the following lemma.

Lemma 2.3. *Let T be a positive operator on H such that $T = \sum_k v_k \otimes v_k$, where the series has either finitely or countably many terms and converges in the strong operator topology. Then $\overline{\operatorname{range}(T)} \;=\; \overline{\operatorname{span}}\{v_k\}$.*

Proof. Let P be the orthogonal projection of H onto $\overline{\operatorname{range}(T)}$, and let $P^\perp = I - P$. Then

$$0 \;=\; P^\perp T P^\perp \;=\; \sum_k P^\perp v_k \otimes P^\perp v_k.$$

But each projection $P^\perp v_k \otimes P^\perp v_k$ is a positive operator, so this implies that $P^\perp v_k = 0$ for all k. Hence $v_k \in P(H) = \overline{\operatorname{range}(T)}$, so $\overline{\operatorname{span}}\{v_k\} \subseteq \overline{\operatorname{range}(T)}$.

For the converse inclusion, suppose that $\overline{\operatorname{span}}\{v_k\}$ was a proper subset of $\overline{\operatorname{range}(T)}$. Then we could find a unit vector $z \in \overline{\operatorname{range}(T)}$ that is perpendicular to each v_k. Let $z_j = Tw_j \in \operatorname{range}(T)$ be such that $z_j \to z$, and let $Q = z \otimes z$ be the orthogonal projection onto the span of z. Then $Qz_j \to Qz = z$, but for each j we also have

$$Qz_j \;=\; QTw_j \;=\; \sum_k \langle w_j, v_k\rangle\, Qv_k \;=\; 0,$$

so this implies $z = 0$, which is a contradiction. □

Now we can characterize the finite-rank operators that are of Type A or Type B.

PROPOSITION 2.4. *Let T be a positive finite-rank operator. Then the following statements are equivalent.*

(a) *T is of Type A.*
(b) *T is of Type B.*
(c) *Each eigenvector of T corresponding to a nonzero eigenvalue belongs to H^1.*
(d) range$(T) \subseteq H^1$.

PROOF. (a) $\Leftrightarrow$ (c) and (a) $\Rightarrow$ (b) are clear.

(b) $\Rightarrow$ (d). Assume that $T = \sum_k v_k \otimes v_k$ where $\sum_k \|v_k\|_1^2 < \infty$. Since T has finite rank, it has closed range. Lemma 2.3 therefore implies that range$(T) = \overline{\mathrm{span}}\{v_k\} \subseteq H^1$.

(d) $\Rightarrow$ (a). Suppose that range$(T) \subseteq H^1$, and let $T = \sum_k h_k \otimes h_k$ be the spectral representation of T as in (2.2). Since T has finite rank, this is a finite sum. Further, Lemma 2.3 implies that $h_k \in \overline{\mathrm{range}(T)} \subset H^1$ for each k. Since there are only finitely many k, it follows that $\sum_k \|h_k\|_1^2 < \infty$, so T is of Type A. □

In particular, the operator of Example 2.2 is neither of Type A nor of Type B.

2.2. Problems. Now we will give a set of problems related to the definition of Type A and Type B operators.

As shown in Proposition 2.4, Type A and Type B are equivalent for positive finite-rank operators. Our first problem asks if this is true in general.

PROBLEM 2.5. If T is of Type B with respect to an orthonormal basis $\mathcal{E}$, must it be of Type A with respect to $\mathcal{E}$? ♢

We expect that the answer to this problem is negative, but this leads immediately to the following problem.

PROBLEM 2.6. Let $\mathcal{E}$ be an orthonormal basis for H. Find a characterization of all positive trace-class operators T that are of Type B with respect to $\mathcal{E}$. ♢

Our next problem comes closer to an operator-theoretic formulation of the question of Feichtinger. This problem asks if a particular class of operators are of Type A. Note that for this class of operators, each eigenvector of T corresponding to a nonzero eigenvalue belongs to H^1 (compare Proposition 2.4).

PROBLEM 2.7. Let $\mathcal{E} = \{e_k\}_{k\in\mathbf{N}}$ be an orthonormal basis for H. Fix scalars $\{c_{mn}\}_{m,n\in\mathbf{N}} \in \ell^1$ such that $c_{mn} = \overline{c_{nm}}$ for all m, n. Each operator $e_m \otimes e_n$ is trace-class, with trace-class norm

$$\|e_m \otimes e_n\|_{\mathcal{I}_1} \;=\; \|e_m\|\,\|e_n\| \;=\; 1.$$

Therefore, we can define $T\colon H \to H$ by

$$T \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\,(e_m \otimes e_n).$$

This series converges in the strong operator topology and also absolutely in trace-class norm, because

$$\|T\|_{\mathcal{I}_1} \;\le\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |c_{mn}|\,\|e_m \otimes e_n\|_{\mathcal{I}_1} \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |c_{mn}| \;<\; \infty. \tag{2.5}$$

Thus $T \in \mathcal{I}_1$. Further, the condition $c_{mn} = \overline{c_{nm}}$ implies that T is self-adjoint, and for simplicity we will also assume that T is positive.

If we write the spectral representation of T as in (2.2), then for each eigenvalue λ_k we have

$$\lambda_k h_k \;=\; T h_k \;=\; \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \langle h_k, e_m \rangle \, e_n.$$

Since $\|h_k\| = \lambda_k^{1/2} > 0$, since $\|e_m\| = 1$, and since $\|e_n\|_1 = 1$, we therefore have

$$\begin{aligned} |\lambda_k| \, \|h_k\|_1 \;=\; \|T h_k\|_1 \;&\le\; \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_{mn}| \, |\langle h_k, e_m \rangle| \, \|e_n\|_1 \\ &\le\; \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_{mn}| \, \|h_k\| \, \|e_m\| \, \|e_n\|_1 \\ &=\; \lambda_k^{1/2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_{mn}| \;<\; \infty. \end{aligned}$$

Thus $\|h_k\|_1 < \infty$ for each k, and in particular we have $h_k \in H^1$ for every k.

Problem: Is this operator of Type A? That is, must we have $\sum_k \|h_k\|_1^2 < \infty$? Unfortunately, the calculation above only tells us that

$$\sum_k \|h_k\|_1^2 \;\le\; C \sum_k \frac{1}{\lambda_k},$$

and the right-hand side above is infinite if there are infinitely many eigenvalues. ♢

A positive solution to Problem 2.7 would imply a positive solution to the original problem of Feichtinger. In particular, the following is an equivalent reformulation of his question.

PROBLEM 2.8. Let $H = L^2(\mathbf{R})$ and let $\mathcal{E}$ be a *Wilson orthonormal basis* for $L^2(\mathbf{R})$ (defined precisely in Section 3.5 below). If T is one of the operators defined in Problem 2.7 with respect to a Wilson basis, must T be of Type A? ♢

Thus, the operator-theoretic problem of Problem 2.7 extracts the essence of Feichtinger's question without reference to the specific structure of a Wilson basis. A counterexample to Problem 2.7 would not necessarily settle the original question, for it could be the case that the particular structure of the Wilson bases plays a role. That is, if this problem really just depends on having an orthonormal basis, then it is a purely operator-theoretic question, while if it depends more explicitly on the particular functional properties of Wilson bases then is becomes more specifically a question about the modulation spaces. In any case, Problem 2.7 establishes a potentially interesting connection between operator theory and modulation space theory.

3. Background and Setting

In the remainder of this article we will attempt to present some background on the modulation spaces and then give the original formulation of Feichtinger's question.

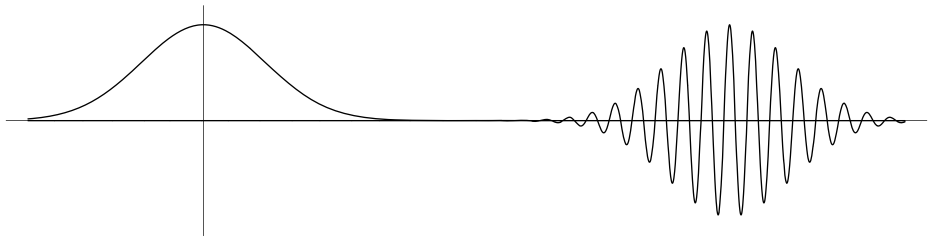

FIGURE 1. Window g and time-frequency shift $g_{x,\omega}$ (real part).

3.1. The STFT. Given a *window function* $g \in L^2(\mathbf{R})$ and given $x,\ \omega \in \mathbf{R}$, let $g_{x,\omega}$ be the *time-frequency shift* of g defined by

$$g_{x,\omega}(t) \;=\; e^{2\pi i\omega t} g(t-x)$$

(see Figure 1). The *short-time Fourier transform* (STFT) or *continuous Gabor transform* of a function $f \in L^2(\mathbf{R})$ with respect to the window g is

$$V_g f(x,\omega) \;=\; \langle f, g_{x,\omega}\rangle \;=\; \int f(t)\,\overline{g(t-x)}\,e^{-2\pi i\omega t}\,dt.$$

A standard fact is that V_g is a multiple of an isometry from $L^2(\mathbf{R})$ into $L^2(\mathbf{R}^2)$. Specifically, if $\|g\|_2 = 1$, then we have the norm equality

$$\|f\|_2 \;=\; \|V_g f\|_2 \;=\; \left(\iint |V_g f(x,\omega)|^2\,dx\,d\omega\right)^{1/2} \;=\; \left(\iint |\langle f, g_{x,\omega}\rangle|^2\,dx\,d\omega\right)^{1/2}, \tag{3.1}$$

and we also have the formal inversion formula

$$f \;=\; \iint V_g f(x,\omega)\, g_{x,\omega}\,dx\,d\omega.$$

The inversion formula represents a function f as a superposition of "notes" $g_{x,\omega}$, with the value of $V_g f(x,\omega)$ determining the "amount" of the note $g_{x,\omega}$ that is present in f. At least qualitatively, $V_g f(x,\omega)$ represents the amount of frequency ω present in f at time x, and hence we say that $V_g f$ is a *time-frequency representation of f*. See [**Grö01**] for precise interpretations of these remarks and the inversion formula.

3.2. Modulation Spaces. The modulation spaces were invented and extensively investigated by Feichtinger, with some of the main references being [**Fei81**], [**Fei89**], [**FG89a**], [**FG89b**], [**FG97**], [**Fei03**]. For a detailed development of the theory of modulation spaces we refer to the original literature mentioned above and to Gröchenig's text [**Grö01**, Ch. 11–13]. For a personal historical account of the development of the modulation spaces, including the Feichtinger algebra in particular, see Feichtinger's recent article [**Fei06**].

The modulation spaces are defined by imposing a different norm on the STFT in place of the usual L^2 norm.

DEFINITION 3.1. Fix $1 \le p \le 2$, and let ϕ be any nonzero Schwartz-class function (for example, the Gaussian function e^{-x^2}). Then the *modulation space*

$M^p(\mathbf{R})$ consists of all functions $f \in L^2(\mathbf{R})$ such that

$$\begin{aligned}\|f\|_{M^p} \;=\; \|V_\phi f\|_p \;&=\; \biggl(\iint |V_\phi f(x,\omega)|^p \, dx \, d\omega\biggr)^{1/p} \\ &=\; \biggl(\iint |\langle f, \phi_{x,\omega}\rangle|^p \, dx \, d\omega\biggr)^{1/p} \;<\; \infty.\end{aligned}$$

The definition of M^p is independent of the choice of ϕ in the sense that each different choice of ϕ defines an equivalent norm for the same set M^p. Each modulation space is a Banach space. Even M^∞, which is "nearly" all of the space of tempered distributions, is a Banach space, whereas the space of tempered distributions $\mathcal{S}'(\mathbf{R})$ is only a Fréchet space.

By equation (3.1), we have that $M^2 = L^2$. For other p, the space M^p is not L^p. Instead, the M^p norm measures the L^p norm *of the STFT of* f, not the L^p norm of f itself. Thus the M^p norm is quantifying the quality of joint time-frequency concentration that f possesses.

REMARK 3.2. a. Because ϕ is taken to be a Schwartz-class function, "inner products" $\langle f, \phi_{x,\omega}\rangle$ are defined not only when $f \in L^2(\mathbf{R})$ but whenever f is a tempered distribution. Thus, we can define modulation spaces not only when $p \le 2$ but also for $p > 2$ by defining

$$M^p(\mathbf{R}) \;=\; \biggl\{ f \in \mathcal{S}'(\mathbf{R}) : \|f\|_{M^p} \;=\; \biggl(\iint |\langle f, \phi_{x,\omega}\rangle|^p \, dx \, d\omega\biggr)^{1/p} \;<\; \infty \biggr\}.$$

When $p > 2$, the space M^p is a superset of L^2. In fact, we have the inclusions

$$\mathcal{S}(\mathbf{R}) \;\subsetneq\; M^1(\mathbf{R}) \;\subsetneq\; M^2(\mathbf{R}) \;=\; L^2(\mathbf{R}) \;\subsetneq\; M^\infty(\mathbf{R}) \;\subsetneq\; \mathcal{S}'(\mathbf{R}).$$

b. The M^p spaces defined above are only the simplest examples of the modulation spaces; we can define other modulation spaces by imposing other norms on the Gabor coefficients. For example, if $1 \le p, q \le \infty$ and $v \colon \mathbf{R} \to (0,\infty)$ is a weight function, then the modulation space $M^{p,q}_v(\mathbf{R})$ consists of all tempered distributions f for which the norm

$$\|f\|_{M^{p,q}_v} = \|V_g f\|_{L^{p,q}_v} = \biggl(\int \biggl(\int |V_g f(x,\omega)|^p \, v(x,\omega)^p \, dx\biggr)^{q/p} d\omega\biggr)^{1/q}$$

is finite, with the usual adjustments if p or q is infinite. These more complicated spaces have many important applications. For example, the space $M^{\infty,1}_v$ plays an important role in modeling transmission channels for wireless communications [**SB03**], [**SH03**], [**Str06**], and in the theory of pseudodifferential operators [**GH99**], [**BO04**].

c. Among the modulation spaces, the space M^1 plays an especially important role. This space is often called the *Feichtinger algebra*, and it corresponds to the space H^1 in the statement of the problems in Section 2. The Feichtinger algebra is sometimes denoted by S_0 instead of M^1, to denote that it is a particular *Segal algebra*. The Feichtinger algebra has many interesting properties. For example, it is a Banach algebra under two operations: pointwise products and convolution. Also, M^1 is invariant under the Fourier transform, as is L^2, the Schwartz space $\mathcal{S}$, the tempered distributions $\mathcal{S}'$, and indeed each of the spaces M^p for $1 \le p \le \infty$. There are many equivalent characterizations of M^1; for example, it is the minimal non-trivial Banach space contained in L^1 that is isometrically invariant under both translations and modulations.

d. If we substitute translations and dilations for time-frequency shifts (i.e., we use the continuous wavelet transform instead of the STFT), then the analogue of the modulation spaces (obtained by imposing norms on the continuous wavelet transform) are the Besov or Triebel–Lizorkin spaces. The norms of the Besov and Triebel–Lizorkin spaces quantify the smoothness properties of functions, while the norms of the modulation spaces quantify the time-frequency concentration of functions. The spaces L^p for $p \neq 2$ are Triebel–Lizorkin spaces, but they are not modulation spaces. Both of the Besov/Triebel–Lizorkin and the modulation space classes are special cases of the general *coorbit theory* developed in [**FG89a**], [**FG89b**].

3.3. Gabor Frames. Gabor frames provide natural basis-like expansions of elements of the Hilbert space $L^2(\mathbf{R})$. We refer to the texts [**Dau92**], [**Grö01**], [**You01**], or [**Chr03**] for more information on frames and Gabor frames in particular.

Fix any *window function* $g \in L^2(\mathbf{R})$ and any $\alpha,\ \beta > 0$. The *Gabor system* generated by g, α, and β is

$$\mathcal{G}(g,\alpha,\beta) \;=\; \{e^{2\pi i\beta nt} g(t-\alpha k)\}_{k,n\in\mathbf{Z}} \;=\; \{g_{\alpha k,\beta n}\}_{k,n\in\mathbf{Z}}.$$

If there exist constants $A,\ B > 0$ such that

$$\forall\, f \in L^2(\mathbf{R}), \quad A\,\|f\|_2^2 \;\le\; \sum_{k,n\in\mathbf{Z}} |\langle f, g_{\alpha k,\beta n}\rangle|^2 \;\le\; B\,\|f\|_2^2,$$

then $\mathcal{G}(g,\alpha,\beta)$ is called a *Gabor frame*, and A, B are *frame bounds*. In this case there exists a *canonical dual window* $\tilde{g} \in L^2(\mathbf{R})$ such that $\mathcal{G}(\tilde{g},\alpha,\beta)$ is a frame and we have the basis-like *frame expansions*

$$\forall\, f \in L^2(\mathbf{R}), \quad f \;=\; \sum_{k,n\in\mathbf{Z}} \langle f, \tilde{g}_{\alpha k,\beta n}\rangle\, g_{\alpha k,\beta n} \;=\; \sum_{k,n\in\mathbf{Z}} \langle f, g_{\alpha k,\beta n}\rangle\, \tilde{g}_{\alpha k,\beta n}, \tag{3.2}$$

where these series converge unconditionally in L^2-norm.

If we can take $A = B = 1$ then we call $\mathcal{G}(g,\alpha,\beta)$ a *Parseval Gabor frame*. This case is especially simple since the dual frame coincides with the original frame, and we have the orthonormal basis-like expansions

$$\forall\, f \in L^2(\mathbf{R}), \quad f \;=\; \sum_{k,n\in\mathbf{Z}} \langle f, g_{\alpha k,\beta n}\rangle\, g_{\alpha k,\beta n}.$$

It is easy to construct Gabor frames, see for example the "painless nonorthogonal expansions" of [**DGM86**]. We have the following facts.

(a) If $\alpha\beta > 1$ then $\mathcal{G}(g,\alpha,\beta)$ cannot be a frame, and in fact is incomplete in $L^2(\mathbf{R})$.

(b) If $\alpha\beta = 1$ then $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame if and only if it is a Riesz basis for $L^2(\mathbf{R})$, i.e., the image of an orthonormal basis under a continuous invertible map. Further, if $\alpha\beta = 1$ then $\mathcal{G}(g,\alpha,\beta)$ is a Parseval Gabor frame if and only if it is an orthonormal basis for $L^2(\mathbf{R})$.

(c) If $\alpha\beta < 1$, then any Gabor frame $\mathcal{G}(g, \alpha, \beta)$ is *overcomplete*, i.e., it contains a complete proper subset. In particular, such a frame is neither orthonormal nor a basis, and the coefficients in the frame expansions in (3.2) are not unique.

These facts are part of the *Density Theorem* for Gabor frames; see [**Hei06**] for a detailed survey of and references for this theorem.

Unfortunately, the *Balian–Low Theorem* implies that if $\mathcal{G}(g, \alpha, \beta)$ is a Riesz basis for $L^2(\mathbf{R})$, then the window g cannot be jointly well-localized in the time-frequency plane—either g or $\hat{g}$ must decay slowly (see [**BHW95**] for a survey of the Balian–Low Theorem). As a consequence, in practice we must usually deal with overcomplete Gabor frames. As long as $\alpha\beta < 1$, we can construct Gabor frames or Parseval Gabor frames with g extremely nice, e.g., Schwartz-class or even infinitely differentiable and compactly supported.

3.4. Gabor Frames and the Modulation Spaces. If $\mathcal{G}(\phi, \alpha, \beta)$ is a frame for $L^2(\mathbf{R})$ and ϕ is a "nice" function (e.g., ϕ Schwartz-class or indeed any $\phi \in M^1$), then the Gabor frame expansions given in (3.2) converge not only in L^2 but *in all the modulation spaces*, as follows (see [**Grö01**, Ch. 11–13] for proofs).

THEOREM 3.3. *Assume that*

(a) $\phi \in M^1(\mathbf{R})$, *and*

(b) $\mathcal{G}(\phi, \alpha, \beta)$ *is a frame for* $L^2(\mathbf{R})$.

Then the following statements hold.

(i) *The dual window* $\tilde{\phi}$ *belongs to* $M^1(\mathbf{R})$.

(ii) *For every* $1 \le p \le \infty$ *we have that*

$$\forall\, f \in M^p(\mathbf{R}), \quad f \;=\; \sum_{k,n\in\mathbf{Z}} \langle f, \tilde{\phi}_{\alpha k,\beta n}\rangle\, \phi_{\alpha k,\beta n} \;=\; \sum_{k,n\in\mathbf{Z}} \langle f, \phi_{\alpha k,\beta n}\rangle\, \tilde{\phi}_{\alpha k,\beta n},$$

where these series converge unconditionally in the norm of M^p *(weak* convergence if* $p = \infty$*).*

(iii) *For every* $1 \le p \le \infty$ *the Gabor frame coefficients provide an* equivalent norm *for the modulation space* M^p*, i.e.,*

$$\|\!|f|\!\|_{M^p} \;=\; \biggl(\sum_{k,n\in\mathbf{Z}} |\langle f, \phi_{\alpha k,\beta n}\rangle|^p\biggr)^{1/p} \tag{3.3}$$

is an equivalent norm for M^p.

Thus, $\mathcal{G}(\phi, \alpha, \beta)$ is a *Banach frame* for M^p in the sense of [**Grö91**], [**CHL99**]. However, we emphasize that rather than constructing a Banach frame for a single particular space, Theorem 3.3 says that any Gabor frame for the Hilbert space $L^2(\mathbf{R})$ whose window lies in M^1 is *simultaneously* a Banach frame for *every modulation space.*

The fact that if $\phi \in M^1$ then $\tilde{\phi} \in M^1$ as well was proved by Gröchenig and Leinert [**GL04**], using deep results about symmetric Banach algebras (for the case that $\alpha\beta$ is rational, this result was obtained earlier by Feichtinger and Gröchenig in [**FG97**]). A different proof based on the concept of *localized frames* was given in [**BCHL06**]. That proof also extends to *irregular Gabor frames*, whose index set is not a lattice, and whose dual frame will not itself be a Gabor frame.

3.5. Wilson Bases. The equivalent norm for M^p given in equation (3.4) suggests that we should have $M^p \cong \ell^p$. While this is true, it does not follow from the facts we have presented so far. The issue is that Gabor frames need not be bases, and hence we cannot use them to define an isomorphism from M^p to ℓ^p. For example, even for the case $p = 2$ we know that for an overcomplete Gabor frame $\mathcal{G}(\phi, \alpha, \beta)$, the range of the *analysis mapping* $f \mapsto \{\langle f, \phi_{\alpha k, \beta n}\rangle\}_{k,n \in \mathbf{Z}}$ is only a proper subspace of ℓ^2. Moreover, because of the Balian–Low Theorem, we cannot get around this by trying to construct a nice ϕ so that the Gabor frame is a Riesz basis—no such nice ϕ exists. And ϕ must be nice (specifically $\phi \in M^1$) in order to apply Theorem 3.3 to conclude that the Gabor frame coefficients will provide an equivalent norm for M^p and that the Gabor frame expansions will converge in M^p. A frame with $\phi \in L^2$ that is not in M^1 will provide frame expansions for L^2, but those frame expansions will not converge in M^p.

Fortunately, there does exist a remarkable construction of an orthonormal basis for L^2, called a *Wilson basis*, which is simultaneously an unconditional basis for every modulation space. Wilson bases were first suggested by Wilson in [**Wil87**]. The fact that they provide orthonormal bases for $L^2(\mathbf{R})$ was rigorously proved by Daubechies, Jaffard, and Journé [**DJJ91**]. Feichtinger, Gröchenig, and Walnut proved that Wilson bases are unconditional bases for all of the modulation spaces [**FGW92**]. We also mention that the *local sine and cosine bases* of Coifman and Meyer [**CM91**] include many examples of Wilson and wavelet bases, and that the lapped transforms of Malvar [**Mal90**] are closely related. We refer to the original literature and to Sections 8.5 and 12.3 of [**Grö01**] for more details and for proofs of the results below.

The construction of a Wilson basis starts with a "twice redundant" Parseval Gabor frame $\mathcal{G}(g, \frac{1}{2}\mathbf{Z} \times \mathbf{Z})$ whose generator satisfies a symmetry condition, then forms linear combinations of elements, namely,

$$M_n T_{\frac{k}{2}} g \pm M_{-n} T_{\frac{k}{2}} g,$$

and finally "magically" extracts from the set of these linear combinations a subset which forms an orthonormal basis for $L^2(\mathbf{R})$. Moreover, if the original window g has sufficient joint concentration in the time-frequency plane, then a Wilson basis will be an unconditional basis not only for $L^2(\mathbf{R})$, but for all the modulation spaces. This is summarized in the following result, see [**Grö01**, Thm. 8.5.1] and [**Grö01**, Thm. 8.5.1] for proof.

THEOREM 3.4. *Assume that $\mathcal{G}(g, \frac{1}{2}\mathbf{Z} \times \mathbf{Z})$ is a Parseval Gabor frame for $L^2(\mathbf{R})$, and that $g(x) = \overline{g(-x)}$. Define*

$$\psi_{k,0} \;=\; T_k g, \qquad k \in \mathbf{Z},$$

and

$$\psi_{k,n}(x) \;=\; \begin{cases} \sqrt{2}\, \cos(2\pi n x)\, g(x - \frac{k}{2}), & \text{if } k+n \text{ is even},\\ \sqrt{2}\, \sin(2\pi n x)\, g(x - \frac{k}{2}), & \text{if } k+n \text{ is odd}, \end{cases}$$

and set

$$\mathcal{W}(g) \;=\; \bigl\{\psi_{k,n}\bigr\}_{k \in \mathbf{Z},\, n \geq 0}.$$

Then $\mathcal{W}(g)$ is an orthonormal basis for $L^2(\mathbf{R})$.

If in addition we have $g \in M^1(\mathbf{R})$, then the following further statements hold.

(i) *For every* $1 \le p \le \infty$ *we have that*

$$\forall\, f \in M^p(\mathbf{R}), \quad f \;=\; \sum_{k\in\mathbf{Z}} \sum_{n\ge 0} \langle f, \psi_{kn}\rangle\, \psi_{kn},$$

where the series converges unconditionally in the norm of M^p *(weak* convergence if* $p=\infty$*).*

(ii) *For every* $1 \le p \le \infty$ *the Wilson basis coefficients provide an* equivalent norm *for the modulation space* M^p*, i.e.,*

$$\|\!|f|\!\|_{M^p} \;=\; \biggl(\sum_{k\in\mathbf{Z}} \sum_{n\ge 0} |\langle f, \phi_{\alpha k,\beta n}\rangle|^p\biggr)^{1/p} \tag{3.4}$$

is an equivalent norm for M^p*.*

Consequently, $f \mapsto \{\langle f, \psi_{kn}\rangle\}_{k\in\mathbf{Z},\, n\ge 0}$ defines an isomorphism of M^p onto ℓ^p. In particular, if $H = L^2(\mathbf{R})$ and $\mathcal{E} = \mathcal{W}(g)$ is a Wilson basis for $L^2(\mathbf{R})$, then the space H^1 defined by equation (2.1) is precisely the modulation space $M^1(\mathbf{R})$. When we consider Wilson bases in this paper, we assume that they are constructed from M^1 windows.

By forming tensor products, the Wilson basis construction can be extended to create unconditional bases for the modulation spaces in higher dimensions.

The definition of the Wilson bases is rather technical, and the procedure behind it is in some sense "magical" and is not well-understood. For example, it is not known if it is possible to start with a "three times" redundant Gabor Parseval frame $\mathcal{G}(g, \frac{1}{3}\mathbf{Z} \times \mathbf{Z})$ and somehow create an orthonormal basis in the spirit of the Wilson bases.

3.6. Integral Operators. Now we return to the setting of Feichtinger's original problem. Given a *kernel function* $k \in L^2(\mathbf{R}^2)$, the corresponding *integral operator* is

$$Tf(x) \;=\; \int k(x,y)\, f(y)\, dy. \tag{3.5}$$

In terms of the kernel, T is self-adjoint if

$$k(x,y) \;=\; \overline{k(y,x)}.$$

Because $k \in L^2(\mathbf{R}^2)$, we know that T is a compact mapping of $L^2(\mathbf{R})$ onto itself. In fact, we have the equivalence that

$$k \in L^2(\mathbf{R}^2) \qquad \Longleftrightarrow \qquad T \text{ is a Hilbert–Schmidt operator.}$$

While such a characterization is not known for the trace-class operators, we will prove below a simple sufficient condition, namely that if k lies in the two-dimensional version of the Feichtinger algebra, i.e., $k \in M^1(\mathbf{R}^2)$, then T is a trace-class operator. This result was proved in [**Grö96**] and [**GH99**], and in fact is only a special case of more general theorems proved in [**GH99**]. For a survey of the role that modulation spaces play in the theory of integral and pseudodifferential operators, see [**Hei03**].

THEOREM 3.5. *If* $k \in M^1(\mathbf{R}^2)$*, then the corresponding integral operator* T *defined by* (3.5) *is trace-class, i.e.,* $T \in \mathcal{I}_1$*.*

Proof. Let $\mathcal{W}(g)$ be a Wilson orthonormal basis for $L^2(\mathbf{R})$ such that $g \in M^1(\mathbf{R})$. By Theorem 3.4, $\mathcal{W}(g)$ is also an unconditional basis for $M^1(\mathbf{R})$. For simplicity of notation, let us index this basis as $\mathcal{W}(g) = \{w_n\}_{n\in\mathbf{N}}$.

Now construct an orthonormal basis for $L^2(\mathbf{R}^2)$ by forming tensor products, i.e., set

$$W_{mn}(x,y) \;=\; w_m(x)\,\overline{w_n(y)}. \tag{3.6}$$

Then $\mathcal{U} = \{W_{mn}\}_{m,n\in\mathbf{N}}$ is both an orthonormal basis for $L^2(\mathbf{R}^2)$ and an unconditional basis for $M^1(\mathbf{R}^2)$. Therefore, since $k \in M^1(\mathbf{R}^2)$, we have that

$$k \;=\; \sum_{m,n\in\mathbf{Z}} \langle k, W_{mn}\rangle\, W_{mn}, \tag{3.7}$$

with convergence of the series in M^1-norm, and furthermore

$$|||k|||_{M^1} \;=\; \sum_{m,n\in\mathbf{Z}} |\langle k, W_{mn}\rangle| \;<\; \infty. \tag{3.8}$$

Substituting the expansion (3.7) into the definition of the integral operator in (3.5) yields

$$\begin{aligned} Tf(x) \;&=\; \int_{\mathbf{R}} \sum_{m,n\in\mathbf{Z}} \langle k, W_{mn}\rangle\, W_{mn}(x,y)\, f(y)\, dy \\ &=\; \sum_{m,n\in\mathbf{Z}} \langle k, W_{mn}\rangle \int_{\mathbf{R}} w_m(x)\,\overline{w_n(y)}\, f(y)\, dy \\ &=\; \sum_{m,n\in\mathbf{Z}} \langle k, W_{mn}\rangle\, \langle f, w_n\rangle\, w_m(x) \\ &=\; \sum_{m,n\in\mathbf{Z}} \langle k, W_{mn}\rangle\, (w_m \otimes w_n)(f)(x). \end{aligned} \tag{3.9}$$

The interchanges in order can all be justified because of the absolute convergence of the series implied by (3.8). Therefore we conclude that

$$T \;=\; \sum_{m,n\in\mathbf{Z}} \langle k, W_{mn}\rangle\, (w_m \otimes w_n). \tag{3.10}$$

Since each operator $w_m \otimes w_n$ belongs to $\mathcal{I}_1$ and the scalars $\langle k, W_{mn}\rangle$ are summable, the series (3.10) converges absolutely in $\mathcal{I}_1$, and therefore $T \in \mathcal{I}_1$. □

Theorem 3.5 can also be formulated in terms of pseudodifferential operators. In particular, the hypothesis that the kernel k of T written as an integral operator belongs to M^1 is equivalent to the hypothesis that the symbol σ of T written as a pseudodifferential operator belongs to M^1. We refer to [**Hei03**] for discussion along these lines.

3.7. The Problem. At last we come to the actual original question of Feichtinger, which is the following question about integral operators whose kernel belongs to $M^1(\mathbf{R}^2)$.

PROBLEM 3.6. Let T be a positive integral operator whose kernel k lies in $M^1(\mathbf{R}^2)$. Let equation (2.2) be the spectral representation of T. Must it be true that

$$\sum_{n=1}^{\infty} \|h_n\|_{M^1}^2 \;<\; \infty? \qquad \diamondsuit$$

We close by showing why Problem 2.8 is an equivalent reformulation of Problem 3.6.

PROPOSITION 3.7. *Problem 2.8 and Problem 3.6 are equivalent.*

PROOF. We simply have to show that every operator of the type considered in Problem 2.8 is an operator of the type considered in Problem 3.6, and vice versa.

Suppose that T is an operator of the type considered in Problem 3.6, i.e., T is be a positive integral operator whose kernel k lies in $M^1(\mathbf{R}^2)$. Then by using a Wilson basis we can, as in (3.10), write

$$T \;=\; \sum_{m,n\in\mathbf{Z}} c_{mn}\,(w_m \otimes w_n), \tag{3.11}$$

where the scalars $c_{mn} = \langle k, W_{mn}\rangle$ are summable. Hence T is exactly the type of operator considered in Problem 2.8.

Conversely, suppose that T is an operator of the type considered in Problem 2.8. That is, T is a positive operator of the form in (3.11) where the scalars c_{mn} are summable and $\{w_n\}_{n\in\mathbf{N}}$ is a Wilson orthonormal basis generated by an M^1 window. Let W_{mn} be the tensor product functions defined in (3.6). Then since $\{W_{mn}\}_{m,n\in\mathbf{N}}$ is an orthonormal basis for $L^2(\mathbf{R}^2)$, we have that

$$k \;=\; \sum_{m,n\in\mathbf{Z}} c_{mn} W_{mn} \in L^2(\mathbf{R}^2). \tag{3.12}$$

As in the calculations in (3.9), it follows that k is the kernel of the operator T (and hence T is a Hilbert–Schmidt operator).

Moreover, each W_{mn} belongs to $M^1(\mathbf{R})$, so since the c_{mn} are summable, we have that the series in (3.12) converges absolutely in M^1-norm. Hence the kernel k belongs to $M^1(\mathbf{R})$, and therefore T is a trace-class operator of exactly the type considered in Problem 3.6. □

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