# SEPARATING VECTORS FOR OPERATORS

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Abstract. It is an open problem whether every one-dimensional extension of a triangular operator admits a separating vector. We prove that the answer is positive for many triangular Hilbert space operators, and in particular, for strictly triangular operators. This is revealing, because two-dimensional extensions of such operators can fail to have separating vectors.

A number of the key motivational examples and pathological counterexamples to open questions in the literature that are contained in the articles [AS1, AS2, HLPW, HLW, LW1, LW3, LW4, W] have a special structural form which can be described in the following elementary way: They are extensions by algebraic operators (in fact often by one or two dimensional operators) of basic types of Hilbert space operators which themselves have especially good (that is, non-pathological) structure. An operator $T \in B(H)$ is called an *extension* of $A$ by $C$ if it has the form

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

with respect to an orthogonal decomposition $H = M \oplus N$ for the underlying Hilbert space. The good types that have been studied (so far) are normal operators and triangular operators.

Let $\mathcal{A}$ be an algebra of bounded linear operators on a separable complex Hilbert space $H$. A vector $x \in H$ is called a *separating vector* for $\mathcal{A}$ if the map $A \to Ax,\ A \in \mathcal{A}$, is injective. We denote by $Sep(\mathcal{A})$ the set of all separating vectors for $\mathcal{A}$. For an operator $T \in B(H)$, we use $\mathcal{W}(T)$ to denote the weakly closed unital subalgebra generated by $T$. An operator $T$ is said to have the *separating vector property*, or simply that *$T$ has a separating vector,* if $\mathcal{W}(T)$ has a separating vector. A vector $x$ in $H$ is called *an algebraic vector* for an operator $T \in B(H)$ if there is a non-zero polynomial $p$ in one variable satisfying $p(T)x = 0$. We use $\mathcal{E}_T$ to denote the set of all algebraic vectors for $T$.

An operator $T \in B(H)$ is called *triangular* if $H$ has an orthonormal basis $\{e_n : n = 1, 2, ...\}$ with the property that $Te_n \in span\{e_1, ..., e_n\}$ for each $n \in \mathbb{N}$. Equivalently, $T$ is triangular if it has an upper triangular matrix representation for some orthonormal basis indexed by the natural numbers. (See the survey article [H].) It is well-known that $T$ is triangular iff the set of algebraic vectors for $T$ is

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dense in $H$. An operator $T \in B(H)$ is called *strictly triangular* if it has a strictly upper triangular matrix representation for some orthonormal basis indexed by the natural numbers. In fact, the normal operators that occur in the interesting examples and counterexamples are often diagonal, so are triangular as well, and the triangular operators that occur are often weighted shifts with operator weights, so are strictly triangular, and are sometimes bitriangular operators (i.e. both $T$ and $T^*$ are triangular). These examples in turn motivated interesting research questions for the entire classes of normal operators and triangular operators.

Separating vectors for operators have played a role in several of the papers in the literature, and in particular some of the work of the authors dealing with counterexamples constructed by finite extensions of operators. Separating vectors for operator algebras and linear spaces of operators played an essential role in the work in [L] on algebraic reflexivity. It was a conjecture for awhile that every operator has a separating vector. The work in [W] settled this conjecture negatively, and also answered several longstanding open questions in single operator theory. It was subsequently proven in [GLW] that arbitrary *triangular* operators have separating vectors, and indeed, that $Sep(\mathcal{W}(A))$ is dense in $H$ for every triangular operator.

The example constructed in [W] showing that $\mathcal{W}(T)$ can fail to have a separating vector can be taken to be a two-dimensional extension of a backward shift operator. Therefore the question arises: Does every one dimensional extension of a triangular operator have the separating vector property? This question was posed in [LW2], and is still open. Let us say that an operator $A$ has *property* $(S_n)$ if $\mathcal{W}(T)$ has a separating vector for all $k$ $(k \leq n)$ dimensional extensions $T$ of $A$. Clearly, property $(S_n)$ implies property $(S_k)$ for all $k \leq n$. In [LW3] and [LW4] it was proven that normal operators and bitriangular operators have property $(S_n), \forall n$.

The main purpose of this paper is to prove that every strictly triangular operator has property $(S_1)$, and to prove some related results, some of which have independent interest.

Our first result gives a sufficient condition for an extension of an operator by a triangular operator to have the separating vector property.

In the construction of the counterexamples in [W], $\mathcal{W}(T)$ contains operators in a *corner* of $B(H)$. That is, there are projections $P$ and $Q$ with $PQ = 0$ and so that $PB(H)Q$ meets $W(T)$. In fact $PW(T)Q$ can be essentially arbitrarily prescribed, and this is the key to constructing the counterexamples. This motivates the following definition:

If $T \in B(H)$, and if $P$ is a projection in $lat(T)$, we say that $T$ is *stable* with respect to $P$ if

$$\mathcal{W}(T) \cap PB(H)P^{\perp} = (0).$$

*Lack of stability* is the obstruction to existence of separating vectors for extensions of triangular operators.

Suppose that an operator $T \in B(H \oplus K)$ and $P_H \in lat(T)$, where $P_H$ is the orthogonal projection from $H \oplus K$ onto $H$. Then $T$ has the form

$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}.$$

**Proposition 1.** *Let $T$, $A$, $B$, $X$ be as above. Suppose that $B$ is a triangular operator and $\mathcal{E}_T = \mathcal{E}_A \oplus 0$. If $T$ is stable with respect to $P_H$ and $A$ has the separating vector property, then so does $T$.*

*Proof.* Suppose $B = (b_{ij})$ such that $b_{ij} = 0$ when $i > j$. Let $u$ be a separating vector for $\mathcal{W}(A)$. We show that $u \oplus 0$ is a separating vector for $\mathcal{W}(T)$. Let $S \in \mathcal{W}(T)$ such that $S(u \oplus 0) = 0$. Then $S$ must have the form of

$$\begin{pmatrix} 0 & Y \\ 0 & C \end{pmatrix} = \begin{pmatrix} 0 & y_1 & y_2 & y_3 & . & . & . \\ & c_{11} & c_{12} & . & . & . & . \\ & & c_{22} & . & . & . & . \\ & & & & . & . & . \\ & & & & & & . \end{pmatrix}$$

Also let $z_i (i = 1, 2, ...)$ be the column vectors of the matrix $\begin{pmatrix} Y \\ C \end{pmatrix}$. We will show that $z_i \in \mathcal{E}_T$ for all $i$. If this is done, then, by the assumption that $\mathcal{E}_A \oplus 0 = \mathcal{E}_T$, we have $c_{ij} = 0$ for all $i, j$. Hence $C = 0$ and therefore $Y = 0$ since $T$ is stable with respect to $P_H$. Since $ST = TS$, an elementary matrix computation shows that

$$b_{1i}z_1 + ... + b_{ii}z_i = Tz_i \quad i = 1, 2, ....$$

Thus $z_1 \in \mathcal{E}_T$ and so $(T - b_{22}I)z_2 \in \mathcal{E}_T$. Hence $z_2 \in \mathcal{E}_T$. If we have checked that $z_1, ..., z_{i-1} \in \mathcal{E}_T$, then $(T - b_{ii}I)z_i = b_{1i}z_1 + ... + b_{i-1,i}z_{i-1} \in \mathcal{E}_T$. So $z_i \in \mathcal{E}_T$, as required. □

**Remark 2.** Our main interest in Proposition 1 is the case that $A$ is triangular, and not algebraic. For this situation the results of [HLW], and also [HLPW], can be used to construct finite dimensional extensions to which the proposition can be applied. But note also that if $A$ is any operator such that $\mathcal{E}_A = \{0\}$ (that is, $A$ has no point spectrum), and if $\mathcal{W}(A)$ has a separating vector, then $\mathcal{W}(T)$ has a separating vector for every triangular extension $T$ of $A$ for which $\mathcal{E}_T = \{0\}$.

As usual, a ring $\mathcal{A}$ (in our work $\mathcal{A}$ will be an operator algebra) is an *integral domain* if $\mathcal{A}$ has *no zero divisors* (i.e. if $A, B \in \mathcal{A}$ and $AB = 0$, then either $A = 0$ or $B = 0$.)

**Proposition 3.** *Let $A \in B(H)$. If $\mathcal{W}(A)$ is an integral domain and has a separating vector, then $A$ has property $(S_1)$.*

*Proof.* Let

$$T = \begin{pmatrix} A & b \\ 0 & t \end{pmatrix}$$

be a one dimensional extension of $A$ with some vector $b \in H$ and some scalar $t \in \mathbb{C}$. Choose $u \in Sep(\mathcal{W}(A))$. We claim that in fact at most one element in

$$\{\begin{pmatrix} \lambda u \\ 1 \end{pmatrix} : \lambda \in \mathbb{C}\}$$

is *not* a separating vector for $\mathcal{W}(T)$. To see this, assume that there exist two different numbers $\lambda_1$ and $\lambda_2$ such that neither $\lambda_1 u \oplus 1$ nor $\lambda_2 u \oplus 1$ is separating for $\mathcal{W}(T)$. Then there exist operators

$$T_1 = \begin{pmatrix} A_1 & b_1 \\ 0 & t_1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} A_2 & b_2 \\ 0 & t_2 \end{pmatrix}$$

in $\mathcal{W}(T)$ such that $T_i \neq 0$ $(i = 1, 2)$, and

$$T_1 \begin{pmatrix} \lambda_1 u \\ 1 \end{pmatrix} = 0, \qquad T_2 \begin{pmatrix} \lambda_2 u \\ 1 \end{pmatrix} = 0.$$

Thus

$$T_1 T_2 \begin{pmatrix} \lambda_1 u \\ 1 \end{pmatrix} = 0, \qquad T_2 T_1 \begin{pmatrix} \lambda_2 u \\ 1 \end{pmatrix} = 0.$$

Note that $T_1T_2 = T_2T_1$ since $\mathcal{W}(T)$ is abelian. Thus, taking the difference yields

$$T_1 T_2 \begin{pmatrix} (\lambda_1 - \lambda_2)u \\ 0 \end{pmatrix} = 0,$$

which implies that $A_1A_2(\lambda_1 - \lambda_2)u = 0$. Since $A_1A_2 \in \mathcal{W}(A)$ and $u \in Sep(\mathcal{W}(A))$, it follows that $A_1A_2 = 0$. Thus either $A_1 = 0$ or $A_2 = 0$ since $\mathcal{W}(T)$ is an integral domain. By assumption we have

$$T_1 \begin{pmatrix} \lambda_1 u \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 A_1 u + b_1 \\ t \end{pmatrix} = 0.$$

So if $A_1 = 0$, then $T_1 = 0$ which contradicts with our assumption on $T_1$. Similarly, the assumption $A_2 = 0$ gives a contradiction. Thus, except possibly for at most one number $\lambda$, $\lambda u \oplus 1 \in Sep(T)$. $\square$

**Remark 4.** Regarding the proof of the above proposition, it is of interest to note that for each $\lambda \in \mathbb{C}$ and each $u \in Sep(\mathcal{W}(A))$, there is an extension $T$ of $A$ such that $\lambda u \oplus 1 \notin Sep(\mathcal{W}(T))$. Namely, take

$$T = \begin{pmatrix} A & -\lambda A u \\ 0 & 0 \end{pmatrix}.$$

We give a natural application of Proposition 3, which we will generalize. For a function $\phi \in L^\infty$, we use $T_\phi$ to denote the Toeplitz operator on $H^2$ defined by

$$T_\phi f = P\phi f, \quad f \in H^2,$$

where $P$ is the projection from $L^2$ onto $H^2$.

**Example 5.** *If $\phi \in H^\infty$, then both $T_\phi$ and $T_\phi^*$ have property $(S_1)$.*

*Proof.* It is well known that

$$\mathcal{W}(T) = \{T_z\}' = \{T_h : h \in H^\infty\}$$

and that the mapping $h \to T_h$, $h \in H^\infty$, is an (isometric) algebra isomorphism of $H^\infty$ onto $\mathcal{W}(T_z)$. Since $H^\infty$ is an integral domain, so is $\mathcal{W}(T_z)$. Clearly each $f \in H^\infty$ with $f \neq 0$ separates $\mathcal{W}(T_z)$. If $\phi \in H^\infty$, then $\mathcal{W}(T_\phi) \subseteq \mathcal{W}(T_z)$. So we can apply Proposition 3 to conclude that $T_\phi$ has property $(S_1)$. Similarly

$$\mathcal{W}(T_\phi^*) = \mathcal{W}(T_\phi)^* \subseteq \mathcal{W}(T_z)^*$$

Thus $\mathcal{W}(T_\phi^*)$ is an integral domain. Since $T_\phi^*$ is triangular, $\mathcal{W}(T_\phi^*)$ has separating vectors [GLW]. Thus, by Proposition 3, $T_\phi^*$ also has property $(S_1)$. $\square$

We next show that $\mathcal{W}(T)$ is an integral domain for any non-nilpotent strictly triangular operator.

**Lemma 6.** *An operator $T \in B(H)$ is strictly triangular if and only if $\cup_{n=1}^{\infty} kerT^n$ is dense in $H$.*

*Proof.* The "only if" direction is clear. The "if" direction follows easily from the observation that the sequence of closed subspaces $\{kerT^n : 0 \leq n < \infty\}$ is a nest of invariant subspaces for $T$ which has closed union $H$, for which the restriction of $T$ to each member is nilpotent. □

**Theorem 7.** *Every strictly triangular operator has property $(S_1)$.*

To prove Theorem 7 we need the following results:

**Proposition 8.** *Let $T$ be a non-nilpotent strictly triangular operator. Then for every operator $A \in \mathcal{W}(T)$, there is a unique formal series*

$$\sum_{k=0}^{\infty} a_k T^k$$

*such that $A|_{kerT^n} = \sum_{k=0}^{n} a_k T^k|_{kerT^n}$ for each $n$.*

*Proof.* Since $T$ is strictly triangular, by Lemma 6, we have $\cup_{n=1}^{\infty} kerT^n$ is dense in $H$. Let $N_1 = kerT$, and $N_{k+1} = kerT^{k+1} \ominus kerT^k$ for all $k \geq 1$. Then $H = \oplus_{k=1}^{\infty} N_k$. If we write $M_n = \oplus_{k=1}^{n} N_k$, then $M_n = kerT^n$ is an invariant subspace for $T$. Thus $T$ has matrix form of

$$T = \begin{pmatrix} 0 & T_{12} & & & & * \\ 0 & 0 & T_{23} & & & \\ 0 & 0 & 0 & T_{34} & & \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}$$

such that $T_{k,k+1}$ is one to one for every $k$ because $T$ is not nilpotent. For each $n$ observe that $T|_{M_n}$ is nilpotent of index $n$.

Let $A \in \mathcal{W}(T)$. Then $A|_{M_n} \in \mathcal{W}(T)|_{M_n} \subseteq \mathcal{W}(T|_{M_n})$. Since $(T|_{M_n})^n = 0$, we have that $A|_{M_n} = p_n(T)|_{M_n}$ for some polynomial $p_n$ of degree $\leq n$. We need to show that for each $k$, the coefficient $a_k^{(n)}$ of $z^k$ in $p_n$ is independent of $n \geq k$.

Suppose that $p(z) = \sum_{k=0}^{m} a_k z^k$ is a polynomial of degree $m \geq n$ and $A|_{M_n} = p(T)|_{M_n}$. Let $P_k$ be the projection from $H$ onto $N_k$. Then

$$P_1 A P_1 = P_1 \sum_{k=0}^{m} a_k T^k P_1 = a_0 P_1.$$

So $a_0$ is uniquely determined. Suppose that $j \leq n$ and $a_0, a_1, ..., a_{j-1}$ have been shown to be independent of $p$. Then

$$P_1 A P_j = P_1 \sum_{0}^{j} a_k T^k P_j = P_1 \sum_{0}^{j-1} a_k T^k P_j + a_j T_{12} T_{23} ... T_{j,j+1}.$$

Since $T_{1,2} T_{2,3} ... T_{j,j+1} \neq 0$, we obtain that $a_j$ is uniquely determined. Thus $A = \sum_{k=0}^{\infty} a_k T^k$, where we interpret this to mean that for any $n$,

$$A|_{M_n} = \sum_{0}^{\infty} a_k T^k|_{M_n} = \sum_{0}^{n} a_k T^k|_{M_n}. \qquad \square$$

**Remark 9.** Proposition 8 shows that if $T$ is strictly triangular, then each element $A \in \mathcal{W}(T)$ is a formal power series

$$A = \sum_{k=0}^{\infty} a_k T^k.$$

In the case that $T$ is a backward unilateral weighted shift with all weights non-zero, it is well known [S] that $\mathcal{W}(T) = \{T\}'$ is an algebra of power series in $T$ and that the Cesaro sums of each such series converges weakly. In particular, $\mathcal{W}(T)$ and $\mathcal{W}(T^*)(= \mathcal{W}(T)^*)$ are integral domains each with a separating vector. Thus both $T$ and $T^*$ have property $(S_1)$. This generalizes Example 5.

In general, if $T$ is strictly triangular relative to an orthonormal basis $\{e_n\}$, and if $A \in \mathcal{W}(T)$, $A = \sum_{k=0}^{\infty} a_k T^k$, then we can interpret the equality as saying that for any $n$,

$$Ae_n = \sum_{k=0}^{\infty} a_k T^k e_n \qquad (*)$$

where the sum on the right side has only finitely many terms which are nonzero. Define

$$\mathcal{F}(T) = \{A \in B(H) : A = \sum a_k T^k \ \textit{formally as in } (*)\}.$$

The following is an easy consequence of Proposition 8.

**Corollary 10.** *Let $T$ be as in Proposition 8.*

*(i) If $A = \sum_0^{\infty} a_k T^k \in \mathcal{W}(T)$, then $T = 0$ if and only if $a_k = 0$ for every $k \geq 0$.*

*(ii) If $A, B \in \mathcal{W}(T)$ such that $A = \sum_0^{\infty} a_k T^k$ and $B = \sum_0^{\infty} b_k T^k$, then $AB = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) T^k$.*

*(iii) $\mathcal{F}(T)$ is a weakly closed abelian algebra.*

*(iv) $\mathcal{W}(T) \subset \mathcal{F}(T) \subset \{T\}''$.*

We make two further observations. First, the series $\sum_k a_k T^k$ need not converge in any sense. Second, operators represented by a formal series need not be in $\mathcal{W}(T)$, and equalities in (iv) may hold in either inclusion. We illustrate these observations with two examples (Examples 12 and 13).

**Corollary 11.** *If $T$ is a strictly triangular operator which is not nilpotent, then $\mathcal{W}(T)$ is an integral domain.*

*Proof.* Let $A, B \in \mathcal{W}(T)$ with formal series $A = \sum_{k=0}^{\infty} a_k T^k$ and $B = \sum_{k=0}^{\infty} b_k T^k$ and such that $A \neq 0$, $B \neq 0$. Assume that $l$ (resp. $j$) is the first nonzero coefficient for $A$ (resp. $B$). Then, by Corollary 10 (ii), $AB$ has a formal series with a non-zero $l+j$-th coefficient. Hence $AB \neq 0$ by Corollary 10(i). Therefore $\mathcal{W}(T)$ has no zero divisors. □

**Proof of Theorem 7.** *Every strictly triangular operator has property $(S_1)$.*

*Proof.* If $A$ is not nilpotent, then it has property $(S_1)$ by Corollary 11 and Proposition 3. If $T$ is nilpotent, then every one dimensional extension $T$ of $A$ is algebraic. Thus $\mathcal{W}(T)$ has separating vectors. □

**Example 12.** Let $G = \{z : \ 1 < |z| < 3\}$, and let $m$ be the area measure on $G$. Define

$$L_a^2(G) = \{f : G \to \mathbb{C}|\ f \text{ is analytic on } G \text{ and } f \in L^2(m)\}$$

If $\phi \in H^\infty(G)$, let $S_\phi f = \phi f, \quad f \in L_a^2(G)$. Then $S_z$ is the Bergman operator on $L_a^2(G)$. Let $T^* = S_z - 2I$. Then $f \in ran(T^{*k})$ if and only if $f$ has a zero of order at least $k$ at 2. Thus $\cap_{k=1}^\infty ran(T^{*k}) = \{0\}$. Thus $\cup_{k=1}^\infty kerT^k$ is dense. Therefore $T$ is strictly triangular. Also

$$\mathcal{W}(T^*) = \mathcal{W}(S_z) = \{S_\phi : \ \phi \text{ is bounded and analytic on } \ \{z : \ |z| < 3\}\}.$$

Hence if $S_\phi \in \mathcal{W}(T^*)$, and formally,

$$S_\phi = \sum_{k=0}^\infty a_k T^{*k} = \sum_{k=0}^\infty a_k S_{(z-2)^k} = S_{\sum_k a_k (z-2)^k}$$

we can only guarantee that $\sum a_k(z-2)^k$ converges to $\phi(z)$ for $|z-2| < 1$. For example, suppose that $\phi$ has a singularity at $z = 3$. Then the radius of convergence of this series must be 1.

We note that $\{T^*\}' = \{S_\phi : \ \phi \in H^\infty(G)\}$. In particular $S_{1/z} \notin \mathcal{W}(T^*)$. Thus $\mathcal{W}(T^*) \subsetneqq \{T^*\}'$, and therefore $\mathcal{W}(T) \subsetneqq \{T\}'$. Furthermore if $kerT^k = M_k$, then $M_k$ is invariant for $\{T\}'$ ( that is, $M_k$ is hyperinvariant) for each $k$, and $T|_{M_k}$ is nilpotent of index $k$. Thus $T|_{M_k}$ is similar to a nilpotent Jordan block $J_k$ on $\mathbb{C}^k$. (Here $J_k e_1 = 0$ and $J_k e_n = e_{n-1}, \ 2 \leq n \leq k$.) But $\{J_k\}' = \{p(J_k) : p$ is a polynomial of degree $\leq k\}$. Thus if $A \in \{T\}'$, then $A|_{M_k} \in \{T|_{M_k}\}'$.

Therefore $A|_{M_k} = p_k(T)|_{M_k}$ for some polynomial $p_k$. It is easy to see, (see the proof of Proposition 8), that the coefficients of $p_k$ are uniquely determined, and that $A = \sum_k a_k T^k$, formally. Thus $\mathcal{W}(T) \subsetneqq \mathcal{F}(T) = \{T\}' = \{T\}''$. □

**Example 13.** In this example we outline the construction of a strictly triangular operator $S$ so that $\mathcal{W}(S) = \mathcal{F}(S)$ is properly contained in $\{S\}'$. Suppose that $S_1$ and $S_2$ are backward weighted shifts on $l^2$ with nonzero weights, and let $S = S_1 \oplus S_2$. Then $S$ is an operator weighted shift, and as in the scalar case (see Remark 9) one has that $\mathcal{W}(S)$ is an algebra of power series in $S$, so that $\mathcal{W}(S) = \mathcal{F}(S)$. Now suppose that the weights have been chosen so that there are no nonzero operators intertwining the two summands of $S$. It follows that $\{S\}' = \{S_1\}' \oplus \{S_2\}' = \{S\}''$, which properly contains $\mathcal{W}(S)$.

One can construct $S_1$ and $S_2$ as follows. We will choose both sequences of weights to be constant on blocks of size $2, 2^2, 2^3, ....$ For $S_1$, let the weights be 1 on all of the odd blocks and 1/2 on the even blocks. For $S_2$, the weights will be 1 on the even blocks and 1/2 on the odd blocks. A matrix computation shows that if $\ AS_1 = S_2 A \ $ or if $\ AS_2 = S_1 A, \ $ then $A = 0$.

Note that with $T$ as in Example 12, the operator $S \oplus T$ is strictly triangular and $\mathcal{F}(S \oplus T)$ lies properly between $\mathcal{W}(S \oplus T)$ and $\{S \oplus T\}''$. □

Theorem 7 has elementary generalizations to operators $T$ which are strictly lower triangular (the adjoint of a strictly upper triangular operator) as well as to operators which have a strict 2-sided triangular form. By the latter we mean that there is an orthonormal basis for the underlying Hilbert space indexed by the integers, $\{e_n : n \in \mathbb{Z}\}$, such that $Te_n \in [e_k : \ k < n]$ for all $n$, where [ ] denotes closed linear span. In [GLW, Corollary 12] it was proven that triangular, lower triangular, and 2-sided triangular operators have the separating vector property.

**Proposition 14.** *If $T \in B(H)$ is either strictly triangular, strictly lower triangular, or has a strict 2-sided triangular form, then either $T$ is nilpotent or $\mathcal{W}(T)$ has no zero divisors. Hence $T$ has property $(S_1)$.*

*Proof.* Suppose that $T$ is *not* nilpotent. If $T$ is strictly triangular, then $\mathcal{W}(T)$ has no zero divisors by Corollary 11. If it is strictly lower triangular then $\mathcal{W}(T^*)$ is an integral domain, hence so is $\mathcal{W}(T) = (\mathcal{W}(T^*))^*$.

The case remains where $T$ has a strict 2-sided triangular form. Let $\{e_n : n \in \mathbb{Z}\}$ be the corresponding orthonormal basis for $H$, and for each $n$ let $P_n$ be the orthogonal projection onto $E_n := [e_k : k \leq n]$. Then each compression $P_n T|_{E_n}$ is strictly lower triangular and each compression $P_n^{\perp} T|_{E_n^{\perp}}$ is strictly triangular. If both of these compressions were nilpotent then $T$ would have the form

$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix},$$

where $A$ and $B$ are nilpotent, and this would imply that $T$ was nilpotent, a contradiction. Thus for each $n$ at least *one* of these two compressions is *not* nilpotent. A similar operator matrix argument shows that if $P_n T|_{E_n}$ is not nilpotent, then $P_m T|_{E_m}$ is not nilpotent for all $m \in \mathbb{Z}$, and if $P_n^{\perp} T|_{E_n^{\perp}}$ is not nilpotent, then $P_m^{\perp} T|_{E_m^{\perp}}$ is not nilpotent for all $m \in \mathbb{Z}$.

Assume, by way of contradiction, that $\mathcal{W}(T)$ is not an integral domain. Then there exist $A_1, A_2 \in \mathcal{W}(T)$, $A_i \neq 0$, $A_1 A_2 = 0$. We have two cases:

**Case I.** $P_0 T|_{E_0}$ is not nilpotent. Since $P_n \to I$ strongly as $n \to \infty$ there exists $k \in \mathbb{Z}$ such that $P_k A_i|_{E_k} \neq 0$, $i = 1, 2$. We also have $P_k \mathcal{W}(T)|_{E_k} \subseteq \mathcal{W}(P_k T|_{E_k})$. Moreover $P_k T|_{E_k}$ is not nilpotent, hence $\mathcal{W}(P_k T|_{E_k})$ is an integral domain by the first paragraph. However,

$$P_k A_1|_{E_k} \cdot P_k A_2|_{E_k} = P_k A_1 A_2|_{E_k} = 0,$$

which is a contradiction. Hence $\mathcal{W}(T)$ must be an integral domain.

**Case II.** $P_0^{\perp} T_{E_0^{\perp}}$ is not nilpotent. Since $P_n^{\perp} \to I$ strongly as $n \to -\infty$ there exists $k \in \mathbb{Z}$ such that $P_k^{\perp} A_i|_{E_k^{\perp}} \neq 0$, $i = 1, 2$. The rest of the argument is analogous to Case I. □

The above result can be extended to the following more general situation.

**Theorem 15.** *Let $\{A_i \in B(H_i) : i = 1, 2, ...\}$ be a sequence of operators such that either $A_i$ is algebraic or $A_i = T_i + c_i I$ for some strictly triangular, strictly lower triangular or strict 2-sided triangular operator $T_i$. Then $A := \oplus_{i=1}^{\infty} A_i$ has property $(S_1)$.*

To prove Theorem 15 we need the following lemma:

**Lemma 16.** *Let $A_i \in B(H_i)$ and*

$$\hat{A}_i = \begin{pmatrix} A_i & x_i \\ 0 & 0 \end{pmatrix} \in B(H_i \oplus \mathbb{C})$$

*such that $0 \oplus 1 \in \overline{Sep(\mathcal{W}(\hat{A}_i))}$ and $X = (x_1, x_2, ...) \in \oplus_i H_i$. Let*

$$\hat{A} = \begin{pmatrix} A & X^t \\ 0 & 0 \end{pmatrix},$$

*where* $A = \oplus_{i=1}^{\infty} A_i$*. Then* $\mathcal{W}(\hat{A})$ *has a separating vector.*

*Proof.* Since $0 \oplus 1 \in \overline{Sep(\mathcal{W}(\hat{A}_i))}$, we can choose $u_i \oplus 1 \in Sep(\mathcal{W}(\hat{A}_i))$ such that $\sum_i ||u_i||^2 \leq \infty$. We claim that $v = (\oplus_i u_i) \oplus 1 \in Sep(\mathcal{W}(\hat{A}))$. To this purpose, let $\hat{B} \in \mathcal{W}(\hat{A})$ such that $\hat{B}v = 0$. We can write $\hat{B} = \begin{pmatrix} B & Y \\ 0 & 0 \end{pmatrix}$ with $B = \oplus_{i=1}^{\infty} B_i$ and $Y = (y_1, y_2, ...)^t$. Let

$$\hat{B}_i = \begin{pmatrix} B_i & y_i \\ 0 & 0 \end{pmatrix}.$$

Then $\hat{B}_i \in \mathcal{W}(\hat{A}_i)$ and $\hat{B}_i(u_i \oplus 1) = 0$, which implies that $\hat{B}_i = 0$. Thus $\hat{B} = 0$. □

**Proof of Theorem 15:.**

*Proof.* Let $\hat{A} = \begin{pmatrix} A & X^t \\ 0 & \lambda \end{pmatrix}$ be any one dimensional extension of $A$. Write $X = (x_1, x_2, ...)$. By replacing $\hat{A}$ by $\hat{A} - \lambda I$, noting that the new operator has the same form, we can assume that $\lambda = 0$. If $A_i$ is algebraic, then so is $\hat{A}_i$, where $\hat{A}_i = \begin{pmatrix} A_i & x_i \\ 0 & 0 \end{pmatrix}$. Thus $Sep(\mathcal{W}(\hat{A}_i))$ is dense in $H_i \oplus \mathbb{C}$. If $A_i = T_i + c_i I$ is not algebraic, then $\mathcal{W}(A_i)$ is an integral domain by Proposition 14. Therefore, by the proof of Proposition 3, $0 \oplus 1 \in \overline{Sep(\mathcal{W}(\hat{A}_i))}$. So the conclusion follows from Lemma 16. □

**Corollary 17.** *Suppose that* $A_i \in B(H_i)$ *such that either* $A_i$ *is algebraic or* $\mathcal{W}(A_i)$ *is an integral domain for all* $i = 1, 2, ....$ *If each* $A_i$ *has property* $(S_1)$*, then so does* $\oplus_{i=1}^{\infty} A_i$*.*

Note that $\mathcal{W}(\oplus_i A_i)$ is not necessarily an integral domain in general. Thus the above corollary is an extension of Proposition 3.

*Proof.* If $A_i$ is algebraic, clearly $0 \oplus 1 \in \overline{Sep(\mathcal{W}(\hat{A}_i))}$. If $\mathcal{W}(A_i)$ is an integral domain, then, from the proof of Proposition 3, $0 \oplus 1 \in \overline{Sep(\mathcal{W}(\hat{A}_i))}$. Hence $\oplus_i A_i$ has property $(S_1)$ by Lemma 16. □

**Remark 18.** Proposition 14 is the best possible general result of its kind for operators with a triangular form modeled on a nest of invariant subspaces. The reason is that strictly triangular operators can fail to have property $(S_2)$, and the example in [W] which points this out (although of course the terminology is different) is a 2-dimensional extension $T$ of a strictly triangular operator $A$ such that $T$ itself has a strict triangular form with respect to the associated nest order-isomorphic to $\mathbb{N}+\{1, 2\}$. So the intermediate operator $B$, which is the corresponding 1-dimensional extension of $A$, has strict triangular form with respect to $\mathbb{N}+\{1\}$, yet *cannot* be an integral domain operator because $B$ has a 1-dimensional extension, namely $T$, which does not have the separating vector property.

We say that a non-empty set $E$ of $H$ is *linearly dense* in $H$ if $E \cap L$ is dense in $L$ for all complex lines $L$ that meet $E$. (By a *complex line* we mean a one-dimensional complex affine subspace of $H$.) This is a stronger property than density. It was proven in [GLW] that if $\mathcal{A}$ is a linear subspace of operators with denumerable Hamel basis, then $Sep(\mathcal{A})$ is either empty or linearly dense. Also. if $\mathcal{A}$ is a m.a.s.a, then

$Sep(\mathcal{A})$ is linearly dense. We conjecture that linear density is common when $Sep(\mathcal{A})$ is nonempty.

**Question 1.** If $\mathcal{A} \subseteq B(H)$ is an integral domain and $Sep(\mathcal{A})$ is non-empty, is $Sep(\mathcal{A})$ linearly dense?

The following elementary result shows the above is true when $\mathcal{A}$ is abelian.

**Proposition 19.** *If $\mathcal{A}$ is an abelian integral domain and $Sep(\mathcal{A})$ is non-empty, then $Sep(\mathcal{A})$ is linearly dense.*

*Proof.* Let $u \in Sep(\mathcal{A})$ and let $x \in H$ be an arbitrary element. Assume that there exist two different scalars $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $u + \lambda_1 x,\ u + \lambda_2 x \notin Sep(\mathcal{A})$. Then there exist non-zero operators $A_1,\ A_2 \in \mathcal{A}$ such that $A_1(u + \lambda_1 x) = 0$ and $A_2(u + \lambda_2 x) = 0$. Note that $A_1 A_2 = A_2 A_1$. We have $(\lambda_1 - \lambda_2) A_1 A_2 u = 0$. Hence $A_1 A_2 = 0$, which implies that either $A_1 = 0$ or $A_2 = 0$. Therefore except for possibly at most one $\lambda$, all $u + \lambda x \in Sep(\mathcal{A})$, which implies that $Sep(\mathcal{A})$ is linearly dense. □

**Remark 20.** The separating vector index was introduced in [HLW] which generalizes the concept of spectrum cardinality for operators acting on finite dimensional Hilbert spaces. Let $\mathcal{A}$ be a linear subspace of operators such that $Sep(\mathcal{A})$ is nonempty. Define

$$i(\mathcal{A}; L) = card\{y \in L : \ y \notin Sep(\mathcal{A})\}$$

for any complex line $L$ meeting $Sep(\mathcal{A})$, and for any $x \in Sep(\mathcal{A})$ define

$$j(\mathcal{A}; x) = sup\{(i(\mathcal{A}; L) : \ L \text{ is a complex line containing } x\}.$$

The *separating vector index* of $\mathcal{A}$ is defined by $k(\mathcal{A}) = sup_x j(\mathcal{A}; x)$. If $T$ acts on a finite dimensional space, then $k(\mathcal{W}(T))$ is the cardinality of the spectrum of $T$ ([GLW]). Proposition 19 tells us that the separating vector index of $\mathcal{A}$ is at most 1 when $\mathcal{A}$ is an abelian integral domain which has a separating vector.

**Question 2.** Let $\mathcal{A}$ be a weakly closed subalgebra of $B(H)$. If $\mathcal{A}$ is an integral domain, does $\mathcal{A}$ have a separating vector?

Note that Wogen's example [W] shows that property $(S_1)$ does not imply property $(S_2)$. We have:

**Question 3.** If $T$ has property $(S_n)(n \geq 2)$, does it have property $(S_{n+1})$ ?

**Question 4.** Assume the hypotheses of Proposition 1, and suppose that $K$ has dimension1. Must $T$ have a separating vector that is not in $H$? A counterexample would provide the first known example of an operator $T$ with a separating vector such that $Sep(\mathcal{W}(\mathcal{T}))$ is not dense. The question is open even for the case that $A$ is triangular. A counterexample in this setting would provide evidence that there may be a one-dimensional extension of a triangular operator that has no separating vector.

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