

Operator-valued Frames on C*-Modules

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ABSTRACT. Frames on Hilbert C*-modules have been defined for unital C*-algebras by Frank and Larson [5] and operator-valued frames on a Hilbert space have been studied in [8]. The goal of this paper is to introduce operator-valued frames on a Hilbert C*-module for a σ -unital C*-algebra. Theorem 1.4 reformulates the definition given in [5] in terms of a series of rank-one operators converging in the strict topology. Theorem 2.2. shows that the frame transform and the frame projection of an operator-valued frame are limits in the strict topology of a series in the multiplier algebra and hence belong to it. Theorem 3.3 shows that two operator-valued frames are right similar if and only if they share the same frame projection. Theorem 3.4 establishes an one-to-one correspondence between Murray-von Neumann equivalence classes of projections in the multiplier algebra and right similarity equivalence classes of operator-valued frames and provides a parametrization of all Parseval operator-valued frames on a given Hilbert C*-module. Left similarity is then defined and Proposition 3.9 establishes when two left unitarily equivalent frames are also right unitarily equivalent.

Introduction

Frames on a Hilbert space are collections of vectors satisfying the condition

$$a\|\xi\|^2 \leq \sum_{j \in \mathbb{J}} |\langle \xi, \xi_j \rangle|^2 \leq b\|\xi\|^2$$

for some positive constants a and b and all vectors ξ . This notion has been naturally extended by Frank and Larson [5] to countable collections of vectors in a Hilbert C*-module for a unital C*-algebra satisfying an analogous defining property (see below 1.1 for the definitions). Most properties of frames on a Hilbert space hold also for Hilbert C*-modules, often have quite different proofs, but new phenomena do arise.

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A different generalization where frames are no longer vectors in a Hilbert space but operators on a Hilbert space is given in [8] with the purpose of providing a natural framework for multiframes, especially for those obtained from a unitary system, e.g, a discrete group representation. Operator-valued frames both generalize vector frames and can be decomposed into vector frames.

The goal of this article is to introduce the notion of operator-valued frame on a Hilbert C*-module. Since the frame transform of Frank and Larson permits to identify a vector frame on an arbitrary Hilbert C*-module with a vector frame on the standard Hilbert C*-module $\ell^2(\mathcal{A})$ of the associated C*-algebra \mathcal{A} , for simplicity's sake we confine our definition directly to frames on $\ell^2(\mathcal{A})$. When the associated C*-algebra is σ -unital, it well known (see [9]) that the algebra of bounded adjointable operators on $\ell^2(\mathcal{A})$ can be identified with the multiplier algebra $M(\mathcal{A} \otimes \mathcal{K})$ of $\mathcal{A} \otimes \mathcal{K}$, about which a good deal is known. Since reference are mainly formulated in terms of right-modules, we treat $\ell^2(\mathcal{A})$ as a right module (i.e., as 'row vectors').

A frame on $\ell^2(\mathcal{A})$ is thus defined as a collection of operators $\{A_j\}_{j \in \mathbb{J}}$ with $A_j \in E_0 M(\mathcal{A} \otimes \mathcal{K})$ for a fixed projection $E_0 \in M(\mathcal{A} \otimes \mathcal{K})$ for which

$$aI \leq \sum_{j \in \mathbb{J}} A_j^* A_j \leq bI$$

for some positive constants a and b , where I is the identity of $M(\mathcal{A} \otimes \mathcal{K})$ and the series converges in the strict topology of $M(\mathcal{A} \otimes \mathcal{K})$.

We will show in Theorem 1.4 how to associate (albeit not uniquely) to a vector frame in the sense of [5] an operator-valued frame. When \mathcal{A} is unital, we will decompose in Section 3.10) every operator-valued frame (albeit not uniquely into vector frames (i.e., a multiframe) Some properties of operator-valued frames on a Hilbert C*-module track fairly well the properties of operator-valued and vector-valued frames on a Hilbert space. Often, the key difference in the proofs is the need to express objects like the *frame transform* or the *frame projection* as series of elements of $M(\mathcal{A} \otimes \mathcal{K})$ that converge in the strict topology, and hence, belong to $M(\mathcal{A} \otimes \mathcal{K})$.

We illustrate some commonalities and differences with the Hilbert space case by considering in particular three topics. That the dilation approach of Han and Larson in [6], which was extended to operator-valued frames on Hilbert spaces in [8], has a natural analog for operator-valued frames on Hilbert C*-modules if the frame transform is defined to have values inside the same Hilbert C*-module instead of into an ampliation of it.

Similarity of frames can also be defined and characterized as in the Hilbert space case, but now there is also a similarity from the left and we compare the two notions.

Finally, there is a natural composition of operator-valued frames - a new operation that has no vector frame analog and that illustrates the 'multiplicity' of operator-valued frames.

In this paper we have explored the analogs of some of the properties of Hilbert space frames and much work remains to be done.

1. Operator-valued frames

1.1. Frames and operator-valued frames on a Hilbert space.

A frame on a Hilbert space \mathcal{H} is a collection of vectors $\{\xi_j\}_{j \in \mathbb{J}}$ indexed by a countable set \mathbb{J} for which there exist two positive constants a and b such that for all $\xi \in \mathcal{H}$,

$$a\|\xi\|^2 \leq \sum_{j \in \mathbb{J}} |\langle \xi, \xi_j \rangle|^2 \leq b\|\xi\|^2.$$

Equivalently,

$$aI \leq \sum_{j \in \mathbb{J}} \xi_j \otimes \xi_j \leq bI,$$

where I is the identity of $\mathcal{B}(\mathcal{H})$, $\eta \otimes \xi$ is the rank-one operator defined by $(\eta \otimes \xi)\zeta := \langle \zeta, \xi \rangle \eta$ and the series converges in the *strong operator topology* (pointwise convergence). The above condition can be rewritten as

$$aI \leq \sum_{j \in \mathbb{J}} A_j^* A_j \leq bI$$

where $A_j := \eta \otimes \xi_j$ for some arbitrary fixed unit vector $\eta \in \mathcal{H}$. It is thus equivalent to the series $\sum_{j \in \mathbb{J}} A_j^* A_j$ converging in the strong operator topology to a bounded invertible operator. Notice that the convergence of the numerical and the operatorial series are unconditional.

This reformulation naturally leads to the more general notion of operator-valued frames $\{A_j\}_{j \in \mathbb{J}}$ on a Hilbert space \mathcal{H} in [8], namely a collection of operators $A_j \in B(H, H_0)$, with ranges in a fixed Hilbert space H_0 (not necessarily of dimension one) for which the series $\sum_{j \in \mathbb{J}} A_j^* A_j$ converges in the strong operator topology to a bounded invertible operator. Operator-valued frames on a Hilbert space can be decomposed into, and hence identified with, multiframe.

Frames with values in a Hilbert C*-module have been introduced in [5] and then studied in [7], [13], and others. So much of the Hilbert space frame theory carries over, that one could argue that frame theory finds a natural general setting in Hilbert C*-modules. We will show in Theorem 1.4 that frames on a Hilbert C*-module can be equivalently defined in terms of rank-one operators on the module. This leads naturally to the definition of general operator-valued frames on a Hilbert C*-module. Before giving the formal definitions, we recall for the readers' convenience some relevant background about Hilbert C*-modules.

1.2. Hilbert C*-modules ([2,Ch.13], [9]).

Let \mathcal{A} be a C*-algebra. Then a Hilbert (right) C*- \mathcal{A} -module is a pair $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, with \mathcal{H} a (right) module over \mathcal{A} and $\langle \cdot, \cdot \rangle$ a binary operation from \mathcal{H} into \mathcal{A} , that satisfies the following six axioms, similar to those of Hilbert spaces, except that for right modules the linearity occurs for the second and not

the first component of the inner product. For $\xi, \eta, \eta_1, \eta_2 \in \mathcal{H}$ and $a \in \mathcal{A}$

- (i) $\langle \xi, \eta_1 + \eta_2 \rangle = \langle \xi, \eta_1 \rangle + \langle \xi, \eta_2 \rangle$;
- (ii) $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$;
- (iii) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$;
- (iv) $\langle \xi, \xi \rangle \geq 0$;
- (v) $\langle \xi, \xi \rangle = 0 \iff \xi = 0$;
- (vi) $(\mathcal{H}, \|\cdot\|)$ is complete, where $\|\xi\| := \|\langle \xi, \xi \rangle\|^{1/2}$

The classic example of Hilbert (right) \mathcal{A} -module and the only one we will consider in this paper is the *standard module* $\mathcal{H}_{\mathcal{A}} := \ell^2(\mathcal{A})$, the space of all sequences $\{a_i\} \subset \mathcal{A}$ such that $\sum_{i=1}^{\infty} a_i^* a_i$ converges in norm to a positive element of \mathcal{A} . $\ell^2(\mathcal{A})$ is endowed with the natural linear (\mathcal{A} -module) structure and right \mathcal{A} -multiplication, and with the \mathcal{A} -valued inner product defined by

$$\langle \{a_i\}, \{b_i\} \rangle = \sum_{i=1}^{\infty} a_i^* b_i,$$

where the sum converges in norm by the Schwartz Inequality ([2] or [9]).

A map T from $\mathcal{H}_{\mathcal{A}}$ to $\mathcal{H}_{\mathcal{A}}$ is called a (linear) bounded operator on $\mathcal{H}_{\mathcal{A}}$, if $T(\lambda\xi) + T(\mu\eta) = T(\lambda\xi + \mu\eta)$, $T(\xi a) = T(\xi)a$ for all $\xi, \eta \in \mathcal{H}_{\mathcal{A}}$, $\lambda, \mu \in \mathbb{C}$, and $a \in \mathcal{A}$, and if

$$\|T\| := \sup\{\|T\xi\| \mid \xi \in \mathcal{H}_{\mathcal{A}}, \|\xi\| \leq 1\} < \infty.$$

Not every bounded operator T has a bounded adjoint T^* , namely

$$\langle T^* \xi, \eta \rangle = \langle \xi, T \eta \rangle \quad \text{for all } \xi, \eta \in \mathcal{H}_{\mathcal{A}},$$

as there is no Riesz Representation Theorem for general Hilbert C^* -modules. Nevertheless, there are abundant operators on $\mathcal{H}_{\mathcal{A}}$ whose adjoints exist and the collection of bounded adjointable operators is denoted by $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$. Then, $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ is a C^* -algebra (see [9] or [2, Ch. 13]). Notice that if $\mathcal{A} = \mathbb{C}$, then $\mathcal{H}_{\mathcal{A}} = \ell^2$ and $\mathcal{B}(\mathcal{H}_{\mathcal{A}}) = \mathcal{B}(\ell^2)$. Some of the properties of $\mathcal{B}(\ell^2)$ extend naturally to $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$. For each pair of elements ξ and η in $\mathcal{H}_{\mathcal{A}}$, a bounded ‘rank-one’ operator is defined by

$$\theta_{\xi, \eta}(\zeta) = \xi \langle \eta, \zeta \rangle \quad \text{for all } \zeta \in \mathcal{H}_{\mathcal{A}}.$$

The closed linear span of all rank-one operators is denoted by $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$. When $\mathcal{A} = \mathbb{C}$, $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$ coincides with the ideal \mathcal{K} of all compact operators on ℓ^2 . $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$ is always a closed ideal of $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$, but contrary to the separable infinite dimensional Hilbert space case, in general it is not unique (e.g., see [2] or [9]).

The analog of the strong*-topology on $\mathcal{B}(\ell^2)$ is the *strict topology* on $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ defined by

$$\mathcal{B}(\mathcal{H}_{\mathcal{A}}) \ni T_\lambda \longrightarrow T \text{ strictly if } \|(T_\lambda - T)S\| \rightarrow 0 \text{ and } \|S(T_\lambda - T)\| \rightarrow 0 \forall S \in \mathcal{K}(\mathcal{H}_{\mathcal{A}}).$$

We will use the following elementary properties: $T_\lambda \longrightarrow T$ strictly iff $T_\lambda^* \longrightarrow T^*$ strictly, and either of these convergences implies $BT_\lambda \longrightarrow BT$ and $T_\lambda B \longrightarrow TB$ strictly for all $B \in \mathcal{B}(\mathcal{H}_{\mathcal{A}})$. Also, if $T_\lambda \longrightarrow T$ strictly and $S_\lambda \longrightarrow S$ strictly, then $T_\lambda S_\lambda \longrightarrow TS$ strictly.

There is an alternative view of the objects $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ and $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$. Embed the tensor product $\mathcal{A} \otimes \mathcal{K}$ into its Banach space double dual $(\mathcal{A} \otimes \mathcal{K})^{**}$, which, as is

well known, is a W*-algebra ([12]). The multiplier algebra of $\mathcal{A} \otimes \mathcal{K}$, denoted by $M(\mathcal{A} \otimes \mathcal{K})$, is defined as the collection

$$\{T \in (\mathcal{A} \otimes \mathcal{K})^{**} : TS, ST \in \mathcal{A} \otimes \mathcal{K} \ \forall S \in \mathcal{A} \otimes \mathcal{K}\}.$$

Equipped with the norm of $(\mathcal{A} \otimes \mathcal{K})^{**}$, $M(\mathcal{A} \otimes \mathcal{K})$ is a C*-algebra. Assuming that \mathcal{A} is σ -unital, we will frequently apply the following two *-isomorphisms without further reference:

$$\mathcal{B}(\mathcal{H}_{\mathcal{A}}) \cong M(\mathcal{A} \otimes \mathcal{K}) \quad \text{and} \quad \mathcal{K}(\mathcal{H}_{\mathcal{A}}) \cong \mathcal{A} \otimes \mathcal{K} \quad [\text{Ka1}].$$

The algebra $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ is technically hard to work with, while $M(\mathcal{A} \otimes \mathcal{K})$ is more accessible due to many established results. More information on the subject can be found in the sample references [9] and [2], among many others. Although most properties hold with appropriate modifications also for left modules, since the original theory was developed by Kasparov for right Hilbert C*-modules ([9]), the results found in the literature are often formulated for right modules. This is the reason why our definition of frames is given for right modules instead of left modules as in [5].

To avoid unnecessary complications, from now on, we assume that \mathcal{A} is a σ -unital C*-algebra.

1.3. Vector Frames on Hilbert C*-modules

According to [5], a (vector) frame on the Hilbert C*-module $\mathcal{H}_{\mathcal{A}}$ of a σ -unital C*-algebra \mathcal{A} is a collection of elements $\{\xi_j\}_{j \in \mathbb{J}}$ in $\mathcal{H}_{\mathcal{A}}$ for which there are two positive scalars a and b such that for all $\xi \in \mathcal{H}_{\mathcal{A}}$,

$$a \langle \xi, \xi \rangle \leq \sum_{j \in \mathbb{J}} \langle \xi, \xi_j \rangle \langle \xi_j, \xi \rangle \leq b \langle \xi, \xi \rangle,$$

where the convergence is in the norm of the C*-algebra \mathcal{A} . The following theorem permits us to reformulate this definition in terms of rank-one operators. Notice that $\langle \xi, \xi_j \rangle \langle \xi_j, \xi \rangle = \theta_{\xi_j, \xi_j} \xi, \xi \rangle$.

1.4. Theorem Let \mathcal{A} be a σ -unital C*-algebra. Then the collection $\{\xi_j\}_{j \in \mathbb{J}}$ in the Hilbert C*-module $\mathcal{H}_{\mathcal{A}}$ is a frame if and only if the series $\sum_{j \in \mathbb{J}} \theta_{\xi_j, \xi_j}$ converges in the strict topology to a bounded invertible operator in $B(\mathcal{H}_{\mathcal{A}})$.

First we need the following elementary facts. For the readers' convenience we present their proofs.

1.5 Lemma Assume that $\eta, \eta', \xi, \xi' \in \mathcal{H}_{\mathcal{A}}$. Then the following hold:

- (i) $\theta_{\xi, \eta} \theta_{\eta', \xi'} = \theta_{\xi \langle \eta, \eta' \rangle, \xi'}$.
- (ii) $\theta_{\xi, \eta}^* = \theta_{\eta, \xi}$.
- (iii) $\theta_{\xi, \eta}^* \theta_{\xi, \eta} = \theta_{\eta \langle \xi, \xi \rangle, \eta} = \theta_{\eta \langle \xi, \xi \rangle^{\frac{1}{2}}, \eta \langle \xi, \xi \rangle^{1/2}}$.
- (iv) If $T \in B(\mathcal{H}_{\mathcal{A}})$, then $T \theta_{\xi, \eta} = \theta_{T\xi, \eta}$.
- (v) $\|\theta_{\xi, \eta}\| = \|\xi \langle \eta, \eta \rangle^{1/2}\| = \|\langle \xi, \xi \rangle^{1/2} \langle \eta, \eta \rangle^{1/2}\|$. In particular, if \mathcal{A} is unital and $\langle \eta, \eta \rangle = I$, then $\|\theta_{\xi, \eta}\| = \|\xi\|$.
- (vi) $\|\theta_{\xi, \eta}\| \leq \|\eta\| \|\xi\|$.
- (vii) The rank-one operator $\theta_{\eta, \eta}$ is a projection if and only if $\langle \eta, \eta \rangle$ is a projection, if and only if $\eta = \eta \langle \eta, \eta \rangle$. [5, Lemma 2.3]

PROOF. (i) For any $\gamma \in \mathcal{H}_{\mathcal{A}}$, one has

$$\begin{aligned}\theta_{\xi, \eta} \theta_{\eta', \xi'} \gamma &= \xi \langle \eta, \theta_{\eta', \xi'} \gamma \rangle \\ &= \xi \langle \eta, \eta' \langle \xi', \gamma \rangle \rangle \\ &= \xi \langle \eta, \eta' \rangle \langle \xi', \gamma \rangle \\ &= \theta_{\xi \langle \eta, \eta' \rangle, \xi'} \gamma.\end{aligned}$$

(ii)

$$\begin{aligned}\langle \theta_{\xi, \eta}^* \xi', \eta' \rangle &= \langle \xi', \theta_{\xi, \eta} \eta' \rangle \\ &= \langle \xi', \xi \langle \eta, \eta' \rangle \rangle \\ &= \langle \xi', \xi \rangle \langle \eta, \eta' \rangle \\ &= \langle \eta \langle \xi, \xi' \rangle, \eta' \rangle \\ &= \langle \theta_{\eta, \xi} \xi', \eta' \rangle.\end{aligned}$$

(iii) The first identity follows from (i) and (ii). Moreover

$$\begin{aligned}\theta_{\eta \langle \xi, \xi \rangle, \eta} \xi' &= \eta \langle \xi, \xi \rangle \langle \eta, \xi' \rangle \\ &= \eta \langle \xi, \xi \rangle^{\frac{1}{2}} \langle \eta \langle \xi, \xi \rangle^{\frac{1}{2}}, \xi' \rangle \\ &= \theta_{\eta \langle \xi, \xi \rangle^{\frac{1}{2}}, \eta \langle \xi, \xi \rangle^{1/2}} \xi'.\end{aligned}$$

(iv) follows directly from the definition.

(v)

$$\begin{aligned}\|\theta_{\xi, \eta}\| &= \sup\{\|\theta_{\xi, \eta} \gamma\| \mid \|\gamma\| = 1\} \\ &= \sup\{\|\langle \xi \langle \eta, \gamma \rangle, \xi \langle \eta, \gamma \rangle \rangle\|^{1/2} \mid \|\gamma\| = 1\} \\ &= \sup\{\|\langle \eta, \gamma \rangle^* \langle \xi, \xi \rangle \langle \eta, \gamma \rangle\|^{1/2} \mid \|\gamma\| = 1\} \\ &= \sup\{\|\langle \xi, \xi \rangle^{1/2} \langle \eta, \gamma \rangle\| \mid \|\gamma\| = 1\} \\ &= \sup\{\|\langle \eta \langle \xi, \xi \rangle^{1/2}, \gamma \rangle\| \mid \|\gamma\| = 1\} \\ &= \|\eta \langle \xi, \xi \rangle^{1/2}\| \\ &= \|\langle \xi, \xi \rangle^{1/2} \langle \eta, \eta \rangle \langle \xi, \xi \rangle^{1/2}\|^{1/2} \\ &= \|\langle \xi, \xi \rangle^{1/2} \langle \eta, \eta \rangle^{1/2}\|\end{aligned}$$

(vi) is obvious

(vii) For completeness we add a short proof. By (ii), $\theta_{\eta, \eta}$ is a projection if and only if

$$0 = \theta_{\eta, \eta} - \theta_{\eta, \eta} \theta_{\eta, \eta} = \theta_{\eta, \eta} - \theta_{\eta \langle \eta, \eta \rangle, \eta} = \theta_{\eta - \eta \langle \eta, \eta \rangle, \eta}$$

and by (v), this condition is equivalent to $(\eta - \eta \langle \eta, \eta \rangle) \langle \eta, \eta \rangle^{1/2} = 0$. If $\langle \eta, \eta \rangle$ is a projection, then

$$\langle \eta - \eta \langle \eta, \eta \rangle, \eta - \eta \langle \eta, \eta \rangle \rangle = \langle \eta, \eta \rangle - 2 \langle \eta, \eta \rangle^2 + \langle \eta, \eta \rangle^3 = 0,$$

hence $\langle \eta - \eta \langle \eta, \eta \rangle, \eta - \eta \langle \eta, \eta \rangle \rangle = 0$, and thus $\theta_{\eta, \eta}$ is a projection. Conversely, if $(\eta - \eta \langle \eta, \eta \rangle) \langle \eta, \eta \rangle^{1/2} = 0$ then

$$\langle \eta, (\eta - \eta \langle \eta, \eta \rangle) \langle \eta, \eta \rangle^{1/2} \rangle = \langle \eta, \eta \rangle^{3/2} - \langle \eta, \eta \rangle^{5/2} = 0,$$

whence $\langle \eta, \eta \rangle$ is a projection. \square

Notice that equality in (vi) may fail. For instance, if $\xi = \{p, 0, 0, \dots\}$ and $\eta = \{q, 0, 0, \dots\}$ where $p, q \in \mathcal{A}$ are orthogonal non-zero projections, then $\langle \xi, \xi \rangle = p = \langle \xi, \xi \rangle^{1/2}$ and $\langle \eta, \eta \rangle = q = \langle \eta, \eta \rangle^{1/2}$ hence $\langle \xi, \xi \rangle^{1/2} \langle \eta, \eta \rangle^{1/2} = 0$.

PROOF OF THEOREM 1.4. Assume first that the series $\sum_{j \in \mathbb{J}} \theta_{\xi_j, \xi_j}$ converges in the strict topology to some operator $D_A \in B(\mathcal{H}_A)$. Set $T_{\mathbb{F}} = \sum_{j \in \mathbb{F}} \theta_{\xi_j, \xi_j} - D_A$ for any finite subset \mathbb{F} of \mathbb{J} . Then the net $\{T_{\mathbb{F}}\}$ converges to 0 in the strict topology. Using the equality $\|T_{\mathbb{F}} \theta_{\xi, \eta}\| = \|T_{\mathbb{F}} \xi \langle \eta, \eta \rangle^{1/2}\|$ for all ξ, η from Lemma 1.5 (v), it follows that $\|T_{\mathbb{F}} \xi a\| \rightarrow 0$ for all positive $a \in \mathcal{A} \otimes \mathcal{K}$. But then,

$$\begin{aligned} \|T_{\mathbb{F}} \xi\| &\leq \|T_{\mathbb{F}} \xi a\| + \|T_{\mathbb{F}}(\xi - a\xi)\| \\ &\leq \|T_{\mathbb{F}} \xi a\| + \sup\{\|T_{\mathbb{F}}\| \|(\xi - a\xi)\| \\ &= \|T_{\mathbb{F}} \xi a\| + \sup\{\|T_{\mathbb{F}}\| \|\langle \xi, \xi \rangle - a \langle \xi, \xi \rangle - \langle \xi, \xi \rangle a + a \langle \xi, \xi \rangle a\|^{1/2} \\ &\leq \|T_{\mathbb{F}} \xi a\| + \sup\{\|T_{\mathbb{F}}\| \|\langle \xi, \xi \rangle - a \langle \xi, \xi \rangle\| + \|a\| \|\langle \xi, \xi \rangle - a \langle \xi, \xi \rangle\|^{1/2}. \end{aligned}$$

Since every C*-algebra \mathcal{A} has a positive approximate identity, one can choose $a > 0$ such that

$$\sup\{\|T_{\mathbb{F}}\| \|\langle \xi, \xi \rangle - a \langle \xi, \xi \rangle\| + \|a\| \|\langle \xi, \xi \rangle - a \langle \xi, \xi \rangle\|^{1/2} < \epsilon.$$

For that a , $\|T_{\mathbb{F}} \xi a\| < \epsilon$ for all $\mathbb{F} \supset G$ for some finite subset G of \mathbb{J} . This shows that $\|T_{\mathbb{F}} \xi\| \rightarrow 0$. Consequently, the series $\sum_{j \in \mathbb{J}} \theta_{\xi_j, \xi_j} \xi$ converges in the norm of \mathcal{H}_A to $D_A \xi$, and hence,

$$\sum_{j \in \mathbb{J}} \langle \theta_{\xi_j, \xi_j} \xi, \xi \rangle = \sum_{j \in \mathbb{J}} \langle \xi, \xi_j \rangle \langle \xi_j, \xi \rangle$$

converges in the norm of \mathcal{A} to $\langle D_A \xi, \xi \rangle$ by the Schwartz Inequality. Now a positive operator D_A is bounded and invertible if and only if $aI \leq D_A \leq bI$ for some constants $a, b > 0$. By [11, 2.1.3], this condition is equivalent to $a \langle \xi, \xi \rangle \leq \langle D_A \xi, \xi \rangle \leq b \langle \xi, \xi \rangle$. Therefore, $\{\xi_j\}_{j \in \mathbb{J}}$ is a frame.

Conversely, assume that $\{\xi_j\}_{j \in \mathbb{J}}$ is a frame. Then $\sum_{j \in \mathbb{J}} \langle \theta_{\xi_j, \xi_j} \xi, \xi \rangle$ converges in the norm of \mathcal{A} to $\langle D_A \xi, \xi \rangle$ for some positive operator $D_A \in B(\mathcal{H}_A)$. For any finite subset $\mathbb{F} \subset \mathbb{J}$, $\langle \sum_{j \in \mathbb{F}} \theta_{\xi_j, \xi_j} \xi, \xi \rangle \leq \langle D_A \xi, \xi \rangle$, hence, again by [11, 2.1.3],

$$0 \leq D_A - \sum_{j \in \mathbb{F}} \theta_{\xi_j, \xi_j} \leq D_A.$$

But then, by (vi) in the above lemma,

$$\begin{aligned} \|(D_A - \sum_{j \in \mathbb{F}} \theta_{\xi_j, \xi_j}) \theta_{\xi, \eta}\| &\leq \|(D_A - \sum_{j \in \mathbb{F}} \theta_{\xi_j, \xi_j})^{1/2}\| \|\theta_{(D_A - \sum_{j \in \mathbb{F}} \theta_{\xi_j, \xi_j})^{1/2} \xi, \eta}\| \\ &\leq \|D_A\|^{1/2} \|(D_A - \sum_{j \in \mathbb{F}} \theta_{\xi_j, \xi_j})^{1/2} \xi\| \|\eta\| \\ &= \|D_A\|^{1/2} \|\eta\| \|(D_A - \sum_{j \in \mathbb{F}} \theta_{\xi_j, \xi_j}) \xi, \xi\|^{1/2} \rightarrow 0. \end{aligned}$$

Since the linear span of rank-one operators is by dense in $\mathcal{A} \otimes \mathcal{K}$, it follows that $\|(D_A - \sum_{j \in \mathbb{F}} \theta_{\xi_j, \xi_j}) S\| \rightarrow 0$ for all $S \in \mathcal{A} \otimes \mathcal{K}$. Since $D_A - \sum_{j \in \mathbb{F}} \theta_{\xi_j, \xi_j}$ is selfadjoint,

this proves that the series $\sum_{j \in \mathbb{J}} \theta_{\xi_j, \xi_j}$ converges to D_A in the strict topology. The same argument as above shows that since D_A is bounded and invertible then

$$a \langle \xi, \xi \rangle \leq \langle D_A \xi, \xi \rangle = \sum_{j \in \mathbb{J}} \langle \xi, \xi_j \rangle \langle \xi_j, \xi \rangle \leq b \langle \xi, \xi \rangle \text{ for all } \xi.$$

□

Many of the results on frames in Hilbert C^* -modules are obtained under the assumption that \mathcal{A} is unital, which is of course the case for Hilbert space frames where $\mathcal{A} = \mathbb{C}$. When \mathcal{A} is unital, in lieu of viewing frames as collections of vectors in $\mathcal{H}_{\mathcal{A}}$, we can view them as collections of rank-one operators on $\mathcal{H}_{\mathcal{A}}$ with range in a submodule \mathcal{H}_0 . Indeed, if $\eta \in \mathcal{H}_{\mathcal{A}}$ is an arbitrary unital vector, i.e., $\langle \eta, \eta \rangle = I$, then by Lemma 1.5 (vii), (i) and (ii), $E_0 := \theta_{\eta, \eta} \in \mathcal{A} \otimes K$ is a projection, actually the range projection of $\theta_{\eta, \xi}$ for every $\xi \in \mathcal{H}_{\mathcal{A}}$. Then $\mathcal{H}_0 := E_0 \mathcal{H}_{\mathcal{A}}$ is a submodule of $\mathcal{H}_{\mathcal{A}}$ and we can identify $E_0 M(\mathcal{A} \otimes K)$ with $B(\mathcal{H}_{\mathcal{A}}, \mathcal{H}_0)$, the set of linear bounded adjointable operators from $\mathcal{H}_{\mathcal{A}}$ to the submodule \mathcal{H}_0 . Notice that all this would hold also under the weaker hypothesis that $\langle \eta, \eta \rangle$ is a projection. Then for every collection $\{\xi_j\}_{j \in \mathbb{J}}$ in $\mathcal{H}_{\mathcal{A}}$, define the rank-one operators $A_j := \theta_{\eta, \xi_j}$. Since $E_0 A_j = A_j$, $A_j \in B(\mathcal{H}_{\mathcal{A}}, \mathcal{H}_0)$. Again, by Lemma 1.5, $A_j^* A_j = \theta_{\xi_j, \langle \eta, \eta \rangle, \xi_j} = \theta_{\xi_j, \xi_j}$. It follows from Theorem 1.4 that $\{\xi_j\}_{j \in \mathbb{J}}$ is a frame if and only if the series $\sum_{j \in \mathbb{J}} A_j^* A_j$ converges in the strict topology to a bounded invertible operator on $\mathcal{H}_{\mathcal{A}}$.

This leads naturally to the following definition.

1.6 Definition Let \mathcal{A} be a σ -unital C^* -algebra and \mathbb{J} be a countable index set. Let E_0 be a projection in $M(\mathcal{A} \otimes K)$. Denote by \mathcal{H}_0 the submodule $E_0 \mathcal{H}_{\mathcal{A}}$ and identify $B(\mathcal{H}_{\mathcal{A}}, \mathcal{H}_0)$ with $E_0 M(\mathcal{A} \otimes K)$. A collection $A_j \in B(\mathcal{H}_{\mathcal{A}}, \mathcal{H}_0)$ for $j \in \mathbb{J}$ is called an operator-valued frame on $\mathcal{H}_{\mathcal{A}}$ with range in \mathcal{H}_0 if the sum $\sum_{j \in \mathbb{J}} A_j^* A_j$ converges in the strict topology to a bounded invertible operator on $\mathcal{H}_{\mathcal{A}}$, denoted by D_A . $\{A_j\}_{j \in \mathbb{J}}$ is called a tight operator-valued frame (resp., a Parseval operator-valued frame) if $D_A = \lambda I$ for a positive number λ (resp., $D_A = I$). If the set $\bigcup \{A_j \mathcal{H}_{\mathcal{A}} : j \in \mathbb{J}\}$ is dense in \mathcal{H}_0 , then the frame is said to be non-degenerate.

From now on, by frame, we will mean an operator-valued frame on a Hilbert C^* -module. Notice that if $\{A_j\}_{j \in \mathbb{J}}$ is a frame with range in \mathcal{H}_0 then it is also a frame with range in any larger submodule.

A minor difference with the definition given in [8] for operator-valued frames on a Hilbert space, is that here, in order to avoid introducing maps between different modules, we take directly \mathcal{H}_0 as a submodule of $\mathcal{H}_{\mathcal{A}}$. For operator-valued frames on a Hilbert space we do not assume in [8] that $\mathcal{H}_0 \subset \mathcal{H}$ and hence we are left with the flexibility of considering $\dim \mathcal{H}_0 > \dim \mathcal{H}$.

1.7 Example Let $\sum_{j \in \mathbb{J}} E_j = I$ be a decomposition of the identity of $M(\mathcal{A} \otimes K)$ into mutually orthogonal equivalent projections in $M(\mathcal{A} \otimes K)$, i.e., $L_j L_j^* = E_j$ and $L_j^* L_j = E_0$ for some collection of partial isometries $L_j \in M(\mathcal{A} \otimes K)$, and the series converges in the strict topology. Let T be a left-invertible element of $M(\mathcal{A} \otimes K)$ and let $A_j := L_j^* T$. Then $A_j \in B(\mathcal{H}_{\mathcal{A}}, E_0 \mathcal{H}_{\mathcal{A}})$ for $j \in \mathbb{J}$, and $\sum_{j \in \mathbb{J}} A_j^* A_j = \sum_{j \in \mathbb{J}} T^* A_j T = T^* T$ is an invertible element of $M(\mathcal{A} \otimes K)$, where

the convergence is in the strict topology. Thus $\{A_j\}_{j \in \mathbb{J}}$ is a frame with range in $E_0\mathcal{H}_A$. The frame is Parseval precisely when T is an isometry. We will see in the next section that this example is generic.

2. Frame Transforms

2.1. Definition Assume that $\{A_j\}_{j \in \mathbb{J}}$ is a frame in $B(\mathcal{H}_A, E_0\mathcal{H}_A)$ for the Hilbert C^* -module \mathcal{H}_A and set $\mathcal{H}_0 := E_0\mathcal{H}_A$. Decompose the identity of $M(\mathcal{A} \otimes \mathcal{K})$, into a strictly converging sum of mutually orthogonal projections $\{E_j\}_{j \in \mathbb{J}}$ in $M(\mathcal{A} \otimes \mathcal{K})$ with $E_j \sim E_{00} \geq E_0$. Let L_j be partial isometries in $M(\mathcal{A} \otimes \mathcal{K})$ such that $L_j L_j^* = E_j$ and $L_j^* L_j = E_{00}$. Define the frame transform θ_A of the frame $\{A_j\}_{j \in \mathbb{J}}$ as

$$\theta_A = \sum_{j \in \mathbb{J}} L_j A_j : \mathcal{H}_A \longrightarrow \mathcal{H}_A.$$

2.2. Theorem Assume that $\{A_j\}_{j \in \mathbb{J}}$ is a frame in $B(\mathcal{H}_A, \mathcal{H}_0)$.

- (a) The sum $\sum_{j \in \mathbb{J}} L_j A_j$ converges in the strict topology, and hence θ_A is an element of $M(\mathcal{A} \otimes \mathcal{K})$.
- (b) $D_A = \theta_A^* \theta_A$, $\theta_A D_A^{-1/2}$ is an isometry, $P_A := \theta_A D_A^{-1} \theta_A^*$ is the range projection of θ_A , and all these three elements belong to $M(\mathcal{A} \otimes \mathcal{K})$.
- (c) $\{A_j\}_{j \in \mathbb{J}}$ is a Parseval frame, if and only if θ_A is an isometry of $M(\mathcal{A} \otimes \mathcal{K})$, and again if and only if $\theta_A \theta_A^*$ is a projection.
- (d) $A_j = L_j^* \theta_A$ for all $j \in \mathbb{J}$.

PROOF. (a) For every finite subset F of \mathbb{J} , let $S_F = \sum_{j \in F} L_j A_j$. We need to show that $\{S_F : F \text{ is a finite subset of } \mathbb{J}\}$ is a Cauchy net in the strict topology of $M(\mathcal{A} \otimes \mathcal{K})$, i.e., for every for any $a \in \mathcal{A} \otimes \mathcal{K}$, $\max\{\|(S_F - S_{F'})a\|, \|a(S_F - S_{F'})\|\} \longrightarrow 0$, in the sense that for every $\varepsilon > 0$ there is a finite set G such that

$$\max\{\|(S_F - S_{F'})a\|, \|a(S_F - S_{F'})\|\} < \varepsilon \quad \text{for any finite sets } F \supset G, F' \supset G.$$

Firstly, since the partial isometries L_j have mutually orthogonal ranges, one has

$$\begin{aligned} \|(S_F - S_{F'})a\| &= \|a^*(S_F - S_{F'})^*(S_F - S_{F'})a\|^{\frac{1}{2}} \\ &= \|a^* \left(\sum_{j \in (F \setminus F') \cup (F' \setminus F)} A_j^* L_j^* L_j A_j \right) a\|^{\frac{1}{2}} \\ &= \|a^* \left(\sum_{j \in (F \setminus F') \cup (F' \setminus F)} A_j^* E_{00} A_j \right) a\|^{\frac{1}{2}} \\ &\leq \|a\|^{\frac{1}{2}} \left\| \sum_{j \in (F \setminus F') \cup (F' \setminus F)} A_j^* A_j a \right\|^{\frac{1}{2}} \longrightarrow 0, \end{aligned}$$

where the last term above converges to 0 because of the assumption that $\sum_{j \in \mathbb{J}} A_j^* A_j$ converges in the strict topology. Secondly, for all $\xi \in \mathcal{H}_A$

$$\begin{aligned}
\|(S_F - S_{F'})\xi\|^2 &= \sum_{j \in (F \setminus F') \cup (F' \setminus F)} \|L_j A_j \xi\|^2 \\
&= \sum_{j \in (F \setminus F') \cup (F' \setminus F)} \|A_j \xi\|^2 \\
&= \sum_{j \in (F \setminus F') \cup (F' \setminus F)} \langle A_j^* A_j \xi, \xi \rangle \\
&= \langle \sum_{j \in (F \setminus F') \cup (F' \setminus F)} A_j^* A_j \xi, \xi \rangle \\
&\leq \langle D_A \xi, \xi \rangle \\
&= \|D_A^{1/2} \xi\|^2
\end{aligned}$$

Thus $\|S_F - S_{F'}\| \leq \|D_A\|^{1/2}$ for any finite sets F and F' . Moreover, $\sum_{j \in (F \setminus F') \cup (F' \setminus F)} E_j (S_F - S_{F'}) = S_F - S_{F'}$, hence for every $a \in \mathcal{A} \otimes \mathcal{K}$,

$$\begin{aligned}
\|a(S_F - S_{F'})\| &= \|a(S_F - S_{F'})(S_F - S_{F'})^* a^*\|^{1/2} \\
&= \|a \sum_{j \in (F \setminus F') \cup (F' \setminus F)} E_j (S_F - S_{F'}) (S_F - S_{F'})^* \sum_{j \in (F \setminus F') \cup (F' \setminus F)} E_j a^*\|^{1/2} \\
&\leq \|S_F - S_{F'}\| \|a \sum_{j \in (F \setminus F') \cup (F' \setminus F)} E_j a^*\|^{1/2} \\
&\leq \|D_A\|^{1/2} \|a\|^{1/2} \left\| \sum_{j \in (F \setminus F') \cup (F' \setminus F)} E_j a^* \right\|^{1/2} \longrightarrow 0
\end{aligned}$$

by the strict convergence of the series $\sum_{j \in \mathbb{J}} E_j$. The statements (b) and (c) are now obvious.

(d) Also obvious since the series $\sum_{i \in \mathbb{J}} L_j^* L_i A_i$ converges strictly to $L_j^* \theta_A$ and

$$L_j^* L_i A_i = \delta_{i,j} E_{00} A_i = \delta_{i,j} A_j.$$

□

The projection P_A is called the *frame projection*.

2.3 Remark (i) Since θ_A is left-invertible as $(D_A^{-1} \theta_A^*) \theta_A = I$, Example 1.7 is indeed generic, i.e., every frame $\{A_j\}_{j \in \mathbb{J}}$ is obtained from partial isometries L_j with mutually orthogonal range projections summing to the identity and same first projection majorizing E_0 . The relation with the by now familiar “dilation” point of view of the theory of frames is clarified in (ii) below.

(ii) For vector frames on a Hilbert space, the frame transform is generally defined as a map from the Hilbert space \mathcal{H} into $\ell^2(\mathbb{J})$ - a dilation of \mathcal{H} . If \mathcal{H} is infinite dimensional and separable and if \mathbb{J} is infinite and countable, which are the most common assumptions, then \mathcal{H} can be identified with $\ell^2(\mathbb{J})$, and hence, the frame transform can be seen as mapping of \mathcal{H} onto a subspace. In the case of Hilbert C*-modules, it is convenient to choose the latter approach, so to identify the frame transform and the frame projection with elements of $M(\mathcal{A} \otimes \mathcal{K})$.

(iii) The range projections of elements of $M(\mathcal{A} \otimes \mathcal{K})$ and even of elements of $\mathcal{A} \otimes \mathcal{K}$ always belong to $(\mathcal{A} \otimes \mathcal{K})^{**}$, but may fail to belong to $M(\mathcal{A} \otimes \mathcal{K})$. As shown above,

however, the frame projection P_A is always in $M(\mathcal{A} \otimes \mathcal{K})$ and $P_A \sim I$ since $\theta_A D_A^{1/2}$ is an isometry.

(iv) When \mathcal{A} is not simple, given an arbitrary nonzero projection $E_0 \in M(\mathcal{A} \otimes \mathcal{K})$ there may be no decomposition of the identity in projections equivalent to E_0 . Nevertheless, there is always a decomposition of the identity into a strictly convergent sum of mutually orthogonal projections that are all equivalent to a projection $E_{00} \geq E_0$, e.g., $E_{00} = I$.

(v) There seem to be no major advantage in considering only non-degenerate frames, i.e., seeking a “minimal” Hilbert module \mathcal{H}_0 that contains the ranges of all the operators A_j or similarly, choosing a frame transform with a minimal projection E_{00} . In fact, if we view the operators A_j as having their ranges in some $E_{00}\mathcal{H}_A$, the ensuing frame transform will, as seen in (d) above and in the next section, carry equally well all the “information” of the frame.

(vi) For the vector case, [5, 4.1] proves that the frame transform θ as a map from a finite or countably generated Hilbert C*-module \mathcal{H} to the standard module \mathcal{H}_A is adjointable. This is obvious in the case that we consider, where $\mathcal{H} = \mathcal{H}_A$ as then θ_A is in the C*-algebra $M(\mathcal{A} \otimes \mathcal{K})$.

(vii) A compact form of the *reconstruction formula* for a frame is simply

$$D_A^{-1} \sum_{j \in \mathbb{J}} A_j^* A_j = D_A^{-1} \theta_A^* \theta_A = I.$$

In the special case that \mathcal{A} is unital and that $\{A_j\}_{j \in \mathbb{J}}$ is a vector frame, we have seen in the course of the proof of Theorem 1.4 that $\sum_{j \in \mathbb{J}} A_j^* A_j$ converges strongly and the same result was obtained in [5, 4.1]. Thus, assuming for the sake of simplicity that the frame is Parseval, the reconstruction formula has the more familiar form

$$\sum_{j \in \mathbb{J}} A_j^* A_j \xi = \sum_{j \in \mathbb{J}} \xi_j \langle \xi_j, \xi \rangle = \xi \text{ for all } \xi \in \mathcal{H}_A$$

where the convergence is in the norm of \mathcal{H}_A .

3. Similarity of frames

3.1. Definition Two frames $\{A_j\}_{j \in \mathbb{J}}$ and $\{B_j\}_{j \in \mathbb{J}}$ in $B(\mathcal{H}_A, \mathcal{H}_0)$ are said to be right-similar (resp. right-unitarily equivalent) if there exists an invertible (resp. a unitary) element $T \in M(\mathcal{A} \otimes \mathcal{K})$ such that $B_j = A_j T$ for all $j \in \mathbb{J}$.

The following facts are immediate and their proofs are left to the reader.

3.2. Lemma

(i) If $\{A_j\}_{j \in \mathbb{J}}$ is a frame and T is an invertible element in $M(\mathcal{A} \otimes \mathcal{K})$, then $\{A_j T\}_{j \in \mathbb{J}}$ is also a frame.

(ii) If $\{A_j\}_{j \in \mathbb{J}}$ and $\{B_j\}_{j \in \mathbb{J}}$ are right-similar and T is an invertible element in $M(\mathcal{A} \otimes \mathcal{K})$ for which $B_j = A_j T$ for all $j \in \mathbb{J}$, then $\theta_B = \theta_A T$. Therefore $T = D_A^{-1} \theta_A^* \theta_B$, hence T is uniquely determined. Moreover, $P_A = P_B$ and $D_B = T^* D_A T$. Conversely, if $\theta_B = \theta_A T$ for some invertible element $T \in M(\mathcal{A} \otimes \mathcal{K})$, then $B_j = A_j T$ for all $j \in \mathbb{J}$.

(iii) Every frame is right-similar to a Parseval frame, i.e., $\{A_j\}_{j \in \mathbb{J}}$ is right-similar to $\{A_j\}_{j \in \mathbb{J}} D_A^{-1/2}$. Two Parseval frame are right-similar if and only if they are right-unitarily equivalent.

3.3. Theorem Let $\{A_j\}_{j \in \mathbb{J}}$ and $\{B_j\}_{j \in \mathbb{J}}$ be two frames in $B(\mathcal{H}_A, \mathcal{H}_0)$. Then the following are equivalent:

- (i) $\{A_j\}_{j \in \mathbb{J}}$ and $\{B_j\}_{j \in \mathbb{J}}$ are right-similar.
- (ii) $\theta_B = \theta_A T$ for some invertible operator $T \in M(\mathcal{A} \otimes \mathcal{K})$.
- (iii) $P_A = P_B$.

PROOF. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are given by Lemma 3.2. (iii) \Rightarrow (i). Assume that $P_A = P_B$. Then by Theorem 2.2 (b)

$$\theta_A D_A^{-1} \theta_A^* = \theta_B D_B^{-1} \theta_B^*.$$

By Theorem 2.2. (d)

$$B_j = L_j^* \theta_B = L_j^* P_A \theta_B = L_j^* \theta_A D_A^{-1} \theta_A^* \theta_B = A_j D_A^{-1} \theta_A^* \theta_B.$$

Let $T := D_A^{-1} \theta_A^* \theta_B$, then $T \in M(\mathcal{A} \otimes \mathcal{K})$ and

$$(D_B^{-\frac{1}{2}} \theta_B^* \theta_A D_A^{-\frac{1}{2}})(D_A^{-\frac{1}{2}} \theta_A^* \theta_B D_B^{-\frac{1}{2}}) = D_B^{-\frac{1}{2}} \theta_B^* P_B \theta_B D_B^{-\frac{1}{2}} = I.$$

Interchanging A and B , one has also

$$(D_A^{-\frac{1}{2}} \theta_A^* \theta_B D_B^{-\frac{1}{2}})(D_B^{-\frac{1}{2}} \theta_B^* \theta_A D_A^{-\frac{1}{2}}) = I.$$

Thus, $D_A^{-\frac{1}{2}} \theta_A^* \theta_B D_B^{-\frac{1}{2}}$ is unitary, hence $\theta_A^* \theta_B$ is invertible, and thus so is $T = D_A^{-1} \theta_A^* \theta_B$. This concludes the proof. \square

As is the case in $B(H)$, all the projections in $M(\mathcal{A} \otimes \mathcal{K})$ that are equivalent to the frame projection of a given frame, are also the frame projection of a frame, which by Theorem 3.3 is unique up to right similarity.

3.4. Theorem Let $\{A_j\}_{j \in \mathbb{J}}$ be a frame in $B(\mathcal{H}_A, \mathcal{H}_0)$ and let P be a projection in $M(\mathcal{A} \otimes \mathcal{K})$. Then $P \sim P_A$ if and only if there exists a frame $\{B_j\}_{j \in \mathbb{J}}$ in $B(\mathcal{H}_A, \mathcal{H}_0)$ such that $P = P_B$.

PROOF. If $P = P_B$, let $V = \theta_B D_B^{-\frac{1}{2}} D_A^{-\frac{1}{2}} \theta_A$. Then $V \in B(\mathcal{H}_A)$, $VV^* = P$, and $V^*V = P_A$, i.e., $P \sim P_A$. Conversely, if there exists $V \in M(\mathcal{A} \otimes \mathcal{K})$ such that $VV^* = P$ and $V^*V = P_A$, then set $B_j = L_j^* V \theta_A$. Then

$$\sum_{j \in \mathbb{J}} B_j^* B_j = \theta_A^* V^* \left(\sum_{j \in \mathbb{J}} L_j^* L_j \right) V \theta_A = \theta_A^* V^* V \theta_A = D_A.$$

Thus $\{B_j\}_{j \in \mathbb{J}}$ is a frame with $D_A = D_B$. Moreover,

$$\theta_B = \sum_{j \in \mathbb{J}} L_j^* L_j V \theta_A = V \theta_A.$$

It follows that $P_B = VV^* = P$. \square

The proof of Theorem 3.4 actually yields a parametrization of all the operator-valued frames with range in $B(\mathcal{H}_A, \mathcal{H}_0)$. For simplicity's sake, we formulate it in terms of Parseval frames

3.5. Corollary Let $\{A_j\}_{j \in \mathbb{J}}$ be a Parseval frame in $B(\mathcal{H}_A, \mathcal{H}_0)$. Then $\{B_j\}_{j \in \mathbb{J}}$ is a Parseval frame in $B(\mathcal{H}_A, \mathcal{H}_0)$ if and only if $B_j = L_j^* V \theta_A$ for some partial isometry $V \in M(\mathcal{A} \otimes \mathcal{K})$ such that $VV^* = P_B$ and $V^*V = P_A$.

3.6. Remark For a given equivalence $P \sim P_A$ the choice of partial isometry $V \in M(\mathcal{A} \otimes \mathcal{K})$ with $VV^* = P$ and $V^*V = P_A$ is determined up to a unitary that commutes with P_A , i.e., $V_1V_1^* = P$ and $V_1^*V_1 = P_A$ implies $V_1 = VU$ for some unitary U that commutes with P_A , or, equivalently, $V_1 = U_1V$ for some unitary U_1 that commutes with P . Notice that if V and U are as above, then $\{L_j^* V \theta_A D_A^{-\frac{1}{2}}\}_{j \in \mathbb{J}}$ and $\{L_j^* V U \theta_A D_A^{-\frac{1}{2}}\}_{j \in \mathbb{J}}$ are two frames having the same frame projections P . It then follows from Theorem 3.3 that the two frames are right-similar, actually, right-unitarily equivalent, because $L_j^* V U \theta_A D_A^{-\frac{1}{2}} = L_j^* V \theta_A D_A^{-\frac{1}{2}} (D_A^{-\frac{1}{2}} \theta_A^* U \theta_A D_A^{-\frac{1}{2}})$ and $D_A^{-\frac{1}{2}} \theta_A^* U \theta_A D_A^{-\frac{1}{2}}$ is a unitary operator on \mathcal{H}_A as $\theta_A D_A^{-\frac{1}{2}}$ is an isometry.

For operator valued frames it is natural to consider also the notion of left similarity.

3.7. Definition Two frames $\{A_j\}_{j \in \mathbb{J}}$ and $\{B_j\}_{j \in \mathbb{J}}$ in $B(\mathcal{H}_A, \mathcal{H}_0)$ are said to be left-similar (resp., left-unitarily equivalent) if there exists an invertible (resp., a unitary) element S in the corner algebra $E_0 M(\mathcal{A} \otimes \mathcal{K}) E_0$ such that $B_j = S A_j$ for all $j \in \mathbb{J}$, where E_0 is the projection in $M(\mathcal{A} \otimes \mathcal{K})$ such that $E_0 \mathcal{H}_A = \mathcal{H}_0$.

The following results are elementary.

3.8. Lemma If $\{A_j\}_{j \in \mathbb{J}}$ is a frame in $B(\mathcal{H}_A, \mathcal{H}_0)$ and S is an invertible element in $E_0 M(\mathcal{A} \otimes \mathcal{K}) E_0$, then $\{A_j\}_{j \in \mathbb{J}}$ is also a frame. Moreover, $\theta_B = \sum_{j \in \mathbb{J}} L_j S A_j$, hence $D_B = \sum_{j \in \mathbb{J}} A_j^* S^* S A_j$ and thus

$$\|S^{-1}\|^{-2} D_A \leq D_B \leq \|S\|^2 D_A.$$

In particular, if S is unitary, then $D_A = D_B$.

3.9. Proposition Given two left-unitarily equivalent frames $\{A_j\}_{j \in \mathbb{J}}$ and $\{B_j\}_{j \in \mathbb{J}}$ in $B(\mathcal{H}_A, \mathcal{H}_0)$ with $B_j = S A_j$ for some unitary element $S \in M(\mathcal{A} \otimes \mathcal{K})$, then the following are equivalent:

- (i) S commutes with $A_j D_A^{-1} A_j^*$ for all $i, j \in \mathbb{J}$.
- (ii) $\{A_j\}_{j \in \mathbb{J}}$ and $\{B_j\}_{j \in \mathbb{J}}$ are right-unitarily equivalent.

PROOF. (i) \implies (ii) One has

$$\begin{aligned} B_j &= SA_j \\ &= SA_j(D_A^{-1}D_A) = \sum_{i \in \mathbb{J}} SA_j D_A^{-1} A_i^* A_i \\ &= \sum_{i \in \mathbb{J}} A_j D_A^{-1} A_i^* S A_i = A_j D_A^{-1} \sum_{i \in \mathbb{J}} A_i^* S A_i \end{aligned}$$

where the series here and below converge in the strict topology. Let

$$U = D_A^{-\frac{1}{2}} \sum_{i \in \mathbb{J}} A_i^* S A_i D_A^{-\frac{1}{2}}.$$

Then

$$\begin{aligned} UU^* &= D_A^{-\frac{1}{2}} \sum_{i,j \in \mathbb{J}} A_i^* S A_i D_A^{-1} A_j^* S^* A_j D_A^{-\frac{1}{2}} \\ &= D_A^{-\frac{1}{2}} \sum_{i,j \in \mathbb{J}} A_i^* A_i D_A^{-1} A_j^* S S^* A_j D_A^{-\frac{1}{2}} \\ &= D_A^{-\frac{1}{2}} \left(\sum_{i \in \mathbb{J}} A_i^* A_i \right) D_A^{-1} \left(\sum_{j \in \mathbb{J}} A_j^* A_j \right) D_A^{-\frac{1}{2}} \\ &= I. \end{aligned}$$

Similarly, $U^*U = I$. Notice that $B_j = A_j D_A^{-\frac{1}{2}} U D_A^{\frac{1}{2}}$, and that $D_A^{-\frac{1}{2}} U D_A^{\frac{1}{2}}$ is invertible. It follows that $\{A_j\}_{j \in \mathbb{J}}$ and $\{B_j\}_{j \in \mathbb{J}}$ are right-unitarily equivalent frames.

(ii) \implies (i) $P_A = P_B$ by Theorem 3.3 and $D_A = D_B$ by Lemma 3.8. Then

$$\begin{aligned} P_A &= \theta_A D_A^{-1} \theta_A^* = \sum_{i,j \in \mathbb{J}} L_i A_i D_A^{-1} A_j^* L_j^* \\ &= P_B = \theta_B D_B^{-1} \theta_B^* = \sum_{i,j \in \mathbb{J}} L_i S A_i D_A^{-1} A_j^* S^* L_j^*. \end{aligned}$$

Multiplying on the left by L_i^* and on the right by L_j , one has

$$A_i D_A^{-1} A_j^* = S A_i D_A^{-1} A_j^* S^* \quad \forall i, j \in \mathbb{J}.$$

□

3.10. Composition of frames Let $\{A_j\}_{j \in \mathbb{J}}$ be a frame in $B(\mathcal{H}_A, \mathcal{H}_0)$ and $\{B_i\}_{i \in \mathbb{I}}$ be a frame in $B(\mathcal{H}_0, \mathcal{H}_1)$. Then it is easy to check that $\{C_{i,j} := B_i A_j\}_{j \in \mathbb{J}, i \in \mathbb{I}}$ is a frame in $B(\mathcal{H}_A, \mathcal{H}_1)$, called the composition of the frames $\{A_j\}_{j \in \mathbb{J}}$ and $B(\mathcal{H}_A, \mathcal{H}_0)$. In symbols, $C = BA$

It is easy to see that the composition of two Parseval frames is also Parseval.

3.11. Remarks (i) If $BA = BA'$, then $A = A'$. Indeed, if for all $i \in \mathbb{I}$ and $j \in \mathbb{J}$ and $B_i A_j = B_i A'_j$, then $\sum_{i \in \mathbb{I}} B_i^* B_i A_j = D_B A_j = D_B A'_j$. Then $A_j = A'_j$, since D_B is invertible.

(ii) If $A := \{A_j\}_{j \in \mathbb{J}}$ is non-degenerate (i.e., the closure of $\bigcup_{j \in \mathbb{J}} A_j \mathcal{H}_2$ is \mathcal{H}_2), then $BA = B'A$ implies $B = B'$.

3.12. Remark Let \mathcal{A} be unital, then as shown in Theorem 1.4, we can identify a vector frame $\{\xi_i\}_{i \in \mathbb{I}}$ on a submodule $\mathcal{H}_1 \subset \mathcal{H}_{\mathcal{A}}$, with the (rank-one) operator valued frame $\{\theta_{\eta, \xi_i}\}_{i \in \mathbb{I}}$ in $B(\mathcal{H}_1, \theta_{\eta, \eta} \mathcal{H}_{\mathcal{A}})$, where $\eta \in \mathcal{H}_{\mathcal{A}}$ is a vector for which $\langle \eta, \eta \rangle = I$. But then, for any operator-valued frame $\{A_j\}_{j \in \mathbb{J}}$ in $B(\mathcal{H}_{\mathcal{A}}, \mathcal{H}_0)$ and $\mathcal{H}_1 \subset \mathcal{H}_0$, the composition $\{\theta_{\eta, \xi_i} A_j = \theta_{\eta, A_j^* \xi_i}\}_{i \in \mathbb{I}, j \in \mathbb{J}}$ is identified with the vector frame $\{A_j^* \xi_i\}_{i \in \mathbb{I}, j \in \mathbb{J}}$. We can view this as a decomposition of the operator valued frame $\{A_j\}_{j \in \mathbb{J}}$ into the collection of (vector-valued) frames $\{A_j^* \xi_i\}_{i \in \mathbb{I}}$ indexed by \mathbb{I} , i.e., a “multiframe”.

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