

# STRUCTURED TRIANGULAR LIMIT ALGEBRAS

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## 1. Introduction

Suppose  $\mathcal{A}$  is a unital Banach algebra which contains an increasing chain  $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$  of finite-dimensional subalgebras, all containing the unit  $I$  of  $\mathcal{A}$ , such that  $\mathcal{A}$  is the closure of the union  $\bigcup_n \mathcal{A}_n$ . Then  $\mathcal{A}$  can be viewed as a direct limit of the algebras  $\mathcal{A}_n$ . It frequently occurs in this setting that there is a sequence of bounded mappings  $\psi_n: \mathcal{A} \rightarrow \mathcal{A}_n$  which satisfy the conditional expectation property  $\psi_n(CAB) = C\psi_n(A)B$  whenever  $C, B \in \mathcal{A}_n$ , and with the further property that  $\{\psi_n\}$  converges to the identity mapping of  $\mathcal{A} \rightarrow \mathcal{A}$  in the strong operator topology (that is,  $\psi_n(A) \rightarrow A$  for  $A \in \mathcal{A}$ ). Indeed, if all the  $\mathcal{A}_n$  are C\*-algebras, so  $\mathcal{A}$  is an AF (approximately finite-dimensional) C\*-algebra, this is always the case. This property holds in many non-selfadjoint settings as well. In some special cases, not only do the  $\{\psi_n\}$  exist but they can be taken to be *multiplicative* on  $\mathcal{A}$ . So the  $\psi_n$  are unital homomorphisms of  $\mathcal{A}$  into itself. This implies very strong structural properties of the limit algebra. Non-trivial instances of this type of *highly-structured* property are not possible in the C\*-setting unless, for instance, the algebras are abelian. However, certain of the more interesting and widely studied of the triangular UHF (uniformly hyperfinite) algebras exhibit precisely this behaviour. The purpose of this article is to investigate and partially classify this family. We begin by establishing a proper framework for this study.

## 2. Preliminaries

A unital C\*-algebra  $\mathcal{B}$  is called an AF algebra if it can be written as a direct limit of finite-dimensional C\*-algebras  $\mathcal{B} = \varinjlim (B_k, \varphi_k)$ . Here each  $B_k$  is a finite-dimensional C\*-algebra and  $\varphi_k: B_k \rightarrow B_{k+1}$  is a unital injective \*-homomorphism. If each  $B_k$  is the algebra of  $n_k \times n_k$  complex matrices, denoted  $M_{n_k}$  or  $M_{n_k}(\mathbb{C})$ , then  $\mathcal{B}$  is called a UHF algebra. Due to the work of Stratila and Voiculescu [11] we know that, given an AF algebra  $\mathcal{B}$ , we can find  $B_k, \varphi_k$  defining  $\mathcal{B}$  as a direct limit, and fix a system of matrix units in each  $B_k$ , as  $B_k$  is isomorphic to a sum of full matrix algebras, in such a way that, for each  $k$ ,  $\varphi_k$  maps the diagonal matrices of  $B_k$  into the diagonal matrices of  $B_{k+1}$  and, for every matrix unit  $e$  in  $B_k$ ,  $\varphi(e)$  is a sum of matrix units in  $B_{k+1}$ . In such a case the maps  $\varphi_k$  will be said to be *canonical* and  $\mathcal{B} = \varinjlim (B_k, \varphi_k)$  is canonical presentation of  $\mathcal{B}$ . This presentation will in general be non-unique.

Given a canonical presentation as above, we can set  $D = \varinjlim (C_k, \varphi_k)$  where  $C_k$  is the algebra of all diagonal matrices in  $B_k$ . Then  $D$  is a masa in  $\mathcal{B}$ . Such a masa

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is called a *canonical masa*. We also call a norm-closed subalgebra  $\mathcal{A}$  of  $\mathcal{B}$  *canonical* if it contains a canonical masa  $D$ . If, in addition  $\mathcal{A} \cap \mathcal{A}^* = D$ ,  $\mathcal{A}$  is said to be a canonical triangular AF algebra. In this paper we will deal only with canonical algebras and, therefore, every triangular AF or triangular UHF algebra will be assumed to be canonical, and we shall simply write TAF or TUHF. In fact, except in the last section when we consider TAF algebras, we will be interested only in TUHF algebras  $\mathcal{A}$  that can be written as

$$\mathcal{A} = \varinjlim (T_{n_k}, \varphi_k)$$

where  $\varinjlim (M_{n_k}, \varphi_k)$  is a canonical presentation of a UHF algebra  $\mathcal{B}$  with the property that each  $\varphi_k$  maps the upper triangular matrices in  $M_{n_k}$ , denoted  $T_{n_k}$ , into the upper triangular matrices in  $M_{n_{k+1}}$ . These algebras are called in the literature *strongly maximal triangular in factors* (see [7, p. 105; 8, Example 2.12]).

Given an AF algebra  $\mathcal{B}$  with a canonical masa  $D \subseteq \mathcal{B}$ , we can express  $\mathcal{B}$  as  $C^*(G)$  for some  $r$ -discrete amenable principal groupoid. In fact the groupoids of the kind we are considering here may be viewed as equivalence relations on  $X$ , the maximal ideal space of  $D$ , having countable equivalence classes. Hence  $G$  can be viewed as a subset of  $X \times X$  with a topology that may be finer than the product topology. For details about groupoids and the associated  $C^*$ -algebras see [10]. Groupoids for AF algebras are treated in [10, Chapter III, § 1]. Elements of the AF algebra  $\mathcal{B}$  are viewed as continuous functions on  $G$  and the multiplication and adjoint operations are defined as follows:

$$(fg)(u, v) = \sum_{\substack{w \\ (w,u) \in G}} f(u, w)g(w, v),$$

$$f^*(u, v) = \overline{f(v, u)}.$$

For  $f \in \mathcal{B}$  we write  $\text{supp } f = \{(u, v) \in G : f(u, v) \neq 0\}$ . We have  $D = \{f \in \mathcal{B} : \text{supp } f \subseteq \Delta\}$  where  $\Delta = \{(u, u) : u \in X\}$ . In fact we shall identify  $\Delta$  with  $X$  (and  $u \in X$  with  $(u, u) \in \Delta \subseteq G$ ).

Given a canonical subalgebra  $D \subseteq \mathcal{A} \subseteq \mathcal{B}$ , there is an open subset  $P \subseteq G$  such that

- (i)  $\Delta \subseteq P$ ;
- (ii)  $P \circ P \subseteq P$ ; that is, if  $(x, y), (y, z)$  are in  $P$  then  $(x, z)$  is in  $P$  and  $\mathcal{A} = \{f \in \mathcal{B} : \text{supp } f \subseteq P\}$  (we then write  $\mathcal{A} = \mathcal{A}(P)$ ) (see [4]).

The algebra  $\mathcal{A} (= \mathcal{A}(P))$  is triangular if and only if

- (iii)  $P \cap P^{-1} = \Delta$  (where  $P^{-1} = \{(u, v) : (v, u) \in P\}$ ).

For a strongly maximal triangular algebra we also have

- (iv)  $P \cup P^{-1} = G$  [5, Theorem 1.3].

Whenever (i)–(iii) are satisfied,  $P$  induces a partial order on every equivalence class (we write  $[u]$  for the equivalence class of  $u \in X$ ). The condition (iv) means that this order is total. When we write  $u \leq v$  it will always be assumed that  $(u, v) \in P$ .

Since we deal mostly with the case where  $\mathcal{B}$  is a UHF algebra, note that in this case each equivalence class is dense in  $X$  (that is, the groupoid is minimal). Given a canonical presentation  $\mathcal{B} = \varinjlim (M_{n_k}, \varphi_k)$  as above, we write the matrix units of  $M_{n_k}$  as  $\{e_{ij}^{(k)}\}$ . Viewing  $\mathcal{B}$  as  $C^*(G)$ , we can view each  $e_{ij}^{(k)}$  as a function on  $G$ . In

fact it is a characteristic function of some open and closed subset  $\hat{e}_{ij}^{(k)} \subseteq G$  (if  $i = j$ ,  $\hat{e}_{ij}^{(k)}$  can be viewed as a subset of  $X$ ). We also write  $\hat{E}$  for the support of  $E$  whenever  $E$  is a sum of matrix units. With this notation  $\varphi_k(e_{ij}^{(k)})$  is a sum of matrix units in  $M_{n_{k+1}}$  and  $e_{l,m}^{(k+1)}$  is one of these matrix units if and only if  $\hat{e}_{l,m}^{(k+1)} \subseteq \hat{e}_{ij}^{(k)}$ . Given a point  $(u, v) \in G$  there are  $e_{i_k, j_k}^{(k)}$  with  $\hat{e}_{i_k, j_k}^{(k)} \supseteq \hat{e}_{i_{k+1}, j_{k+1}}^{(k+1)}$  such that  $\{(u, v)\} = \bigcap_k \hat{e}_{i_k, j_k}^{(k)}$ , and *vice versa*. Every such intersection with  $\hat{e}_{i_k, j_k}^{(k)} \supseteq \hat{e}_{i_{k+1}, j_{k+1}}^{(k+1)}$  defines a point in  $G$ . For more details about triangular AF algebras and the associated partial orders see [9].

Finally, let us recall that UHF algebras  $\varinjlim(M_{n_k}, \varphi_k)$  are characterized by the supernatural number associated with the sequence  $\{n_k\}$  [2]. Given a supernatural number (also called a generalized integer)  $n$ , we write  $M(n)$  for the associated UHF algebra.

### 3. Structured algebras

In this section we study a class of triangular UHF algebras. In order to define this class we first introduce the following definitions.

DEFINITIONS. A map  $\varphi: M_k(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  is called a *canonical embedding* if

- (1) it is an injective unital  $*$ -homomorphism;
- (2)  $\varphi(T_k) \subseteq T_n$  (where  $T_n \subseteq M_n(\mathbb{C})$  is the subalgebra of all upper triangular matrices);
- (3)  $\varphi$  maps each matrix unit of  $M_k(\mathbb{C})$  into a sum of matrix units in  $M_n(\mathbb{C})$ .

Then  $\varphi$  is called a *structured canonical embedding* if, in addition,

- (4) there is a contractive conditional expectation  $\psi$  from  $M_n(\mathbb{C})$  onto  $\varphi(M_k(\mathbb{C}))$  such that  $\psi$ , restricted to  $T_n$ , is multiplicative.

LEMMA 3.1. *For a canonical embedding  $\varphi: M_k(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ , condition (4) is equivalent to*

- (4') *there is a contractive map  $g: M_n(\mathbb{C}) \rightarrow M_k(\mathbb{C})$  such that  $g \circ \varphi = \text{id}_{M_k}$  and  $g|_{T_n}$  is multiplicative.*

*Proof.* Given  $\psi$  as in (4) we define  $g = \varphi^{-1} \circ \psi$  (here  $\varphi^{-1}: \varphi(M_k(\mathbb{C})) \rightarrow M_k(\mathbb{C})$ ). It clearly satisfies (4'). Conversely, given  $g$  as in (4') we let  $\psi = \varphi \circ g$  and then  $\psi \circ \psi = \varphi \circ g \circ \varphi \circ g = \varphi \circ \text{id} \circ g = \psi$ , so that  $\psi$  is indeed a conditional expectation onto  $\varphi(M_k(\mathbb{C}))$ .

If the above holds, we call  $g$  the *back map* of  $\varphi$ .

LEMMA 3.2. *For a canonical embedding  $\varphi: M_k(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ ,  $\varphi$  is structured (that is, satisfies condition (4) above) if and only if it satisfies*

- (4'') *there is some  $0 \leq l \leq n$  such that, writing*

$$N = \{i: l < i \leq l + k\},$$

we have, for every  $A \in M_k(\mathbb{C})$ ,

$$\varphi(A)_{p,m} = \begin{cases} 0 & \text{if } (p, m) \notin (N \times N) \cup (N^c \times N^c), \\ A_{p-l, m-l} & \text{if } (p, m) \in N \times N. \end{cases}$$

(Here  $N^c$  is the complement of  $N$ .)

If this is satisfied, we shall say that the identity is an *interval summand* of  $\varphi$ .

*Proof.* Let  $\varphi: M_k(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  be a canonical embedding satisfying (4). We shall show that it satisfies (4''). We now write  $\{e_{ij}\}$  for the matrix units of  $M_k(\mathbb{C})$  and  $\{f_{pq}\}$  for the ones in  $M_n(\mathbb{C})$ . Also set  $v_{ij} = \varphi(e_{ij})$ . Since  $v_{ij}$  is a sum of some of the matrix units of  $M_n(\mathbb{C})$ , we can define  $\text{supp } v_{ij} = \{(p, q) \in \{1, \dots, n\}^2: f_{pq} \text{ is one of the matrix units of } M_n(\mathbb{C}) \text{ whose sum is } v_{ij}\}$ . If  $(i, j) \neq (c, d)$  then either  $e_{ij}e_{cd}^* = 0$  or  $e_{cd}^*e_{ij} = 0$ , and hence either  $v_{ij}v_{cd}^* = 0$  or  $v_{cd}^*v_{ij} = 0$ . Hence, if  $(i, j) \neq (c, d)$ , then  $\text{supp } v_{ij} \cap \text{supp } v_{cd} = \emptyset$ . For  $a \in \varphi(M_k) \cap T_n$ ,  $a = \sum a_{ij}\varphi(e_{ij}) \in T_n$ . Hence  $v_{ij} \in T_n$  whenever  $a_{ij} \neq 0$ . Since  $\varphi(T_k) \subseteq T_n$ , this shows that  $\varphi(T_k) = \varphi(M_k) \cap T_n$ . Similarly  $\varphi(D_k) = \varphi(M_k) \cap D_n$  (where  $D_k$  is the algebra of diagonal matrices in  $M_k(\mathbb{C})$ ).

Since each  $v_{ij}$  is a partial isometry with  $v_{ij}v_{ij}^* = v_{ii}$  and  $v_{ij}^*v_{ij} = v_{jj}$  in  $D_n$ , we see that there are disjoint subsets  $J_i \subseteq \{1, \dots, n\}$ , for  $1 \leq i \leq k$ , and one-to-one functions  $\tau_{ij}: J_i \rightarrow J_j$  such that

$$v_{ij} = \sum_{m \in J_i} f_{m, \tau_{ij}(m)}$$

and  $\tau_{ii}(m) = m$  for  $m \in J_i$ . Let  $\psi$  be the conditional expectation of (4). Then  $\psi(v_{ij}) = v_{ij}$ . Hence

$$\sum_{m \in J_i} \psi(f_{m, \tau_{ij}(m)}) = \sum_{J_i} f_{m, \tau_{ij}(m)}. \tag{1}$$

For  $i = j$  we get

$$\sum_{m \in J_i} \psi(f_{m,m}) = \sum_{J_i} f_{m,m} (= v_{ii} = \varphi(e_{ii})).$$

But every  $\psi(f_{m,m})$  is a projection in  $\varphi(M_k)$  (since  $\psi$  is multiplicative on  $D_n$  and selfadjoint) and  $\varphi(e_{ii})$  is a minimal projection in  $\varphi(M_k)$ . Hence, for every  $i$  there is some  $m_i \in J_i$  such that

$$\psi(f_{m_i, m_i}) = \varphi(e_{ii})$$

and

$$\psi(f_{jj}) = 0 \quad \text{if } j \in J_i \text{ with } j \neq m_i.$$

Since  $\psi$  is multiplicative on  $T_n$ , we have

$$\psi(f_{pq}) = \psi(f_{pp})\psi(f_{pq})\psi(f_{qq})$$

for  $p \leq q$  (and since  $\psi$  is selfadjoint, for all  $p, q$ ). Write  $N = \{m_i: 1 \leq i \leq k\}$ . Then it follows that, if  $(p, q) \notin N \times N$ , we have  $\psi(f_{pq}) = 0$ . Using (1), we have  $\tau_{ij}(m_i) = m_j$  and

$$\psi(f_{pq}) = \begin{cases} 0 & \text{if } (p, q) \notin N \times N, \\ \varphi(e_{ij}) & \text{if } p = m_i, q = m_j. \end{cases}$$

Suppose  $p, q \in N$  and  $p < r < q$ . Then

$$0 \neq \psi(f_{pq}) = \psi(f_{pr}f_{rq}) = \psi(f_{pr})\psi(f_{rq}).$$

Hence  $r \in N$ . We conclude that  $N$  is a subinterval of  $\{1, \dots, n\}$  and write  $N = \{l + 1, \dots, l + k\}$  ( $0 \leq l < n$ ). It is clear that  $m_i = l + i$ . Now take  $(p, m) \in N \times N^c$ . Then  $p = m_i$  for some  $i$  and then  $\tau_{ij}(p) = m_j \neq m$  (as  $m \notin N$ ). It follows that  $(p, m)$  is not in  $\bigcup_{i,j} \text{supp } v_{ij}$ . Hence for all  $A \in M_k$ ,  $\varphi(A)_{p,m} = 0$ . On the other hand, if  $(p, m) \in N \times N$ , then  $p = l + i$ ,  $m = l + j$  and  $\tau_{ij}(p) = m$ , so that  $\varphi(e_{ij})_{p,m} = 1$  and, for  $A \in M_k$ ,  $\varphi(A)_{p,m} = A_{i,j} = A_{p-l,m-l}$ .

This proves that, for a canonical embedding, (4) implies (4''). For the other direction, suppose  $\varphi$  satisfies (4''). Then we can define  $g: M_n(\mathbb{C}) \rightarrow M_k(\mathbb{C})$  by  $g(A)_{ij} = A_{l+i,l+j}$  for  $1 \leq i, j \leq k$  (with  $l$  as in (4'')). Then  $g \circ \varphi = \text{id}_{M_k}$  and  $g|_T$  is multiplicative since

$$\begin{aligned} g(AB)_{ij} &= (AB)_{l+i,l+j} = \sum_{l+i \leq k \leq l+j} A_{l+i,k} B_{k,l+j} \\ &= \sum_{l+i \leq k \leq l+j} g(A)_{i,k-l} g(B)_{k-l,j} = (g(A)g(B))_{ij}. \end{aligned}$$

Lemma 3.1 completes the proof.

We now define a *structured* triangular subalgebra of a UHF algebra  $\mathcal{B}$  to be an algebra of the form

$$A = \varinjlim (T_{n_k}, \varphi_k)$$

where  $\varphi_k: T_{n_k} \rightarrow T_{n_{k+1}}$  can be extended to a structured embedding  $\tilde{\varphi}_k$  of  $M_{n_k}(\mathbb{C})$  into  $M_{n_{k+1}}(\mathbb{C})$  for every  $k$  and  $\mathcal{B} = \varinjlim (M_{n_k}, \tilde{\varphi}_k)$ .

**THEOREM 3.3.** *A subalgebra  $\mathcal{A}$  of a UHF algebra  $\mathcal{B}$  is structured if and only if we have subalgebras  $B_k \subseteq \mathcal{B}$  and  $A_k \subseteq \mathcal{A}$  and contractive conditional expectations  $\psi_k: \mathcal{B} \rightarrow B_k$  such that*

- (i)  $B_k \subseteq B_{k+1}$ ,  $\overline{\bigcup B_k} = \mathcal{B}$  and  $B_k$  is isomorphic to  $M_{n_k}(\mathbb{C})$ ;
- (ii)  $A_k \subseteq B_k \cap A_{k+1}$ ,  $\mathcal{A} = \overline{\bigcup A_k}$ , and the isomorphism of (i) maps  $A_k$  onto  $T_{n_k}$ ;
- (iii) every matrix unit of  $B_n$  is a sum of matrix units of  $B_{n+1}$ ; here the matrix units of  $\mathcal{B}_n$  are the standard matrix units induced by the isomorphism of (i);
- (iv)  $\psi_k$  is surjective onto  $B_k$ ,  $\psi_k \circ \psi_{k+1} = \psi_k$  and  $\psi_k|_{\mathcal{A}}$  is multiplicative.

*Proof.* Assume that  $\mathcal{A}$  and  $\mathcal{B}$  satisfy the conditions of the theorem. Using (i)–(iii) we can write

$$\begin{aligned} \mathcal{B} &= \varinjlim (M_{n_k}, \varphi_k), \\ \mathcal{A} &= \varinjlim (T_{n_k}, \varphi_k|_{T_{n_k}}), \end{aligned}$$

for some canonical embeddings  $\{\varphi_k\}$ . More precisely, we have isomorphisms  $\eta_k: B_k \rightarrow M_{n_k}(\mathbb{C})$  mapping  $A_k$  onto  $T_{n_k}$  such that  $\varphi_k = \eta_{k+1} \circ i_k \circ \eta_k^{-1}$  (where  $i_k: B_k \rightarrow B_{k+1}$  is the inclusion map) is a canonical embedding for every  $k$ . Fix  $k$

and define  $\psi: M_{n_{k+1}} \rightarrow M_{n_{k+1}}$  by  $\psi = \eta_{k+1} \circ \psi_k \circ \eta_{k+1}^{-1}$ . Then  $\psi$  is a contractive conditional expectation onto

$$\begin{aligned} \eta_{k+1}(\psi_k(B_{k+1})) &= \eta_{k+1}(\psi_k(\psi_{k+1}(B))) = \eta_{k+1} \circ \psi_k(B) = \eta_{k+1}(B_k) \\ &= \varphi_k(M_{n_k}) \subseteq M_{n_{k+1}}. \end{aligned}$$

Since  $\psi_k$  is multiplicative on  $\mathcal{A}$  and  $\eta_{k+1}^{-1}(T_{n_{k+1}}) = A_{k+1} \subseteq \mathcal{A}$ ,  $\psi$  is multiplicative on  $T_{n_{k+1}}$ . Hence  $\varphi_k$  is structured for every  $k$ . For the other direction, let  $B = \varinjlim (M_{n_k}, \varphi_k)$  and  $\mathcal{A} = \varinjlim (T_{n_k}, \varphi_k|_{T_{n_k}})$  with  $\varphi_k$  structured embeddings. From Lemma 3.1 we get maps  $g_k: M_{n_{k+1}} \rightarrow M_{n_k}$  satisfying  $g_k \circ \varphi_k = \text{id}_{M_{n_k}}$  and  $g_k|_{T_{n_{k+1}}}$  is multiplicative. Fix  $k$ . For  $l \geq k$  we define  $g_{l,k} = g_k \circ g_{k+1} \circ \dots \circ g_l: M_{n_{l+1}} \rightarrow M_{n_k}$ . If  $r > l \geq k$  then

$$g_{r,k} \circ \varphi_r \circ \dots \circ \varphi_{l+1} \circ \varphi_l = g_{l,k}. \tag{*}$$

Now, the maps  $\{\varphi_k\}$  give rise to embeddings  $\tilde{\varphi}_k: M_{n_k} \rightarrow \mathcal{B}$  and then (\*) shows that, if we set

$$g'_{l,k} = \tilde{\varphi}_k \circ g_{l,k} \circ \tilde{\varphi}_{l+1}^{-1}: \tilde{\varphi}_{l+1}(M_{n_{l+1}}) \rightarrow \tilde{\varphi}_k(M_{n_k}),$$

we get  $g_{r,k}^*|_{\tilde{\varphi}_{l+1}(M_{n_{l+1}})} = g'_{l,k}$ . Also  $\|g'_{r,k}\| \leq 1$  for all  $r \geq k$ . Hence, for a given  $k$ , the family  $\{g'_{r,k}\}_{r=k}^\infty$  defines a map

$$\psi_k: B \rightarrow \tilde{\varphi}_k(M_{n_k})$$

by  $\psi_k|_{\tilde{\varphi}_{r+1}(M_{n_{r+1}})} = g'_{r,k}$ . Write  $B_k = \tilde{\varphi}_k(M_{n_k})$  and  $A_k = \tilde{\varphi}_k(T_{n_k})$ . Then (i)–(iii) are satisfied and it is left to show that  $\psi_k$  is a conditional expectation onto  $B_k$  which is multiplicative on  $\mathcal{A}$ . To show that it is a conditional expectation onto  $B_k$  it suffices to show that  $\psi_k|_{B_k} = \text{id}$ . But this follows from the definition since

$$\psi_k|_{B_{k+1}} = g'_{k,k} = \tilde{\varphi}_k \circ g_{k,k} \circ \tilde{\varphi}_{k+1}^{-1} = \tilde{\varphi}_k \circ g_k \circ \tilde{\varphi}_{k+1}^{-1},$$

and for  $x = \tilde{\varphi}_k(a)$  and  $a \in M_{n_k}$ , we have  $\tilde{\varphi}_{k+1}(\varphi_k(a)) = \tilde{\varphi}_k(a) = x$ ; hence  $\varphi_k(a) = \tilde{\varphi}_{k+1}^{-1}(x)$  and

$$\psi_k(x) = \tilde{\varphi}_k(g_k \circ \varphi_k(a)) = \tilde{\varphi}_k(a) = x.$$

To show that  $\psi_k|_{\mathcal{A}}$  is multiplicative, it is enough to use the fact that  $g_k|_{T_{n_{k+1}}}$  is multiplicative for every  $k$ .

Let  $\mathcal{B}$  be a UHF algebra and  $D \subseteq B$  be a canonical masa. Let  $G$  be the equivalence relation associated with  $B$  on the space  $X$ , the maximal ideal space of  $D$ , as in § 2. Given a subalgebra  $D \subseteq \mathcal{A} \subseteq B$  let  $P \subseteq G$  be the open partial order supporting  $\mathcal{A}$ . Here we will deal with algebras  $\mathcal{A}$  that are triangular (that is,  $\mathcal{A} \cap \mathcal{A}^* = D$ ) and strongly maximal ( $\mathcal{A} + \mathcal{A}^*$  is dense in  $\mathcal{B}$ ) and then  $P$  becomes a total order. More precisely, for every  $u \in X$ ,  $P$  induces a total order on the equivalence class,  $[u]$ , of  $u$ . For a UHF algebra  $G$  is minimal; that is, for every  $u \in X$ ,  $[u]$  is dense in  $X$ . Excluding finite-dimensional UHF algebras we find that this implies that  $[u]$  is infinite and countable. If  $[u]$  has the property that, for every  $v \leq w$  in  $[u]$ , the interval  $\{z \in [u]: v \leq z \leq w\}$  is finite, then we say that  $[u]$  is *locally finite*. That means that  $[u]$  is order isomorphic to a subset of  $\mathbb{Z}$ .

PROPOSITION 3.4. *If  $\mathcal{A}$  is structured then there is some  $u \in X$  such that  $[u]$  is locally finite.*

*Proof.* We can write

$$\mathcal{B} = \varinjlim(M_{n_k}, \varphi_k) \quad \text{and} \quad \mathcal{A} \varinjlim(T_{n_k}, \varphi_k)$$

and assume that this presentation is structured (that is,  $\varphi_k$  is structured for all  $k$ ). Each  $\varphi_k$  has some  $l = l(k)$  such that (4'') holds (see Lemma 3.2). Write  $E_k = \sum_{i=1}^{n_k} e_{l(k)+i, l(k)+i}^{(k+1)} \in M_{n_{k+1}}$ . Then

- (1) for every  $e_{ij}^{(k)}, \hat{e}_{ij}^{(k)} \subseteq (\hat{E}_k \times \hat{E}_k) \cup (X \setminus \hat{E}_k) \times (X \setminus \hat{E}_k)$ ,
- (2) for every  $e_{ij}^{(k)}, \hat{e}_{ij}^{(k)} \cap (\hat{E}_k \times \hat{E}_k) = \hat{e}_{l(k)+i, l(k)+j}^{(k+1)}$ .

This follows from Lemma 3.2. In particular, it follows that for every  $k$  and  $1 \leq i \leq n_k$ ,

$$\hat{e}_{l(k)+i, l(k)+i}^{(k+1)} \subseteq \hat{e}_{i,i}^{(k)}.$$

Hence  $\hat{e}_{1,1}^{(1)} \supseteq \hat{e}_{l(1)+1, l(1)+1}^{(2)} \supseteq \hat{e}_{l(2)+l(1)+1, l(2)+l(1)+1}^{(3)} \supseteq \dots$  and, if we write  $m(k) = \sum_{j=1}^{k-1} l(j)$ , we can define

$$u = \bigcap_{k=1}^{\infty} \hat{e}_{m(k)+1, m(k)+1}^{(k)}.$$

From (2) above we can conclude that for every  $k$  and every  $1 \leq i, j \leq n_k$ ,

$$\hat{e}_{l(k)+i, l(k)+j}^{(k+1)} \subseteq \hat{e}_{ij}^{(k)}$$

and, thus, for every  $1 \leq i, j \leq n_1$ ,  $\bigcap_{k=1}^{\infty} \hat{e}_{m(k)+i, m(k)+j}^{(k)}$  is a point in  $G$ . Hence the points  $\bigcap_{k=1}^{\infty} \hat{e}_{m(k)+i, m(k)+i}^{(k)}$  are all equivalent in  $X$ . Write

$$C_1 = \left\{ \bigcap_{k=1}^{\infty} \hat{e}_{m(k)+i, m(k)+i}^{(k)} : 1 \leq i \leq n_1 \right\} \subseteq [u].$$

Similarly we can write, for  $p \geq 1$ ,

$$C_p = \left\{ \bigcap_{k=p}^{\infty} \hat{e}_{m(k)-m(p)+i, m(k)-m(p)+i}^{(k)} : 1 \leq i \leq n_p \right\} \subseteq [u].$$

Note that we have  $u \in C_p$ , by taking  $i = m(p) + 1$ . In fact, for  $p \leq q$ ,  $C_p \subseteq C_q$ . To see this take  $w = \bigcap_{k=p}^{\infty} \hat{e}_{m(k)-m(p)+i, m(k)-m(p)+i}^{(k)}$  in  $C_p$ . Write it as  $\bigcap_{k=q}^{\infty} \hat{e}_{m(k)-m(q)+j, m(k)-m(q)+j}^{(k)}$  by setting  $j = m(q) - m(p) + i$ . Thus  $\bigcup_{p=1}^{\infty} C_p \subseteq [u]$ . We shall show that in fact  $[u] = \bigcup_{p=1}^{\infty} C_p$ . This will complete the proof because it is easy to see that, for  $p \leq q$ ,  $C_p$  forms an interval in  $C_q$ . Indeed, as was shown above, the points in  $C_p$  correspond to values of  $j$  ranging from  $m(q) - m(p) + i$  to  $m(q) - m(p) + n_p$  and it is clear that

$$j \mapsto \bigcap_{k=q}^{\infty} \hat{e}_{m(k)-m(q)+j, m(k)-m(q)+j}^{(k)}$$

is an order isomorphism of  $\{1, \dots, n_q\}$  onto  $C_q$ .

So to complete the proof we now take some  $w \in [u]$ . For some  $p$ , there are  $i, j$  with  $1 \leq i, j \leq n_p$  such that  $(u, w) \in \hat{e}_{i,j}^{(p)}$ . In particular,  $u \in \hat{e}_{ii}^{(p)}$  and  $w \in \hat{e}_{jj}^{(p)}$ . But

$u \in \hat{e}_{m(p)+1, m(p)+1}^{(p)}$ . Hence  $i = m(p) + 1$ . Also  $u \in \hat{E}_p$ ; hence, by (1) above,  $w \in \hat{E}_p$ . It then follows from (2) that

$$\begin{aligned} w \in \hat{e}_{jj}^{(p)} \cap (\hat{E}_p \times \hat{E}_p) &= \hat{e}_{l(p)+j, l(p)+j}^{(p+1)} \\ &= \hat{e}_{m(p+1)-m(p)+j, m(p+1)-m(p)+j}^{(p+1)}. \end{aligned}$$

Using (2) again and the fact that  $w \in \hat{E}_{p+1}$ , because  $u \in \hat{E}_{p+1}$ , we get

$$w \in \hat{e}_{m(p+2)-m(p)+j, m(p+2)-m(p)+j}^{(p+2)}$$

Continuing this way we find that  $w \in C_p \subseteq [u]$ .

The proof of the proposition is constructive and we shall refer to the equivalence class obtained in this way from a given structured presentation with a choice of  $\{l(k)\}$  as the structured equivalence class associated with a given presentation and a choice of  $\{l(k)\}$ . If, for example, each  $\varphi_k$  is the standard embedding (cf. [9]) then we can choose  $l(k)$  in many ways and different choices might give rise to different equivalence classes. In fact, in this case every equivalence class will be associated with some choice of  $\{l(k)\}$ . On the other hand, if each  $\varphi_k$  is the refinement embedding (cf. [9]) then it is not structured and, in fact, we do not have any locally finite equivalence class.

However, the converse of Proposition 3.4 is false. As we show in the example below, the existence of a locally finite equivalence class does not imply the structured property.

EXAMPLE. We now present an example of a strongly maximal triangular AF algebra that has a locally finite equivalence class but the algebra is not structured. The algebra is a subalgebra of  $M(2 \cdot 3^\infty)$  and is defined as

$$\mathcal{C} = \varinjlim (T_{2 \cdot 3^k}, \tau_k)$$

where  $\tau_k: M_{2 \cdot 3^k} \rightarrow M_{2 \cdot 3^{k+1}}$  are defined below. First define  $\tau_1: M_6 \rightarrow M_{18}$  as in Fig. 1. This describes  $\tau_1$  on  $T_6$  and it can be extended to  $M_6$  in the obvious way.

Note that, if we consider the matrices as  $2 \times 2$  block matrices as shown in the diagram, then the blocks are ‘invariant’ and, on the (1, 1) block, the embedding is the standard embedding. On the (2, 2) block the embedding is a mixture of the standard and refinement embeddings. It can be described as being the standard embedding on the 4th and 5th diagonal matrix units and ‘stretching’ the 6th one. For  $\tau_2$ , the map on the (1, 1) block will be the standard embedding, and the map on the (2, 2) block will be the standard embedding on the 10th–17th diagonal matrix units and it will ‘stretch’ the 18th one. On the (1, 2) block,  $\tau_2$  will be defined in a way similar to the definition of  $\tau_1$  on the (1, 2) block. The embeddings  $\tau_k$  for  $k \geq 3$ , are defined similarly. The formal definition of  $\tau_k$  is as follows.

(1) For the (1, 1) block: with  $1 \leq i, j \leq 3^k$ ,

$$\tau_k(e_{ij}^{(k)}) = e_{i,j}^{(k)} + e_{3^k+i, 3^k+j}^{(k+1)} + e_{2 \cdot 3^k+i, 2 \cdot 3^k+j}^{(k+1)}.$$

This is the standard embedding of  $M_{3^k}$  into  $M_{3^{k+1}}$ .

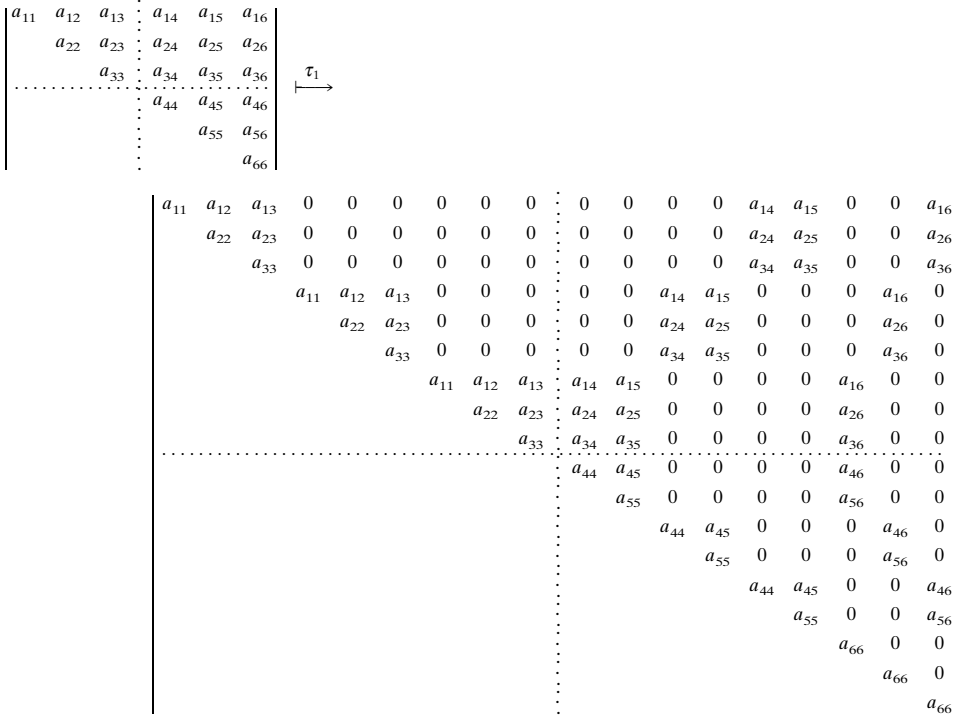


FIG. 1

(2) For the (1, 2) block: with  $1 \leq i, j \leq 3^k$ ,

$$\tau_k(e_{i,j+3^k}^{(k)}) = \begin{cases} e_{i,j+2(3^k-1)+3^{k+1}} + e_{i+3^k,j+(3^k-1)+3^{k+1}} + e_{i+2 \cdot 3^k,j+3^{k+1}} & \text{if } j \neq 3^k, \\ e_{i,2 \cdot 3^{k+1}} + e_{i+3^k,2 \cdot 3^{k+1}-1} + e_{1+2 \cdot 3^k,2 \cdot 3^{k+1}-2} & \text{if } j = 3^k. \end{cases}$$

(3) For the (2, 2) block: with  $1 \leq i, j \leq 3^k$ ,

$$\tau_k(e_{i+3^k,j+3^k}^{(k)}) = \begin{cases} e_{i+2(3^k-1)+3^{k+1},j+2(3^k-1)+3^{k+1}} + e_{i+(3^k-1)+3^{k+1},j+(3^k-1)+3^{k+1}} + e_{i+3^{k+1},j+3^{k+1}} & \text{if } i, j \neq 3^k, \\ e_{i+2(3^k-1)+3^{k+1},2 \cdot 3^{k+1}} + e_{i+(3^k-1)+3^{k+1},2 \cdot 3^{k+1}-1} + e_{i+3^{k+1},2 \cdot 3^{k+1}-2} & \text{if } j = 3^k > i, \\ e_{2 \cdot 3^{k+1},j+2(3^k-1)+3^{k+1}} + e_{2 \cdot 3^{k+1}-1,j+(3^k-1)+3^{k+1}} + e_{2 \cdot 3^{k+1}-2,j+3^{k+1}} & \text{if } i = 3^k > j, \\ e_{2 \cdot 3^{k+1},2 \cdot 3^{k+1}} + e_{2 \cdot 3^{k+1}-1,2 \cdot 3^{k+1}-1} + e_{2 \cdot 3^{k+1}-2,2 \cdot 3^{k+1}-2} & \text{if } i = j = 3^k. \end{cases}$$

(4) The definition for the (2, 1) block can be derived from the definition for the (1, 2) block.

Since the blocks are invariant, the limit algebra,  $\mathcal{C}$  has the form

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ 0 & \mathcal{C}_{22} \end{pmatrix},$$

where  $\mathcal{C}_{11}$  is isomorphic to the standard embedding algebra in  $M(3^\infty)$  and  $\mathcal{C}_{22}$  is isomorphic to the subalgebra of  $M(3^\infty)$  defined by  $\varinjlim(T_n, \tilde{\tau}_n)$  where  $\tilde{\tau}_n$  is  $\tau_n$ , restricted to the (2, 2) block. Note also that

$$\mathcal{C}_{11} = F\mathcal{C}F \quad \text{and} \quad \mathcal{C}_{22} = (I - F)\mathcal{C}(I - F),$$

where  $F = e_{11}^{(1)} + e_{22}^{(1)} + e_{33}^{(1)}$  is a projection in  $(\text{Lat } \mathcal{C}) \cap \mathcal{C}$ . We shall show that  $\mathcal{C}$  is not structured, although it has a locally finite equivalence class. We suspect that this might be true also for  $\mathcal{C}_{22}$  but are unable to show it.

We shall now write  $G$  for the groupoid associated with  $\varinjlim(M_{2 \cdot 3^k}, \tau_k)$ ,  $P$  for the partial order supporting  $\mathcal{C}$  and  $X$  for the unit space of  $G$  (equal to the spectrum of  $\varinjlim(D_{2 \cdot 3^k}, \tau_k)$ ). For every matrix unit  $e_{ij}^{(k)}$  we shall write  $\hat{e}_{ij}^{(k)}$  for the corresponding closed and open subset of  $G$  (the support set of  $e_{ij}^{(k)}$ ). If  $i = j$ , then  $\hat{e}_{ii}^{(k)} \subseteq X$ . Also write  $X_1$  for the subset of  $X$  associated with  $F$ , that is,  $X_1 = \bigcup_{i=1}^3 \hat{e}_{i,i}^{(1)}$ . Let  $X_2$  be  $X \setminus X_1$ . Note that  $G \cap (X_1 \times X_1)$  and  $P \cap (X_1 \times X_1)$  are the groupoid and the partial order of the standard embedding algebra, and  $G \cap (X_2 \times X_2)$  and  $P \cap (X_2 \times X_2)$  are the ones for  $\mathcal{C}_{22}$ .

Now fix

$$u_1 = \bigcap_k \hat{e}_{3^k, 3^k}^{(k)} \in X_1 \quad \text{and} \quad v_1 = \bigcap_k \hat{e}_{3^{k+1}, 3^{k+1}}^{(k)} \in X_2.$$

Note that for every  $k$ ,  $e_{3^{k+1}, 3^{k+1}+1}^{(k+1)} \leq \tau_k(e_{3^k, 3^k+1}^{(k)})$ . Hence  $\hat{e}_{3^{k+1}, 3^{k+1}+1}^{(k+1)} \subseteq \hat{e}_{3^k, 3^k+1}^{(k)}$  and  $\bigcap_k \hat{e}_{3^k, 3^k+1}^{(k)}$  defines a point in  $P$  ( $\subseteq G$ ). This point is clearly  $(u_1, v_1)$  and this shows that  $[u_1] = [v_1]$ .

It is known that  $[u_1] \cap X_1$  is order isomorphic to  $-\mathbb{N}$  and  $n_1$  is the maximal element there. This follows from the analysis of equivalence classes in the standard embedding algebra. In fact, it is known that this is the only equivalence class in  $X_1$  that has a maximal element. Note also that for every  $u \in X$ ,

$$[u] = ([u] \cap X_1) \cup ([u] \cap X_2)$$

and for every  $w_1 \in [u] \cap X_1$  and  $w_2 \in [u] \cap X_2$ , we have  $(w_1, w_2) \in P$  since  $(X_2 \times X_1) \cap P = \emptyset$ . Thus, if  $[u]$  is locally finite,  $[u] \cap X_1$  should have a maximal element; hence  $[u] = [u_1]$ . We conclude that, if  $\mathcal{C}$  has a locally finite equivalence class, then it is  $[u_1]$  ( $= [v_1]$ ). We shall now show that  $[v_1] \cap X_2$  is order isomorphic to  $\mathbb{N}$  and  $v_1$  is the minimal element there. For this, fix  $v \in [v_1] \cap X_2$ . Then  $(v_1, v) \in (X_2 \times X_2) \cap G$  and it can be written as

$$\bigcap_{k \geq m} \hat{e}_{1+3^k, j_k}^{(k)}$$

for some  $j_k \geq 3^k$  (as  $v_1 = \bigcap \hat{e}_{1+3^k, 1+3^k}^{(k)}$ ) where  $\hat{e}_{1+3^{k+1}, j_{k+1}}^{(k+1)} \subseteq \hat{e}_{1+3^k, j_k}^{(k)}$ ; that is,

$$e_{1+3^{k+1}, j_{k+1}}^{(k+1)} \leq \tau_k(e_{1+3^k, j_k}^{(k)})$$

for all  $k \geq m$ .

Examining the definition of  $\tau_k$ , on the  $(2, 2)$  block, one finds that, for  $k = m$ , we have either  $j_m = j + 3^m$  (and then  $j_k = j + 3^k$ , for all  $k \geq m$ ) for some  $j$  or  $j_m = 3^m$  and  $j_{m+1} = 2 \cdot 3^{m+1} - 2$ . In the latter case we get  $j_{m+2} = (3^{m+1} - 2) + 3^{m+2}$  and  $j_k = (3^{m+1} - 2)3^k$  for all  $k \geq m + 1$ .

Replacing  $m$  by  $m + 1$  we see that, in either case,

$$v = \bigcap_{k \geq m} \hat{e}_{j+3^k, j+3^k}^{(k)}$$

for some  $j$  (in the second case  $j = 3^{m+1} - 2$ ). We shall write  $v_j$  for this element and it is now clear that

$$X_2 \cap [v_1] = \{v_j : j \geq 1\}$$

and  $j \mapsto v_j$  is an order isomorphism from  $\mathbb{N}$  onto  $X_2 \cap [v_1]$ . Since

$$\begin{aligned} [v_1] &= (X_1 \cap [v_1]) \cup (X_2 \cap [v_1]) \\ &= (X_1 \cap [u_1]) \cup (X_2 \cap [v_1]) \end{aligned}$$

and since  $P \cap (X_2 \times X_1) = \emptyset$ , we see that  $[v_1]$  is order isomorphic to  $\mathbb{Z}$ . In particular, it is locally finite.

It is now left to show that  $[v_1]$  is not a structured equivalence class. This will complete the proof that  $\mathcal{C}$  is not a structured algebra. Suppose, by negation, that  $\mathcal{C}$  is structured with a structured class  $[v_1]$ . First note that for a given structured presentation of  $\mathcal{C}$ , say  $\varinjlim(T_{m_k}, l_k)$  with a structured class  $[v_1]$ , there will be some  $K_0$  such that, for all  $k \geq K_0$ ,  $F$  is a sum of diagonal matrix units in  $T_{m_k}$ . That will enable us to write the matrix algebras in a block form. Considering

$$\varinjlim(F^\perp T_{m_k} F^\perp, l_k | F^\perp T_{m_k} F^\perp),$$

we get a structured presentation for  $\mathcal{C}_{22}$ . It is also clear that, as  $[v_1]$  is the structured class for  $\mathcal{C}$ ,  $[v_1] \cap X_2$  is a structured class for  $\mathcal{C}_{22}$ . It is left, therefore, to prove the following lemma.

LEMMA 3.5. *The algebra  $\mathcal{C}_{22}$  does not have a structured presentation with structured class  $\{v_j: j \geq 1\}$ .*

*Proof.* Assume, by negation, that there is such a presentation, namely,

$$\mathcal{C}_{22} = \varinjlim(T_{3^{n_k}}, \sigma_k).$$

When we refer to this presentation we shall write  $f_{ij}^{(k)}$  for the matrix units. We also have the presentation

$$\mathcal{C}_{22} = \varinjlim(T_{3^k}, \tilde{\tau}_k)$$

where  $\tilde{\tau}_k$  is the restriction of  $\tau_k$ , defined above, to the  $(2, 2)$  block. For example, if  $i, j \neq 3^k$ ,

$$\tilde{\tau}_k(e_{i,j}^{(k)}) = e_{i+2(3^k-1), j+2(3^k-1)}^{(k+1)} + e_{i+(3^k-1), j+(3^k-1)}^{(k+1)} + e_{i,j}^{(k+1)}. \tag{1}$$

The definition of  $\tilde{\tau}_k$  for the other cases is a similar modification of the definition of  $\tau_k$  (that is, ‘drop “ $+3^{k+1}$ ”’). With this modification we now have

$$v_j = \bigcap_{k \geq m_j} \hat{e}_{j,j}^{(k)} \quad \text{for } j \geq 1, \tag{2}$$

where  $m_j$  is the smallest integer with  $1 \leq j < 3^{m_j}$ . Note that given  $m$  and  $j$  with  $1 \leq j < 3^m$ , it is clear that  $\bigcap_{k \geq m} \hat{e}_{j,j}^{(k)}$  is well defined and is the immediate successor of  $\bigcap_{k \geq m} \hat{e}_{j-1, j-1}^{(k)}$ ; we thus get  $v_1, \dots, v_{3^m-1}$ .

We now assume that  $\{v_j\}$  is the structured class associated with the structured presentation. Using the construction of the structured class and the fact that here it has a minimal element  $(v_1)$ , we see that there is some  $k_0$  such that for all  $k \geq k_0$ , and every  $1 \leq i \leq 3^{n_k}$ ,

$$v_i \in \hat{f}_{i,i}^{(k)} \tag{3}$$

In fact, by starting the structured presentation from  $M_{3^{n_{k_0}}}$  we can assume that for every  $k$ ,

$$\sigma_k(f_{ii}^{(k)}) \geq f_{ii}^{(k+1)} \quad \text{for } 1 \leq i \leq 3^{n_k}.$$

In particular  $f_{11}^{(k)} \leq \sigma_k \circ \dots \circ \sigma_1(f_{11}^{(1)})$  and

$$\hat{f}_{11}^{(k)} \leq \hat{f}_{11}^{(1)} \quad \text{for } k \geq 1. \tag{4}$$

Note that  $f_{11}^{(1)}$  is a projection in the diagonal of  $\mathcal{C}_{22}$ ; hence it is a diagonal projection in  $T_{3^n}$  for some large enough  $n$  (as  $e_{22} = \varinjlim(T_{3^n}, \tilde{\tau}_n)$ ). Therefore it is the sum of some projections from the set  $\{e_{i,i}^{(n)}: 1 \leq i \leq 3^n\}$ . Since  $v_1 \in \hat{e}_{1,1}^{(n)} \cap \hat{f}_{1,1}^{(1)}$ , we see that, for  $n$  large enough, say  $n \geq N$ ,

$$\hat{e}_{1,1}^{(n)} \subseteq \hat{f}_{1,1}^{(1)}. \tag{5}$$

Note that  $v_{3^n} = \bigcap_{m \geq n+1} e_{3^m, 3^m}^{(m)}$  (recall (2)) and

$$\hat{e}_{3^n, 3^n}^{(n+1)} \subseteq \hat{e}_{1,1}^{(n)}$$

as  $\tilde{\tau}_n(e_{1,1}^{(n)}) \geq e_{3^n, 3^n}^{(n+1)}$  (see (1) with  $i = j = 1$ ). For  $n \geq N$  we see that  $v_{3^n} \in \hat{e}_{1,1}^{(n)} \subseteq f_{1,1}^{(1)}$  (see (5)). On the other hand,  $v_{3^n} \in \hat{f}_{3^n, 3^n}^{(k)}$  whenever  $n \leq n_k$ . (See (3).)

Now fix  $l$  such that  $n_l \geq N$ . Then, for  $n = n_l$  we have

$$v_{3^n} \in \hat{f}_{11}^{(1)} \cap \hat{f}_{3^n, 3^n}^{(l)} \quad (n = n_l). \tag{6}$$

But, since  $\{\sigma_k\}$  preserve the upper triangular matrices, it follows that

$$\hat{f}_{3^{nk+1}, 3^{nk+1}}^{(k+1)} \leq \sigma_k(f_{3^{nk}, 3^{nk}}^{(k)}) \quad \text{for all } k.$$

Hence  $\hat{f}_{3^{nk+1}, 3^{nk+1}}^{(k+1)} \subseteq \hat{f}_{3^{nk}, 3^{nk}}^{(k)} \subseteq \dots \subseteq \hat{f}_{3^{n_1}, 3^{n_1}}^{(1)}$  and for  $n = n_l$ ,  $\hat{f}_{3^n, 3^n}^{(l)} \subseteq \hat{f}_{3^{n_1}, 3^{n_1}}^{(1)}$ , which contradicts (6). The contradiction completes the proof of the lemma.

#### 4. The inverse limit associated with a structured presentation

Let  $\mathcal{A}$  be a structured algebra and let

$$\mathcal{A} = \varinjlim(T_{n_k}, \varphi_k)$$

be a given structured presentation. This means that we have contractive maps  $g_k: M_{n_{k+1}} \rightarrow M_{n_k}$  such that  $g_k|_{T_{n_{k+1}}}$  is multiplicative,  $g_k \circ \varphi_k = \text{id}_{M_{n_k}}$  and  $g_k \circ g_{k+1} = g_k$  ( $g_k$  are the back maps). We can write

$$T_{n_1} \xleftarrow{g_1} T_{n_2} \xleftarrow{g_2} T_{n_3} \xleftarrow{\dots} \dots$$

With such a diagram we can associate an inverse limit algebra as indicated in the following proposition.

**PROPOSITION 4.1.** *Let  $(T_{n_k}, g_k)$  be as above. Define  $C = \{\bigoplus A_k \in \sum_{k=1}^\infty \bigoplus T_{n_k}: g_k(A_{k+1}) = A_k\}$ . With the norm and the algebraic structure inherited from  $\sum \bigoplus M_{n_k}$ ,  $C$  is a Banach algebra. Moreover, it has the following property. Given a Banach algebra  $E$  and homomorphisms  $\theta_k: E \rightarrow T_{n_k}$  satisfying  $\|\theta_k\| \leq 1$  and  $g_k \circ \theta_{k+1} = \theta_k$ , there is a contractive homomorphism  $\theta: E \rightarrow C$  such that  $\theta(x) = \sum \bigoplus \theta_k(x)$ .*

*Proof.* The proof that  $C$  is a Banach algebra with the norm

$$\|\bigoplus A_k\| = \sup \|A_k\|$$

and the product

$$(\bigoplus A_k)(\bigoplus B_k) = \bigoplus A_k B_k$$

is straightforward and uses the fact that  $g_k$  is a contractive homomorphism on  $T_{n_{k+1}}$ .

Now assume that  $E$  and  $\{\theta_k\}$  are as in the statement of the proposition and define  $\theta: E \rightarrow C$  by

$$\theta(x) = \sum \bigoplus \theta_k(x).$$

Then  $\theta$  is clearly multiplicative since each  $\theta_k$  is, and contractive since  $\|\theta(x)\| = \sup_k \|\theta_k(x)\| \leq \|x\|$ .

Note that if we define  $j_k: C \rightarrow T_{n_k}$  by  $j_k(\bigoplus A_i) = A_k$  then  $j_k$  is a contractive homomorphism and  $g_k \circ j_{k+1} = j_k$ . In fact  $(C, \{j_k\})$  has the following uniqueness property.

LEMMA 4.2. *Let  $C'$  be a Banach algebra and  $\{j'_k\}$  be contractive homomorphisms  $j'_k: C' \rightarrow T_{n_k}$  such that  $g_k \circ j'_{k+1} = j'_k$  for  $k \geq 1$  and  $\bigcap \ker j'_k = \{0\}$ . Assume that  $(C', \{j'_k\})$  has the property indicated in Proposition 4.1. That is, for every Banach algebra  $E$  and contractive homomorphisms  $\theta_k: E \rightarrow T_{n_k}$  with  $g_k \circ \theta_{k+1} = \theta_k$  there is a contractive homomorphism  $\theta: E \rightarrow C'$  with  $j'_k \circ \theta = \theta_k$ . Then there is an isometric isomorphism  $\psi: C' \rightarrow C$  such that  $j_k \circ \psi = j'_k$  for  $k \geq 1$ .*

*Proof.* If  $C'$  is as above, we use the properties of  $C$  and  $C'$  to get contractive homomorphisms  $\psi: C' \rightarrow C$  and  $\psi': C \rightarrow C'$  satisfying  $j_k \circ \psi = j'_k$  and  $j'_k \circ \psi' = j_k$ . Hence  $j_k \circ \psi \circ \psi' = j_k$  and  $j'_k \circ \psi' \circ \psi = j'_k$  and, since  $\bigcap \ker j_j = \bigcap \ker j'_k = \{0\}$ ,  $\psi \circ \psi' = \text{id}_C$  and  $\psi' \circ \psi = \text{id}_{C'}$ . Since  $\|\psi\| \leq 1$  and  $\|\psi'\| \leq 1$ ,  $\psi$  is an isometric isomorphism.

Now, let  $\mathcal{A}$  be a structured algebra with a structured presentation  $\mathcal{A} = \varinjlim (T_{n_k}, \varphi_k)$  and let  $C$  be the inverse limit associated with it and with a choice of back maps. Write  $\tilde{g}_k$  for the contractive homomorphism of  $\mathcal{A}$  onto  $T_{n_k}$  as in the proof of Theorem 3.3 ( $\tilde{g}_k$  is  $\tilde{\varphi}_k^{-1} \circ \psi_k$  in the notation there). Then we know that  $\tilde{g}_k \circ g_{k+1} = \tilde{g}_k$ ; hence, from Proposition 4.1 we get a contractive homomorphism  $\theta: \mathcal{A} \rightarrow C$  satisfying

$$\theta(x) = \sum \bigoplus \tilde{g}_k(x).$$

Since  $\bigcap \ker \tilde{g}_k = \{0\}$ ,  $\theta$  is injective. In fact,  $\|x\| = \sup \|\tilde{g}_k(x)\|$ ; hence  $\theta$  is an isometric embedding of  $\mathcal{A}$  into  $C$ .

Note that, given  $u \in X$ , with equivalence class  $[u]$ , there is a representation  $\pi_u$  of  $B$  defined as follows. Let  $H = l^2([u])$  and define  $\pi_u(f) \in B(H)$  for  $f \in \mathcal{B}$  by

$$(\pi_u(f)g)(v) = \sum_{w \in [u]} f(v, w)g(w).$$

Now let  $\mathcal{A} \subseteq B$  be a structured UHF algebra and fix a structured presentation  $\mathcal{A} = \varinjlim (T_{n_k}, \varphi_k)$  and a sequence  $\{l(k)\}$ , where  $l(k)$  is associated with  $\varphi_k$  as in Lemma 3.2. Then we get a unique equivalence class  $[u]$  associated with this data as in Proposition 3.4. In fact  $[u]$  has been constructed as an increasing union  $[u] = \bigcup_{k=1}^\infty C_k$  of subintervals. With each  $C_k$  we now associate a projection  $F_k \in B(H) = B(l^2([u]))$  which is the orthogonal projection onto  $\text{span}\{e_v: v \in C_k\}$ , where  $\{e_v: v \in [u]\}$  is the standard orthonormal basis in  $l^2([u])$ . Similarly, for

every subset  $\Omega \subseteq [u]$  we can define  $F(\Omega)$  to be the projection onto  $\overline{\text{span}\{e_v : v \in \Omega\}}$ . Let

$$\mathcal{N} = \{F(\Omega) : \Omega \subseteq [u] \text{ is decreasing}\}.$$

Then  $\mathcal{N}$  is a nest of projections and  $\text{Alg } \mathcal{N} (= \{T \in B(H) : (1 - N)TN = 0 \text{ for all } N \in \mathcal{N}\})$  is the algebra of operators whose matrix with respect to the basis  $\{e_v\}$  is upper triangular.

LEMMA 4.3. *Let  $C$  be the inverse limit defined above. Then  $C$  is isometrically isomorphic to  $\text{Alg } \mathcal{N}$ .*

*Proof.* Recall that

$$C = \left\{ \bigoplus A_k \in \sum_{k=1}^{\infty} \bigoplus T_{n_k} : g_k(A_{k+1}) = A_k \right\} \quad \text{and} \quad \|\bigoplus A_k\| = \sup \|A_k\|.$$

Let  $\rho : \text{Alg } \mathcal{N} \rightarrow C$  be defined by

$$\rho(A) = \bigoplus F_k A F_k.$$

Here we use the projection  $F_k$  defined above and note that we can identify  $F_k A F_k$  with  $T_{n_k}$  in the obvious way since  $C_k \subseteq [u]$  is an interval of length  $n_k$ . Using this identification, we see that the map  $g_k$  (the back map for  $\varphi_k$ ) is simply  $F_{k+1} A F_{k+1} \rightarrow F_k F_{k+1} A F_{k+1} F_k = F_k A F_k$ . Hence  $\rho(A) \in C$ . Conversely, take some  $\bigoplus A_k \in C$ . For each  $k$  we associate an operator  $\tilde{A}_k \in F_k \text{Alg } \mathcal{N} F_k$  in the obvious way. That is, its matrix is  $\langle \tilde{A}_k e_v, e_w \rangle = (A_k)_{w,v}$  if  $v, w \in C_k$  and 0 otherwise. Note that by identifying  $C_k$  with  $\{1, \dots, n_k\}$  in an order-preserving way, we see that  $(A_k)_{w,v}$  makes sense. Since  $g_k(A_{k+1}) = A_k$ , we have  $F_k \tilde{A}_{k+1} F_k = \tilde{A}_k$ . Clearly,  $\|\tilde{A}_k\| \leq \sup_j \|A_j\|$  and, thus, there is some subnet  $\tilde{A}_{k'}$  that converges  $\sigma$ -weakly to some  $A \in B(H)$ . Clearly,  $A \in \text{Alg } \mathcal{N}$  and  $F_j A F_j = \lim_{k'} F_j \tilde{A}_{k'} F_j$ . But for  $k' \geq j$ ,  $F_j \tilde{A}_{k'} F_j = A_j$ , and hence  $F_j A F_j = A_j$ . It follows that  $\rho(A) = \bigoplus A_k$  and  $\rho$  is surjective onto  $C$ . Clearly  $\|A\| = \sup \|F_k A F_k\| = \sup \|A_k\| = \|\bigoplus A_k\|$  so that  $\rho$  is an isometry from  $\text{Alg } \mathcal{N}$  onto  $C$ . Since each  $C_k$  is an interval of  $[u]$ , the maps  $A \rightarrow F_k A F_k$  are homomorphisms on  $\text{Alg } \mathcal{N}$  and, thus,  $\rho$  is a homomorphism.

COROLLARY 4.4.  $\rho^{-1} \circ \theta = \pi_u$ .

*Proof.* We have  $\rho^{-1}(\theta(f)) = \rho^{-1}(\bigoplus \tilde{g}_k(f))$ . For  $v, w \in C_k$  the matrix coefficients of  $\rho^{-1}(\theta(f))$  are

$$(\rho^{-1}(\theta(f)))_{v,w} = (\tilde{g}_k(f))_{v,w} = f(v, w).$$

Since  $\bigcup C_k = [u]$ , we have  $\rho^{-1}(\theta(f)) = \pi_u(f)$ .

Note that  $\pi_u$  is a well-defined representation for every  $u \in X$ . What we saw in the above corollary is that when  $[u]$  is structured,  $\pi_u$  can be factored through a subalgebra of  $\sum \bigoplus T_{n_k}$ .

5. *Residually finite-dimensional triangular algebras*

A Banach algebra is called *residually finite-dimensional* (or simply RFD) if for every  $x \neq y$  in the algebra there is a finite-dimensional contractive representation  $\rho$  such that  $\rho(x) \neq \rho(y)$ . For C\*-algebras this property has been studied in [1, 3, 6] and elsewhere. If  $\mathcal{A} = \varinjlim (T_{n_k}, \varphi_k)$  is a structured algebra with back maps  $g_k$  and associated maps  $\tilde{g}_k: \mathcal{A} \rightarrow T_{n_k}$  as in the discussion following Lemma 4.2, then  $\{\tilde{g}_k\}$  are finite-dimensional contractive representations that separate points in  $\mathcal{A}$ . Hence  $\mathcal{A}$  is RFD.

In fact it suffices to assume that the triangular UHF algebra has a locally finite equivalence class in order to conclude that the algebra is RFD. For this simply write this equivalence class, say  $[u]$ , as a union of increasing intervals,  $[u] = \bigcup_{k=1}^\infty L_k$ , and consider the representations  $\pi_k$  defined by compressing  $\pi_u$  (defined in § 4) into  $\text{span}\{e_v: v \in L_k\}$ .

We now introduce a necessary and sufficient condition, in terms of the supporting total order  $P \subseteq G$ , for a triangular AF algebra to be RFD.

In order to discuss contractive representations of a strongly maximal TAF algebra  $\mathcal{A}$  note first that each such representation  $\rho$  is completely contractive [4, Theorem 5.4]; hence it is a compression of some C\*-representation of the C\*-algebra  $\mathcal{B}$ , generated by  $\mathcal{A}$ , to a semi-invariant subspace.

From the work of Renault [10, Theorem 1.21] we know that a \*-representation  $\pi$  of  $\mathcal{B}$  is the integrated form of a representation  $(\mu, U, K)$  of the equivalence relation  $G$  associated with the AF algebra  $\mathcal{B}$ . Here  $K = \{K(u)\}_{u \in X}$  is a measurable field of Hilbert spaces,  $\mu$  is a quasi-invariant measure on  $X$  (here quasi-invariant means that if we let  $\theta(u, v) = (v, u)$  then  $\mu \sim \mu \circ \theta$ ) and  $U(u, v)$  is an isometry from  $K(v)$  to  $K(u)$  (for  $(u, v) \in G$ ) satisfying  $U(u, u) = I$  and  $U(u, v)U(v, w) = U(u, w)$  almost everywhere. To say that  $\pi$  is the integrated form of  $(\mu, U, K)$  means that, up to unitary equivalence, the space of  $\pi$  is  $K = \int_X^\oplus K(u) d\mu(u)$  and  $\pi(f)$  is defined on  $K$  by the formula

$$\langle \pi(f)\xi, \eta \rangle = \int f(u, v) \langle U(u, v)\xi(v), \eta(u) \rangle \Delta^{-1/2}(u, v) dv(u, v),$$

where  $\Delta = d\mu / (d\mu \circ \theta)$  and  $\nu$  is the measure on  $G$  associated with  $\mu$ ; that is,  $\nu = \int_X \lambda^u d\mu(u)$  where  $\lambda^u$  is the counting measure on  $\{(v, u): v \in [u]\}$ .

If  $\rho$  is a contractive representation of  $\mathcal{A}$  then, as mentioned above, there is some C\*-representation  $\pi$  associated with  $(\mu, U, K)$  as above and a subspace  $H \subseteq K = \int_X^\oplus K(u) d\mu(u)$  that is semi-invariant for  $\pi(\mathcal{A})$  (that is, is the difference of two invariant subspaces) such that

$$\rho(a) = P_H \pi(a)|_H,$$

where  $P_H$  is the projection onto  $H$ . Since  $H$  is semi-invariant for  $\pi(D)$  (where  $D = \mathcal{A} \cap \mathcal{A}^*$ ), it is invariant for  $\pi(D)$ . As  $\pi(D)$  is the algebra of all diagonal operators with respect to the decomposition  $K = \int_X^\oplus K(u) d\mu(u)$ , we can decompose  $H = \int_X^\oplus H(u) d\mu(u)$  where  $H(u) \subseteq K(u)$  for all  $u \in X$ .

Now assume that  $\rho$  is finite-dimensional, and hence  $\dim H < \infty$ . Then  $H = \int_X^\oplus H(u) d\mu(u)$  becomes

$$H = \sum_{i=1}^n H(w_i)$$

and  $\dim H(w_i) < \infty$ . If we write  $T_{ij}$  for  $\Delta^{-\frac{1}{2}}(w_i, w_j)P_{H(w_j)}U(w_i, w_j)|_{H(w_i)}$ , where  $P_{H(w_j)}$  is the orthogonal projection of  $K(w_j)$  onto  $H(w_j)$ , then we get

$$(\rho(f)\xi)_j = \sum_{k=1}^n f(w_j, w_k)T_{jk}\xi_k \quad \text{for } f \in \mathcal{A}, \xi \in H.$$

We can write  $\rho(f) = (f(w_j, w_k)T_{jk})$ . If  $(w_j, w_k) \notin G$ , we understand  $f(w_j, w_k)$  to be 0. Write  $T(u, v) = \Delta^{-\frac{1}{2}}(u, v)P_{H(u)}U(u, v)|_{H(v)}$  and  $Q(\rho) = \{(u, v): T(u, v) = 0\}$ . Then  $P \circ Q(\rho) \circ P \subseteq Q(\rho)$  and for every  $(u, v) \notin Q(\rho)$ ,  $(u, v) = (w_i, w_j)$  for some  $1 \leq i, j \leq n$ . Given  $(u, v) \notin Q(\rho)$ , we have  $(u, v) = (w_i, w_j)$  and for every  $u \leq w \leq v$ ,  $(u, w)$ ,  $(w, v)$  are not in  $Q(\rho)$  as  $P \circ Q(\rho) \circ P \subseteq Q(\rho)$ . Hence  $w = w_k$  for some  $1 \leq k \leq n$ . It follows that, if  $T(u, v) \neq 0$ , then the interval  $[u, v]$  is finite. We can now prove the following.

**THEOREM 5.1.** *A triangular AF algebra is RFD if and only if*

$$\bigcup \{J \times J: J \text{ is a finite interval in some equivalence class}\}$$

*is dense in G.*

*Proof.* First assume that the set  $\bigcup \{J \times J: J \subseteq [u]\}$  is a finite interval} is dense in  $G$  and fix  $f \in \mathcal{A}$ . Since  $f$  is a continuous function on  $G$ , there is some  $(u, v)$  in this dense set such that  $f(u, v) \neq 0$ . Since  $(u, v)$  is in this set, there is some finite interval in  $[u]$  containing both  $u$  and  $v$ . Hence the set  $\{w: u \leq w \leq v\}$  is finite, say of cardinality  $n$ . Define  $\rho: \mathcal{A} \rightarrow M_n(\mathbb{C})$  by

$$\rho(f) = (f(w_i, w_j))$$

where  $\{w: u \leq w \leq v\} = \{w_1, \dots, w_n\}$  as ordered sets. Then  $\rho$  is a contractive representation of  $\mathcal{A}$ . In fact, it is a compression of  $\pi_u$  to the semi-invariant subspace of  $L^2([u])$  associated with  $\{w: u \leq w \leq v\}$ . Since  $\rho(f) \neq 0$ , we see that for every  $f \in \mathcal{A}$  there is some contractive finite-dimensional representation  $\rho$  with  $\rho(f) \neq 0$ . This proves that  $\mathcal{A}$  is RFD.

For the other direction, assume that the set in the statement of the theorem is not dense. Then there is some non-zero  $f \in \mathcal{A}$  such that  $f(u, v) = 0$  whenever  $[u, v] (= \{w: u \leq w \leq v\})$  is finite. We shall show that for every contractive finite-dimensional representation  $\rho$  of  $\mathcal{A}$ ,  $\rho(f) = 0$ . So fix such  $\rho$  and write

$$(\rho(f)\xi)_j = \sum_{k=1}^n f(w_j, w_k)T_{jk}\xi_k$$

as in the discussion preceding the theorem. Here  $T_{jk} = T(w_j, w_k)$ . As was shown there, if  $T(w_j, w_k) \neq 0$  then  $[w_j, w_k]$  is finite. But then  $f(w_j, w_k) = 0$  as assumed above and, thus,  $\rho(f) = 0$ . This completes the proof.

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