

Riesz Wavelets and Multiresolution Structures

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ABSTRACT

Multiresolution structures are important in applications, but they are also useful for analyzing properties of associated wavelets. Given a nonorthogonal (multi-) wavelet in a Hilbert space, we construct a core subspace. Subsequently, the dilates of the core subspace defines a ladder of nested subspaces. Of fundamental importance are two questions: 1) when is the core subspace shift invariant; and if yes, then 2) when is the core subspace generated by shifts of a single vector, i.e. there exists a scaling vector. If the wavelet generates a Riesz basis then the answer to question 1) is yes if and only if the wavelet is a biorthogonal wavelet. Additionally, if the wavelet generates a tight frame of arbitrary frame constant, then the core subspace is shift invariant. Question 1) is still open in case the wavelet generates a non-tight frame.

We also present some known results to question 2) and provide some preliminary improvements. Our analysis here arises from investigating the dimension function and the multiplicity function of a wavelet. These two functions agree if the wavelet is orthogonal.

Finally, we discuss how these questions are important for considering linear perturbation of wavelets. Utilizing the idea of the local commutant of a unitary system developed by Dai and Larson, we show that nearly all linear perturbations of two orthonormal wavelets form a Riesz wavelet. If in fact these wavelets correspond to a von Neumann algebra in the local commutant of a “base” wavelet, then the interpolated wavelet is biorthogonal. Moreover, we demonstrate that in this case the interpolated wavelets have a scaling vector if the base wavelet has a scaling vector.

Keywords: wavelet, Riesz basis, multiresolution, perturbation.

1. INTRODUCTION

Multiresolution Analysis plays a key role in the theory of wavelets. Indeed, the classical construction of wavelets arises from multiresolution analyses. However, Journè produced an example of a wavelet which did not arise from a multiresolution analysis; a more general concept is needed.

Several people have worked on the concept of a *multiresolution structure* in various forms. Benedetto and Li¹ studied frame multiresolution analysis, which requires the low frequency space to be principally shift invariant. Papadakis² studied generalized frame multiresolution analysis (GFMRA), which requires that the low frequency space be shift invariant along with the existence of a frame generated by shifts of several (possibly infinitely many) vectors. Baggett, Medina and Merrill³ introduced the notion of a generalized multiresolution analysis, where the low frequency space is only required to be shift invariant. As it turns out, the GMRA’s of Baggett et. al. and GFMRA’s of Papadakis are equivalent on $L^2(\mathbb{R}^d)$. Moreover, both have demonstrated that filter banks can be constructed from GFMRA’s.^{4,5} Our focus will be on the GMRA concept.

Define on $L^2(\mathbb{R}^d)$ the dilation operator $D := D_A f(x) = |\det A|^{1/2} f(Ax)$ and the translation operator $T_z f(x) = f(x - z)$, where A is a $d \times d$ integral dilation matrix and $z \in \mathbb{Z}^d$. Note that since A preserves the integer lattice, we have the commutation relation $T_z D = D T_{Az}$.

Definition 1. A *generalized multiresolution analysis* (GMRA) is a sequence of closed, nested subspaces $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$ of $L^2(\mathbb{R}^d)$ which has the following properties:

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1. $\cup V_j$ is dense in $L^2(\mathbb{R}^d)$; $\cap V_j = \{0\}$.
2. $DV_j = V_{j+1}$.
3. $T_z V_0 = V_0$ for all $z \in \mathbb{Z}^d$.

The space V_0 is called the core space, or the low frequency space. It is well known that these conditions are significant in the construction of filter banks, though historically the shift invariance of V_0 is replaced by the much stronger condition that there exists a Riesz basis of V_0 given by the shifts of a finite number of vectors.^{6,7}

Given an orthonormal wavelet Ψ , define the subspaces

$$V_j = V_j(\Psi) := \overline{\text{span}}\{D^n T_z \psi : n < j, z \in \mathbb{Z}^d, \psi \in \Psi\}. \quad (1)$$

It is not hard to show that these spaces define a GMRA.³ Also shown³ is when a GMRA gives rise to an orthonormal wavelet; we shall use that idea in section 3. With this in mind, we announce the first result of the present paper.

Theorem 1. Let $\Psi = \{\psi_1, \dots, \psi_r\}$ be a Riesz wavelet. The following are equivalent:

1. Ψ is a biorthogonal wavelet;
2. the subspaces $V_j := \overline{\text{span}}\{D^n T_z \psi : n < j, z \in \mathbb{Z}^d, \psi \in \Psi\}$ form a GMRA of $L^2(\mathbb{R}^d)$;
3. there exists an orthonormal wavelet $\Phi = \{\phi_1, \dots, \phi_r\}$ which generates the same GMRA as Ψ .

A second concept that will be of importance to us is the local commutant of a unitary system.⁸ By a unitary system \mathcal{U} (on the Hilbert space H) we mean a collection of unitary operators which contains the identity; note that we do not require any algebraic structure on the set \mathcal{U} . If A is a bounded operator on H and $x \in H$, then A is in the local commutant of \mathcal{U} at x if $AUx = UAx$ for all $U \in \mathcal{U}$. The collection of all such operators A is denoted by $\mathcal{C}_x(\mathcal{U})$,

$$\mathcal{C}_x(\mathcal{U}) = \{A \in B(H) : AUx = UAx \ \forall U \in \mathcal{U}\}. \quad (2)$$

It should be noted that like \mathcal{U} , $\mathcal{C}_x(\mathcal{U})$ may lack certain algebraic structure, i.e. the local commutant is a weakly closed linear subspace of $B(H)$, but may not be closed under multiplication or the taking of adjoints.

The power of the local commutant is that it allows us to parametrize all orthonormal bases given by the action of the unitary system by the unitary operators in the local commutant at one such orthonormal basis generator. Likewise, Riesz bases are parametrized by bounded invertible operators in the local commutant.

For example, suppose $\{U\eta : U \in \mathcal{U}\}$ is an orthonormal basis and $\{U\zeta : U \in \mathcal{U}\}$ is a Riesz basis of H . Then define the operator $S : H \rightarrow H$ by $SU\eta = U\zeta$; this operator is well defined and invertible by the basis properties of both collections. Moreover, $S \in \mathcal{C}_\eta(\mathcal{U})$; this follows because $I \in \mathcal{U}$, whence $S\eta = \zeta$. Conversely, if S is a bounded invertible operator in the local commutant at η , then if $S\eta = \zeta$, $\{U\zeta : U \in \mathcal{U}\}$ is a Riesz basis for H .

The dual basis of $\{U\zeta : U \in \mathcal{U}\}$ has the form $\{U\tilde{\zeta} : U \in \mathcal{U}\}$ if and only if S^{*-1} is again in the local commutant at η . Indeed, the dual basis is $\{S^{*-1}U\eta : U \in \mathcal{U}\}$, since

$$\langle S^{*-1}V\eta, U\zeta \rangle = \langle V\eta, S^{-1}US\eta \rangle = \langle V\eta, S^{-1}SU\eta \rangle = \delta_{V,U}. \quad (3)$$

Whence, if S^{*-1} is in the local commutant at η , the dual basis is $\{US^{*-1}\eta : U \in \mathcal{U}\} = \{U\tilde{\zeta} : U \in \mathcal{U}\}$. Conversely, if the dual basis is of the form $\{U\tilde{\zeta} : U \in \mathcal{U}\} = \{S^{*-1}U\eta : U \in \mathcal{U}\}$, then clearly S^{*-1} is in the local commutant at η .

This paper is organized as follows: section 2 proves the equivalence of conditions 1. and 2. in Theorem 1. Section 3 proves the equivalence of conditions 1. and 3. in Theorem 1, using the wavelet multiplicity function and the wavelet dimension function. Much of the material in these sections is new and will not appear elsewhere. Section 4. presents some preliminary results regarding linear perturbation of orthonormal wavelets.

Remark. During the write up of this paper, we became aware of a paper by Kim, et. al.⁹ in which they prove the equivalence of conditions 1 and 2 in Theorem 1 for singly generated wavelets with a dilation factor of 2 in one dimension.

2. RIESZ WAVELETS

We shall say that $\Psi = \{\psi_1, \dots, \psi_r\}$ is a Riesz wavelet if the collection $\{D^n T_z \psi : n \in \mathbb{Z}, z \in \mathbb{Z}^d, \psi \in \Psi\}$ is a Riesz basis for $L^2(\mathbb{R}^d)$. Moreover, we shall say that Ψ is a biorthogonal wavelet if the dual basis is again a Riesz wavelet. Here we shall prove the equivalence of 1 and 2 above. For a Riesz wavelet Ψ , denote the space V_j above by $V_j(\Psi)$. Define a second sequence of subspaces $W_j(\Psi) = \overline{\text{span}}\{D^j T_z \psi : z \in \mathbb{Z}^d, \psi \in \Psi\}$. Note that $V_{j+1} = V_j \oplus W_j$, a non-orthogonal direct sum.

By the definition of $V_j(\Psi)$, it is clear that they satisfy the nested property and properties 1 and 2 in the definition of a GMRA. The only property we need to check is the shift invariance of $V_0(\Psi)$.

Proposition 1. Let Ψ be a Riesz wavelet. Then Ψ is a biorthogonal wavelet if and only if Ψ generates a GMRA.

Proof. The forward implication is apparently well known, but we include it here for completeness. Suppose Ψ is a biorthogonal wavelet, whose dual wavelet is $\tilde{\Psi} = \{\tilde{\psi}_1, \dots, \tilde{\psi}_r\}$. First observe that the orthogonal complement of $V_0(\Psi)$ is $M = \overline{\text{span}}\{D^n T_z \tilde{\psi} : n \geq 0, z \in \mathbb{Z}^d, \tilde{\psi} \in \tilde{\Psi}\}$. Also, since $T_z D^n = D T_{A^n z}$ and $A^n z \in \mathbb{Z}^d$, we have $T_z M = M$ whence $T_z V_0(\Psi) = V_0(\Psi)$ as required.

For the reverse implication, suppose Ψ generates a GMRA. Let Φ be any orthonormal wavelet with the same number of generators. Let S be the bounded invertible operator in the local commutant of Φ . As noted above, the theory of the local commutant yields that Ψ is a biorthogonal wavelet if and only if S^{*-1} is again in the local commutant at Φ . Also from the theory, we have that $S \in \{D\}'$, whence $S^{*-1} \in \{D\}'$; thus we only need to show that $S^{*-1} T_k \phi_i = T_k S^{*-1} \phi_i$ for all $\phi_i \in \Phi$ and for all $k \in \mathbb{Z}^d$.

We need the following observations: 1) $SV_0(\Phi) = V_0(\Psi)$ and 2) $T_z V_0(\Psi) = T_z V_0(\Psi)$, whence $S^{-1} T_z S V_0(\Phi) = V_0(\Phi)$. We make the following claim:

$$\langle S^* T_k S^{*-1} \phi_i, D^n T_l \phi_j \rangle = \delta_{0,n} \delta_{k,l} \delta_{i,j} = \langle T_k \phi_i, D^n T_l \phi_j \rangle. \quad (4)$$

By the above observation, for $n \leq 0$ we have

$$\langle S^* T_k S^{*-1} \phi_i, D^n T_l \phi_j \rangle = \langle \phi_i, S^{-1} T_k S D^n T_l \phi_j \rangle = 0. \quad (5)$$

For $n > 0$,

$$\langle S^* T_k S^{*-1} \phi_i, D^n T_l \phi_j \rangle = \langle \phi_i, S^{-1} T_k D^n T_l S \phi_j \rangle = \langle \phi_i, S^{-1} D^n T_{A^{n+k+l}} S \phi_j \rangle = \langle \phi_i, D^n T_{A^{n+k+l}} \phi_j \rangle = 0. \quad (6)$$

Finally, for $n = 0$, we have

$$\langle S^* T^k S^{*-1} \phi_i, T^l \phi_j \rangle = \langle \phi_i, S^{-1} T^{-k} S T^l \phi_j \rangle = \langle \phi_i, S^{-1} T^{l-k} S \phi_j \rangle = \langle \phi_i, \phi_j \rangle, \quad (7)$$

since S is in the local commutant at ϕ_j . It follows from the claim that $S^* T^l S^{*-1} \phi_i = \phi_i$, so that S^{*-1} is in the local commutant of the affine system at Φ , which establishes the statement. \square

3. THE SPECTRAL MULTIPLICITY OF BIORTHOGONAL WAVELETS

We now set out to prove the equivalence of items 2 and 3 in Theorem 1. Clearly 3 implies 2, since an orthonormal wavelet automatically has a shift invariant core space. It remains to prove the converse implication.

We begin by making several remarks. First, the result is well known, and not hard to prove, if we assume the wavelet Ψ is MRA, i.e. there exist scaling vectors $\phi_1, \dots, \phi_q \in V_0(\Psi)$ whose shifts form a Riesz basis for V_0 . Indeed, one can orthogonalize the shifts of the scaling vectors, then use the standard construction techniques for generating an orthonormal wavelet. The assumption that Ψ is MRA, however, is quite restrictive, since it is well known that many wavelets are not MRA, and moreover, most dilation matrices do not admit *any* MRA wavelets.

Second, if Ψ is a Riesz wavelet such that $V_0(\Psi)$ is shift invariant, then it follows from the commutation relation of D and T_z that $V_1(\Psi)$ is also shift invariant. Thus, if we take the orthogonal complement W'_0 of V_0 in V_1 , i.e.

$$V_1 = V_0 \oplus W'_0, \quad (8)$$

then this subspace is shift invariant, and its dilates form an orthogonal direct sum decomposition of $L^2(\mathbb{R}^d)$. Therefore, since W'_0 is shift invariant, it follows that there are vectors $\phi_1, \dots, \phi_k \in W'_0$ whose shifts form a tight frame for

W'_0 , whence $\{D^n T_z \phi_i : n \in \mathbb{Z}, z \in \mathbb{Z}^d, i = 1, \dots, k\}$ is a tight frame (indeed, semiorthogonal) wavelet which generates our original GMRA.

In light of this remark, fundamentally, our problem can be rephrased as: how “big” is the subspace W'_0 with respect to the operators $\{T_z : z \in \mathbb{Z}^d\}$? In other words, is W'_0 big enough to have generating vectors with linearly independent shifts? We shall show that the answer is yes using techniques from abstract harmonic analysis, in particular Stone’s theorem and the theory of spectral multiplicity.

If Ψ is a Riesz wavelet such that $V_0(\Psi)$ is shift invariant, then we have a unitary representation of \mathbb{Z}^d associated to Ψ given by the action of the shift operators on V_0 . This is called the core representation of Ψ . Additionally, by the commutation relation of D and T_z and the fact that $V_1(\Psi) = DV_0(\Psi)$, it follows that V_1 is also shift invariant. Let W'_0 be the shift invariant subspace as above. Note that $W_0(\Psi) = W'_0$ if and only if the wavelet Ψ is semiorthogonal (i.e. the dilates of W_0 are orthogonal). Note further that if $\Phi \subset W'_0$ is such that $\{T_z \phi : z \in \mathbb{Z}^d, \phi \in \Phi\}$ is an orthonormal basis of W'_0 , then Φ is an orthonormal wavelet.

Our analysis is as follows: we use Stone’s theorem for representations of locally compact Abelian groups along with the spectral multiplicity to get information about the “size” of the shift invariant subspace W'_0 . Indeed, if the multiplicity is constant, then there exist vectors in W'_0 whose translates form an orthonormal basis; these will be our orthonormal wavelets. In order to compute the multiplicity, we introduce the dimension function of Auscher.¹⁰ We shall establish that the multiplicity of the shifts on $V_0(\Psi)$ agrees with the dimension function, it will then be a straight forward computation of the multiplicity of W'_0 .

Theorem (Stone). Let G be a locally compact Abelian group, and let $\pi(g)$ be a unitary representation of G . There exists a projection valued measure (p.v.m.) p on the (Pontrjagin) dual group \widehat{G} such that

$$\pi(g) = \int_{\widehat{G}} g(\chi) dp(\chi). \quad (9)$$

Theorem (Stone-Mackey). Let p be a projection valued measure on a measurable space $\{\mathcal{S}, \mathcal{B}\}$, with separable Hilbert space H . There exists a finite measure ν and a multiplicity function $m : \mathcal{S} \rightarrow \{0, 1, 2, \dots, \infty\}$ such that:

1. ν is equivalent to p ,
2. there exists a unitary operator

$$U : H \rightarrow \bigoplus_{j=1}^{\infty} L^2(E_j, \nu|_{E_j}, \mathbb{C}^j) \oplus L^2(E_{\infty}, \nu|_{E_{\infty}}, l^2(\mathbb{Z})) \quad (10)$$

such that U intertwines p with the canonical projection valued measure. Here, $E_j = m^{-1}(j)$. Furthermore, the measure ν and multiplicity function m are unique, up to measure class and modulo null sets, respectively.

We wish to apply these theorems to the core representation of a wavelet. In this setting, the group is the integer lattice \mathbb{Z}^d ; the dual group is the d -torus, which we shall associate to $[-\pi, \pi)^d$ in the standard way. The unitary operator described intertwines the p.v.m. p with the canonical p.v.m. (i.e. multiplication by characteristic functions). In other words, the unitary U changes shifts by integers into multiplication by the function $g(\chi)$, a unimodular function.

In general the unitary operator U is not computable. But in our case where the representation is given by shifts, we know this operator quite well: it is the Fourier transform (with a bit of a twist). One consequence of this is that the measure ν is actually the restriction of Lebesgue (Haar) measure to E_1 .

Furthermore, since the measure associated to the left regular representation of a group G is Haar (Lebesgue) measure, from our perspective m describes the core representation as a subrepresentation of a multiple of the regular representation. For example, the regular representation has multiplicity identically 1; the regular representation can also be described as having an orthonormal basis of shifts under the representation. More importantly, the multiplicity function measures the size of the representation; if $m(\xi) \equiv r$, then there exist r vectors in the representation space whose shifts form an orthonormal basis.

Let us first consider the multiplicity theory of a refinable space. Suppose V is a closed linear subspace of $L^2(\mathbb{R})$ which is invariant under shifts by the integer lattice \mathbb{Z}^d . Suppose further that V is refinable with respect to the dilation matrix A , i.e. $V \subset D_A V$. For the remainder of the paper B will denote the matrix A^T . The following is a result of Baggett and Merrill.¹¹

Theorem (Baggett-Merrill). If V is refinable as above, and $m(\xi)$ is the multiplicity function of V , then $m(\xi)$ satisfies the inequality

$$m(\xi) \leq \sum_{B\eta \equiv \xi \pmod{2\pi\mathbb{Z}^d}} m(\eta). \quad (11)$$

Furthermore, if we add the assumption that $m(\xi)$ is finite almost everywhere, we have the equality

$$m(\xi) + m_1(\xi) = \sum_{B\eta \equiv \xi \pmod{2\pi\mathbb{Z}^d}} m(\eta). \quad (12)$$

If we define the closed shift invariant subspace W to be the orthogonal complement of V in DV , i.e. $V \oplus W = DV$, then the function $m_1(\xi)$ describes the structure of W as above.

Since we have established that the structure of the core representation is given by the multiplicity function, it would be nice to be able to compute it. We actually can compute it explicitly for biorthogonal wavelets, utilizing the dimension function of Auscher.¹⁰

Theorem 2. If $\Psi = \{\psi_1, \dots, \psi_r\}$, $\tilde{\Psi} = \{\tilde{\psi}_1, \dots, \tilde{\psi}_r\}$ are biorthogonal wavelets, then the multiplicity function $m(\xi)$ associated to the core representation on $V_0(\Psi)$ is given by

$$m(\xi) = D_{\Psi, \tilde{\Psi}}(\xi) = \sum_{i=1}^r \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^d} \hat{\psi}_i(B^j(\xi + 2\pi k)) \overline{\hat{\psi}_i(B^j(\xi + 2\pi k))}. \quad (13)$$

The significance for us here is that the dimension function, which is the multiplicity function, satisfies the consistency equation

$$D_{\Psi, \tilde{\Psi}}(x) + r = \sum_{\substack{y \in [-\pi, \pi]^d \\ By \equiv x \pmod{2\pi\mathbb{Z}^d}}} D_{\Psi, \tilde{\Psi}}(y). \quad (14)$$

The matrix B has integral entries, so B defines a homomorphism of \mathbb{Z}^d into itself. If we consider the quotient group $\Gamma = \mathbb{Z}^d / B\mathbb{Z}^d$, we get a correspondence between its elements and the y 's in the above sum. In fact, we have that for every element p in the quotient group Γ , there is a representative of $q \in \mathbb{Z}^d$ of $p \in \Gamma$ such that $By = x + 2\pi q$.

Thus we have

$$\sum_{\substack{y \in [-\pi, \pi]^d \\ By \equiv x \pmod{2\pi\mathbb{Z}^d}}} D_{\Psi, \tilde{\Psi}}(y) = \sum_{q \in \Gamma} D_{\Psi, \tilde{\Psi}}(B^{-1}(x + 2\pi q)) \quad (15)$$

$$= \sum_{q \in \Gamma} \sum_{i=1}^r \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^d} \hat{\psi}_i(B^j(B^{-1}(x + 2\pi q) + 2\pi k)) \overline{\hat{\psi}_i(B^j(B^{-1}(x + 2\pi q) + 2\pi k))} \quad (16)$$

$$= \sum_{q \in \Gamma} \sum_{i=1}^r \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^d} \hat{\psi}_i(B^{j-1}(x + 2\pi q + 2\pi Bk)) \overline{\hat{\psi}_i(B^{j-1}(x + 2\pi q + 2\pi Bk))} \quad (17)$$

$$= \sum_{i=1}^r \sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}^d} \hat{\psi}_i(B^j(x + 2\pi l)) \overline{\hat{\psi}_i(B^j(x + 2\pi l))} \quad (18)$$

$$= \sum_{i=1}^r \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^d} \hat{\psi}_i(B^j(x + 2\pi k)) \overline{\hat{\psi}_i(B^j(x + 2\pi k))} + \sum_{i=1}^r \sum_{l \in \mathbb{Z}^d} \hat{\psi}_i(x + 2\pi l) \overline{\hat{\psi}_i(x + 2\pi l)} \quad (19)$$

$$= D_{\Psi, \tilde{\Psi}}(x) + r \quad (20)$$

since the shifts of Ψ and $\tilde{\Psi}$ are biorthogonal. Therefore, the multiplicity function $m_1(\xi)$ of W'_0 is $m_1(\xi) \equiv r$. Thus, by the above discussion, there exists $\Phi = \{\phi_1, \dots, \phi_r\} \subset W'_0$ whose translates form an orthonormal basis. This Γ is in fact an orthonormal wavelet which clearly generates the GMRA we started with.

The above discussion completes the proof of Theorem 1. The remainder of this section is the proof of Theorem 2. It may actually be skipped without loss of continuity.

Remark. After we obtained Theorem 2, we learned from Ron and Shen that it was also proven independently in their preprint¹² using different methods.

We begin by decomposing the core representation into orthogonal cyclic subrepresentations. Cyclic representations have multiplicity functions that take on only the values 0 and 1, and the multiplicity function of a direct sum of representations is the sum of the individual multiplicity functions. Let

$$\psi_{i,j} = D^{-j}\psi_i, \quad i = 1, \dots, r, \quad j \geq 1, \quad (21)$$

$g_{1,1} = \psi_{1,1}$, and let $Y_{1,1}$ be the cyclic subspace generated by $g_{1,1}$, i.e. $\overline{\text{span}}\{T_z g_{1,1} : z \in \mathbb{Z}^d\}$. Let $g_{2,1} = \psi_{2,1} - P_{Y_{1,1}}\psi_{2,1}$, and let $Y_{2,1} = \overline{\text{span}}\{T_z g_{2,1} : z \in \mathbb{Z}^d\}$. Recursively define

$$g_{i,j} = \psi_{i,j} - \sum_{(k,l) < (i,j)} P_{Y_{k,l}}\psi_{i,j}, \quad (22)$$

and $Y_{i,j}$ to be the cyclic subspace generated by $g_{i,j}$. By definition, each $Y_{i,j}$ determines a cyclic subrepresentation, with cyclic vector $g_{i,j}$.

Notation. For convenience and ease of notation we define $(k,l) < (i,j)$ if either $l < j$ or $l = j$ and $k < i$.

Lemma 1. With the above notation,

1. $V_0 = \bigoplus_{i=1}^r \bigoplus_{j>0} Y_{i,j}$,
2. $m = \sum_{i=1}^r \sum_{j=1}^{\infty} m_{i,j}$, where $m_{i,j}$ is the multiplicity function of the cyclic representation on $Y_{i,j}$.

Proof. By definition, the $Y_{i,j}$'s are orthogonal and are subspaces of V_0 . Since the shifts of the $\psi_{i,j}$'s span V_0 , it suffices to show that they are contained in this direct sum. But note that ψ_1 is in $Y_{1,1}$, and then $\psi_{2,1}$ can be written as $g_{2,1} + f_{2,1}$, where $f_{2,1} \in Y_{1,1}$, since $g_{2,1}$ is obtained by a projection. By the recursive definition of the $g_{i,j}$'s, we get that $\psi_{i,j}$ is in the direct sum, and item 1 is established.

Item 2 follows from the general fact that the multiplicity function for a representation is the sum of the multiplicity functions for orthogonal subrepresentations. \square

Since $g_{i,j}$ is a cyclic vector, it generates a positive definite function, $p_{i,j}(l) = \langle T^l g_{i,j}, g_{i,j} \rangle$. By Bochner's theorem, there exists a measure $\mu_{i,j}$ whose Fourier-Stieltjes transform is $p_{i,j}$. Since μ is absolutely continuous with respect to Lebesgue measure, the measure $\mu_{i,j}$ is also. Let $h_{i,j}$ be the Radon-Nikodym derivative of $\mu_{i,j}$ with respect to Lebesgue measure. Since the subrepresentation on $Y_{i,j}$ is cyclic, $m_{i,j}$ takes on only the values 0 and 1; in fact, $m_{i,j} = \chi_{\text{supp}(h_{i,j})}$. Furthermore, since $m = \sum_{i=1}^r \sum_{j=1}^{\infty} m_{i,j}$ we have then that $m = \sum_{i=1}^r \sum_{j=1}^{\infty} \chi_{\text{supp}(h_{i,j})}$.

Lemma 2. Let $h_{i,j}$ be as above. Then:

$$h_{i,j}(\xi) = \|\widehat{g_{i,j}}(\xi)\|^2 \quad (23)$$

Proof. We have:

$$\int_{[-\pi, \pi]^d} e^{-in\xi} h_{i,j}(\xi) d\lambda = \int_{[-\pi, \pi]^d} e^{-in\xi} d\mu_{i,j} \quad (24)$$

$$= \widehat{\mu_{i,j}}(n) \quad (25)$$

$$= p_{i,j}(n) \quad (26)$$

$$= \langle T^n g_{i,j}, g_{i,j} \rangle \quad (27)$$

$$= \int_{\mathbb{R}^d} e^{-in\xi} \widehat{g_{i,j}}(\xi) \overline{\widehat{g_{i,j}}(\xi)} d\lambda \quad (28)$$

$$= \int_{[-\pi, \pi]^d} e^{-in\xi} \|\widehat{g_{i,j}}(\xi)\|^2 d\lambda. \quad (29)$$

□

By definition,

$$g_{i,j}(x) = \psi_{i,j}(x) - w_{i,j}(x) \quad (30)$$

where $w_{i,j}$ is the unique element in $\bigoplus_{(k,l) < (i,j)} Y_{k,l}$ such that $g_{i,j} \perp \bigoplus_{(k,l) < (i,j)} Y_{k,l}$. Additionally, since $w_{i,j}$ can be expressed in terms of the translates of the $g_{k,l}$'s, by taking the Fourier Transform of both sides of 30, we get:

$$\widehat{g}_{i,j}(\xi) = \widehat{\psi}_{i,j}(\xi) - \sum_{(k,l) < (i,j)} \eta_{i,j,k,l}(\xi) \widehat{g}_{k,l}(\xi) \quad (31)$$

where the $\eta_{i,j,k,l}$ are measurable functions and periodic with respect to the $2\pi\mathbb{Z}^d$ lattice. For convenience, given a function $f \in L^2(\mathbb{R}^d)$ we shall define for almost every $\xi \in \mathbb{R}^d$ a $l^2(\mathbb{Z}^d)$ sequence $\vec{f}(\xi)$ by

$$\vec{f}(\xi)[k] = f(\xi + 2\pi k), \quad k \in \mathbb{Z}^d. \quad (32)$$

Lemma 3. If $\eta_{i,j,k,l}$ is as above, then

$$\eta_{i,j,k,l}(\xi) = \frac{\langle \vec{\psi}_{i,j}(\xi), \vec{g}_{k,l}(\xi) \rangle}{\|\vec{g}_{k,l}(\xi)\|^2} \quad (33)$$

where this is interpreted to be 0 when the denominator is 0.

Proof. First notice that since $\eta_{i,j,k,l} \widehat{g}_{k,l} \in L^2(\mathbb{R})$, $\eta_{i,j,k,l}(\xi) \|\vec{g}_{k,l}(\xi)\|^2 \in L^1([0, 2\pi])$. Furthermore, $\eta_{i,j,k,l} \widehat{g}_{k,l} \in \widehat{Y}_{k,l}$, indeed, $\eta_{i,j,k,l} \widehat{g}_{k,l}$ is the function such that $\widehat{\psi}_{i,j} - \eta_{i,j,k,l} \widehat{g}_{k,l} \in \widehat{Y}_{k,l}^\perp$. Hence, $\langle \eta_{i,j,k,l} \widehat{g}_{k,l}, e^{-in \cdot} \widehat{g}_{k,l} \rangle = \langle \widehat{\psi}_{i,j}, e^{-in \cdot} \widehat{g}_{k,l} \rangle$. Hence,

$$\int_{[-\pi, \pi]^d} \eta_{i,j,k,l}(\xi) \|\vec{g}_{k,l}(\xi)\|^2 e^{-in\xi} d\lambda = \int_{\mathbb{R}^d} \eta_{i,j,k,l}(\xi) \widehat{g}_{k,l}(\xi) \overline{\widehat{g}_{k,l}(\xi)} e^{-in\xi} d\lambda \quad (34)$$

$$= \langle \eta_{i,j,k,l} \widehat{g}_{k,l}, e^{-in \cdot} \widehat{g}_{k,l} \rangle \quad (35)$$

$$= \langle \widehat{\psi}_{i,j}, e^{-in \cdot} \widehat{g}_{k,l} \rangle \quad (36)$$

$$= \int_{\mathbb{R}^d} \widehat{\psi}_{i,j}(\xi) \overline{\widehat{g}_{k,l}(\xi)} e^{-in\xi} d\lambda \quad (37)$$

$$= \int_{[-\pi, \pi]^d} \langle \vec{\psi}_{i,j}(\xi), \vec{g}_{k,l}(\xi) \rangle e^{-in\xi} d\lambda. \quad (38)$$

□

This gives us that

$$\widehat{g}_{i,j}(\xi) = \widehat{\psi}_{i,j}(\xi) - \sum_{(k,l) < (i,j)} \frac{\langle \vec{\psi}_{i,j}(\xi), \vec{g}_{k,l}(\xi) \rangle}{\|\vec{g}_{k,l}(\xi)\|^2} \widehat{g}_{k,l}(\xi). \quad (39)$$

Since the inner product in equation 39 is invariant under 2π translations, we have:

$$\vec{g}_{i,j}(\xi) = \vec{\psi}_{i,j}(\xi) - \sum_{(k,l) < (i,j)} \langle \vec{\psi}_{i,j}(\xi), \vec{u}_{k,l}(\xi) \rangle \vec{u}_{k,l}(\xi) \quad (40)$$

where

$$\vec{u}_{k,l}(\xi) = \frac{\vec{g}_{k,l}(\xi)}{\|\vec{g}_{k,l}(\xi)\|_2} \quad (41)$$

if the norm is non-zero.

As we have mentioned above, the multiplicity function is the sum of the multiplicity functions for each cyclic subspace $Y_{i,j}$, each of which is the characteristic function of the support of $h_{i,j}$. Hence, by equation 23, the multiplicity function is precisely the number of non-zero sequences $\widehat{g_{i,j}}(\xi)$.

Now, let us examine more closely equation 40. Note that $g_{1,1} = \psi_{1,1}$, so $\widehat{g_{1,1}}(\xi) = \widehat{\psi_{1,1}}(\xi)$ for almost all ξ , and $u_{1,1}$ is the normalization of that vector. Furthermore, $\widehat{g_{1,2}}(\xi) = \widehat{\psi_{1,2}}(\xi) - \langle \widehat{\psi_{1,2}}(\xi), u_{1,1}(\xi) \rangle u_{1,1}(\xi)$ which is the Gram-Schmidt orthogonalization of $\widehat{\psi_{1,1}}(\xi)$ and $\widehat{\psi_{1,2}}(\xi)$, with $\vec{u}_{1,1}(\xi)$ and $\vec{u}_{1,2}(\xi)$ being normalized. By the recursive definition of the $g_{i,j}$'s, equation 40 is actually the Gram-Schmidt orthogonalization of the $\widehat{\psi_{i,j}}(\xi)$'s. Hence, $h_{i,j}(\xi) = 0$ if and only if $\widehat{\psi_{i,j}}(\xi)$ is in the linear span of the previous $\widehat{\psi_{i,j}}(\xi)$'s. Therefore, $m(\xi)$ is the number of linearly independent vectors in the collection $\{\widehat{\psi_{i,j}}(\xi)\}$,

$$m(\xi) = \dim\{\widehat{\psi_{i,j}} : i = 1, \dots, r, j > 0\}. \quad (42)$$

Now, turning our attention back to the dimension function, note that $D_{\Psi, \vec{\Psi}}$ is a sum of inner products. Indeed, define the $l^2(\mathbb{Z}^d)$ sequence $\vec{\phi}_{i,j}(\xi)$ by

$$\vec{\phi}_{i,j}(\xi)[k] = \widehat{\psi_i}(B^j(\xi + 2\pi k)) \quad (43)$$

and likewise for $\vec{\phi}_{i,j}(\xi)$. It is shown in¹³ that the dimension function $D_{\Psi, \vec{\Psi}}$ is the dimension of the subspace of $l^2(\mathbb{Z}^d)$ spanned by $\{\vec{\phi}_{i,j}(\xi) : i = 1, \dots, r, j > 0\}$.

We have that both the dimension function and the multiplicity function describe the dimension of some subspace of $l^2(\mathbb{Z}^d)$. Note, however, that

$$\vec{\phi}_{i,j}(\xi)[k] = \widehat{\psi_i}(B^j(\xi + 2\pi k)) = |\det B|^{-j/2} \widehat{D^j \psi_i}(\xi + 2\pi k) = |\det B|^{-j/2} \widehat{\psi_{i,j}}(\xi + 2\pi k) = |\det B|^{-j/2} \widehat{\psi_{i,j}}(\xi)[k] \quad (44)$$

whence,

$$m(\xi) = \dim\{\vec{\psi}_{i,j}(\xi) : i = 1, \dots, r, j > 0\} = \dim\{\vec{\phi}_{i,j}(\xi) : i = 1, \dots, r, j > 0\} = D_{\Psi, \vec{\Psi}}(\xi). \quad (45)$$

4. LINEAR PERTURBATION OF ORTHONORMAL WAVELETS

For the remainder of the paper, we shall restrict our attention to wavelets with a single generator. The expansive matrix and spatial dimension remain A and d , respectively, as above. We make this restriction for ease of notation and to allow for more transparent proofs of our results. Again, these results are preliminary.

Given two orthonormal wavelets ψ and η , there exists a unitary operator $V \in \mathcal{C}_\psi(D, T)$ such that $V\psi = \eta$. Here $\mathcal{C}_\psi(D, T)$ denotes the local commutant of the affine unitary system $\{D^n T_z : n \in \mathbb{Z}, z \in \mathbb{Z}^d\}$. The unitary V is called the interpolation unitary between ψ and η .

Define the *linear perturbation* of ψ and η by $\psi_t = \psi + t\eta$. It's not hard to show that ψ_t is almost always a Riesz wavelet. Indeed, define the operator $V_t = I + tV$, which is in the local commutant $\mathcal{C}_\psi(D, T)$. Note that $V_t \psi = \psi_t$, whence, ψ_t is a Riesz wavelet if V_t is invertible. By the spectral mapping theorem, the spectrum of V_t is precisely the spectrum of V under the mapping $1 + t$. So, since the spectrum of V lies on the unit circle, for $t \notin \{-1, 1\}$, the spectrum of V_t does *not* contain 0 and hence is invertible.¹⁴

We ask the questions: when is ψ_t biorthogonal? If so, when is ψ_t associated to an MRA? For the first question, by using the local commutant, ψ_t is biorthogonal if the operator $V_t^{*-1} = (I + tV)^{* -1}$ is in $\mathcal{C}_\psi(D, T)$. By theorem 1, ψ_t is biorthogonal if $V_0(\psi_t)$ is shift invariant. It turns out that neither of these conditions are necessarily true (see our remarks below), nor are they easily checked. Additionally, the interpolated wavelet will be associated to an MRA if

$$\dim\{\widehat{D^{-j} \psi_t}(\xi) : j > 0\} = 1 \quad (46)$$

almost everywhere, another condition which is not easily checked.

Therefore, we present the following theorem, which yields sufficient conditions to answer both questions positively.

Theorem 3. Suppose ψ and η are orthonormal wavelets such that the interpolation unitary V between ψ and η has the property that V^n is in the local commutant at ψ for all integers n . Then for $t \notin \{-1, 1\}$, $\psi_t = \psi + t\eta$ is a biorthogonal wavelet. Moreover, if ψ is a MRA wavelet, then so is ψ_t .

The proof uses techniques from operator algebras and will appear elsewhere. It is a technical modification of the following result which appears in.¹⁵

Theorem (Weber). Let U be a unitary operator such that $U^n \in \mathcal{C}_\psi(D, T)$ for all $n \in \mathbb{Z}$. Then, we have a sequence of wavelets, $U^n\psi$, and they are all core equivalent.

We conclude with several remarks about this result. The first is that the assumption in Theorem 3. may seem strong, and indeed it is. However, many examples exist where linear perturbation results in non-biorthogonal Riesz wavelets. Indeed, see the work of Chui and Shi,¹⁶ Zalik,¹⁷ or Example A.6. in.⁸

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